A Convex Optimization Approach to Model (In)validation of Switched ARX Systems with Unknown Switches

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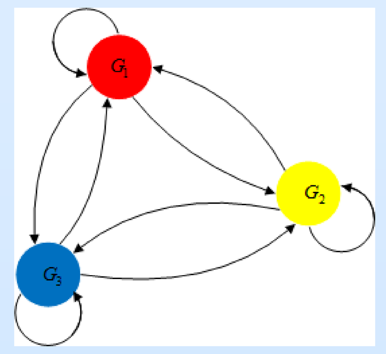
Motivation

Switched ARX System Identification

Air Conditioner

- **Cooling**
- **Power Saving**
- **Heating**



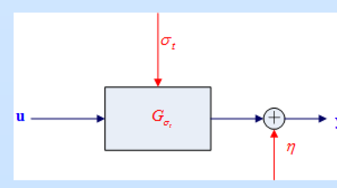
Given:

Order of Submodels n_a ; Number of submodels n_s ;

A priori bound on noise ; Experimental data $\{\mathbf{u}_t, \mathbf{y}_t\}_{t=0}^{T-1}$

Find:

A piecewise linear affine model such that



$$\xi_t = \sum_{k=1}^{n_a} \mathbf{A}_k(\sigma_t)\xi_{t-k} + \sum_{k=1}^{n_c} \mathbf{C}_k(\sigma_t)\mathbf{u}_{t-k} + \mathbf{f}(\sigma_t)$$
$$\mathbf{y}_t = \xi_t + \eta_t$$

Continuous Variables: η_t
Integer Variables: σ_t

Heuristics
Optimization
Probabilistic Priors
Convex Relaxation

NP hard

Solutions

Reliable ?

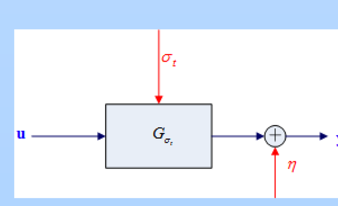
Model Invalidation Problem

Problem Statement
Model Invalidation of Switched ARX System

Given:

Parameters of n_s submodels of a hybrid system :

$$G_i = \left\{ \left[\mathbf{A}_k(i)\right]_{k=1}^{n_a}, \left[\mathbf{C}_k(i)\right]_{k=1}^{n_c}, \mathbf{f}(i) \right\}, i = 1, \cdots, n_s$$



A priori bound ε on noise $\|\eta_t\|_\infty = \varepsilon$

Experimental data $\{\mathbf{u}_t, \mathbf{y}_t\}_{t=0}^{T-1}$

Determine:

Whether the consistency set $\Gamma(\eta,\sigma)$ is nonempty, where

$$\Gamma(\eta,\sigma) = \left\{ \begin{array}{c} \|\eta_t\|_\infty \le \varepsilon,\ \sigma_t \in \mathbf{N}_s \\ \sum_{k=1}^{n_a} \mathbf{A}_k(\sigma_t)(\mathbf{y}_{t-k} - \eta_{t-k}) - (\mathbf{y}_t - \eta_t) + \sum_{k=1}^{n_c} \mathbf{C}_k(\sigma_t)\mathbf{u}_{t-k} + \mathbf{f}(\sigma_t), \forall t \in [0, T-1] \\ \doteq \mathbf{g}_{\sigma_t,t} - \mathbf{h}_{\sigma_t}\eta_{t:t-n_a} = 0 \end{array} \right\}$$

$\{G_1, \cdots, G_{n_s}\}$ not invalidated at t $\Leftrightarrow$ $\exists \sigma_t \in \mathbf{N}_s \doteq \{1, \cdots, n_s\}, \|\eta_{t:t-n_a}\|_\infty \le \varepsilon$, such that $\mathbf{g}_{\sigma_t,t} - \mathbf{h}_{\sigma_t}\eta_{t:t-n_a} = 0$

$\exists \sigma_t \in \mathbf{N}_s \doteq \{1, \cdots, n_s\}, \|\eta_{t:t-n_a}\|_\infty \le \varepsilon$, such that $\mathbf{g}_{\sigma_t,t} - \mathbf{h}_{\sigma_t}\eta_{t:t-n_a} = 0$

G_1: $s_{1,t}(\mathbf{g}_{1,t} - \mathbf{h}_1\eta_{t:t-n_a}) = 0$

G_2: $s_{2,t}(\mathbf{g}_{2,t} - \mathbf{h}_2\eta_{t:t-n_a}) = 0$

G_3: $s_{3,t}(\mathbf{g}_{3,t} - \mathbf{h}_3\eta_{t:t-n_a}) = 0$

$s_{i,t}(\mathbf{g}_{i,t} - \mathbf{h}_i\eta_{t:t-n_a}) = 0$
$s_{i,t} \in \{0,1\},\ \sum_{i=1}^{n_s} s_{i,t} = 1$
$\|\eta_{t:t-n_a}\|_\infty \le \varepsilon$
is feasible.

Invalidated $\{s_{i,t}, \eta_{t:t-n_a}\} = \Phi$

Not Invalidated $\{s_{i,t}, \eta_{t:t-n_a}\} \ne \Phi$

NONCONVEX ?

Sparsification Based Approach

The model is not invalidated $\Leftrightarrow$

$s_{i,t}(\mathbf{g}_{i,t} - \mathbf{h}_i\eta_{t:t+n_a}) = 0$
$s_{i,t} \in \{0,1\},\ \sum_{i=1}^{n_s} s_{i,t} = 1$
$\|\eta_{t:t+n_a}\|_\infty \le \varepsilon$
is feasible.

$\Leftrightarrow$

$s_{i,t}\mathbf{g}_{i,t} - \mathbf{h}_i\eta_{i,t:t+n_a} = 0$
$s_{i,t} \in \{0,1\}$, and $\sum_{i=1}^{n_s} s_{i,t} = 1$
$\|\eta_{i,t:t+n_a}\|_\infty \le s_{i,t}\varepsilon$, and $\sum_{i=1}^{n_s}\eta_{i,t:t+n_a} = \eta_{t:t+n_a}$
is feasible.

$$\mathbf{s} = \begin{array}{c} \begin{array}{cccc} 0 & 1 & \cdots & T-1 \end{array} \\ \begin{bmatrix} s_{1,0} & s_{1,1} & \cdots & s_{1,T-1} \\ s_{2,0} & s_{2,2} & \cdots & s_{2,T-1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n_s,0} & s_{n_s,2} & \cdots & s_{n_s,T-1} \end{bmatrix} \end{array}$$

$\Updownarrow$

$s_{i,t}\mathbf{g}_{i,t} - \mathbf{h}_i\eta_{i,t:t+n_a} = 0$
$0 \le s_{i,t} \le 1,\ \sum_{i=1}^{n_s} s_{i,t} = 1,\quad \min\|\mathbf{s}\|_0 = T$
$\|\eta_{i,t:t+n_a}\|_\infty \le s_{i,t}\varepsilon$, and $\sum_{i=1}^{n_s}\eta_{i,t:t+n_a} = \eta_{t:t+n_a}$
is feasible.

ℓ_1 Relaxation of cardinality $\|\cdot\|_0$
Re-weighted heuristic

Sparsification Based (In)validation Certificates

Algorithm

Solve iteratively

$$\begin{aligned} &\min_{s,\eta} \quad \sum_{i,t} w_{i,t}^{(k)} s_{i,t} \\ &\text{subject to} \\ &s_{i,t}\mathbf{g}_{i,t} - \mathbf{h}_i\eta_{i,t:t-n_a} = 0 \ \forall i,t \\ &0 \le s_{i,t} \le 1 \ \forall i,t \qquad (1) \\ &\sum_{i=1}^{n_s} s_{i,t} = 1 \ \forall t \\ &\|\eta_{i,t:t-n_a}\|_\infty \le s_{i,t}\varepsilon \quad \forall i,t \\ &\sum_{i=1}^{n_s}\eta_{i,t:t-n_a} = \eta_{t:t-n_a} \quad \forall t \end{aligned}$$

Update the weight by

$$w_{i,t}^{(k+1)} = \left(s_{i,t}^{(k)} + \delta\right)^{-1}, w_{i,t}^{(0)} = [1, 1, \cdots, 1]^T$$

Analysis

(1) is infeasible $\Rightarrow$ **invalidated**

(1) is feasible, $s_{i,t}^* \in \{0,1\}$ $\Rightarrow$ **not invalidated**

$$\mathbf{s} = \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & \boxed{1} & 0 & \boxed{1} \\ 0 & 0 & 0 & \boxed{1} & 0 \end{bmatrix}$$

$G_1 \to G_2 \to G_2 \to G_3 \to G_2$

(1) is feasible, $s_{i,t}^* \notin \{0,1\}$?

Moments Based Approach

The model is not invalidated $\Leftrightarrow$

$s_{i,t}(\mathbf{g}_{i,t} - \mathbf{h}_i\eta_{t:t+n_a}) = 0$
$s_{i,t} \in \{0,1\},\ \sum_{i=1}^{n_s} s_{i,t} = 1$
$\|\eta_{t:t+n_a}\|_\infty \le \varepsilon$
is feasible.

$\Leftrightarrow$

$$\begin{aligned} p^* = &\min_{s_{i,t},\eta_{t:t-n_a}} \sum_{t=0}^{T-1}\sum_{i=1}^{n_s} s_{i,t}^2\left(\mathbf{g}_{i,t} - \mathbf{h}_i\eta_{t:t-n_a}\right)^2 \\ &\text{subject to} \\ &s_{i,t}^2 = s_{i,t},\ \sum_{i=1}^{n_s} s_{i,t} = 1 \quad \forall t \qquad (2) \\ &\|\eta_{t:t-n_a}\|_\infty \le \varepsilon \quad \forall t \end{aligned}$$

$p^* = 0$

The model is invalidated $\Leftrightarrow$

$s_{i,t}(\mathbf{g}_{i,t} - \mathbf{h}_i\eta_{t:t+n_a}) = 0$
$s_{i,t} \in \{0,1\},\ \sum_{i=1}^{n_s} s_{i,t} = 1$
$\|\eta_{t:t+n_a}\|_\infty \le \varepsilon$
is infeasible.

$\Leftrightarrow$

$$\begin{aligned} p^* = &\min_{s_{i,t},\eta_{t:t-n_a}} \sum_{t=0}^{T-1}\sum_{i=1}^{n_s} s_{i,t}^2\left(\mathbf{g}_{i,t} - \mathbf{h}_i\eta_{t:t-n_a}\right)^2 \\ &\text{subject to} \\ &s_{i,t}^2 = s_{i,t},\ \sum_{i=1}^{n_s} s_{i,t} = 1 \quad \forall t \qquad (2) \\ &\|\eta_{t:t-n_a}\|_\infty \le \varepsilon \quad \forall t \end{aligned}$$

$p^* > 0$

A Moments-based Relaxation

$$\begin{aligned} p_N^* = &\min_{\mathbf{m}} \sum_{t=0}^{T-1}\sum_{i=1}^{n_a} l_{i,t}(\mathbf{m}) \\ &\text{subject to} \\ &M_N(\mathbf{m}) \succeq \mathbf{0} \\ &L_N(\mathbf{m}) \succeq \mathbf{0} \end{aligned}$$

$p^* \ge p_N^*$

Further Relaxation due to Running Intersection Property

$$\begin{aligned} p_N^* = &\min_{\mathbf{m}} \sum_{t=0}^{T-1}\sum_{i=1}^{n_a} l_{i,t}(\mathbf{m}_{t:t-n_a}) \\ &\text{subject to} \\ &M_N(\mathbf{m}_{t-n_a:t}) \succeq \mathbf{0} \quad \forall t \in [0, T-1] \\ &L_N(\mathbf{m}_{t-n_a:t}) \succeq \mathbf{0} \quad \forall t \in [0, T-1] \end{aligned}$$

$p^* \ge p_N^*$

Analysis

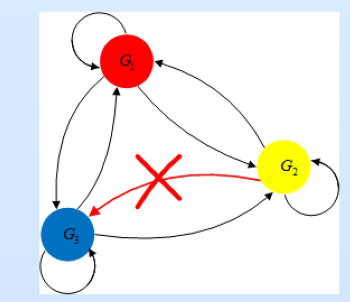
The model is **invalidated** if and only of there exists an **N** such that $p_N^* > 0$

The model is **not invalidated** if and only of there exists an **N** such that $p^* = p_N^* = 0$

Specifically, in this problem, T+1 is a choice of N.

Extension to Systems with Structural Constraints

Submodels cannot switch from G_2 to G_3

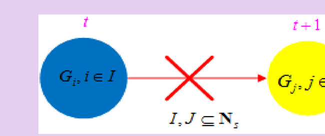


$\Leftrightarrow$

$$\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & T-2 & T-1 \end{array} \\ \begin{bmatrix} s_{1,0} & s_{1,1} & s_{1,2} & \cdots & s_{1,T-2} & s_{1,T-1} \\ s_{2,0} & s_{2,1} & s_{2,2} & \cdots & s_{2,T-2} & s_{2,T-1} \\ s_{3,0} & s_{3,1} & s_{3,2} & \cdots & s_{3,T-2} & s_{3,T-1} \end{bmatrix} \end{array}$$

$s_{2,0} + s_{3,1} \le 1$ $\quad$ $s_{2,1} + s_{3,2} \le 1$ $\quad \cdots \quad$ $s_{2,T-2} + s_{3,T-1} \le 1$

The structural constraints are imposed as linear inequality constraints on the indicator variables



$\Leftrightarrow$ $s_{i,t} + s_{j,t+1} \le 1, \forall i \in I, \forall j \in J$

Examples

Academic Example

Given submodels without structural constraints

$$\begin{aligned} \xi_t &= 0.2\xi_{t-1} + 0.24\xi_{t-2} + 2u_{t-1} && (G_1) \\ \xi_t &= -1.4\xi_{t-1} - 0.53\xi_{t-2} + u_{t-1} && (G_2) \\ \xi_t &= 1.7\xi_{t-1} - 0.72\xi_{t-2} + 0.5u_{t-1} && (G_3) \end{aligned}$$

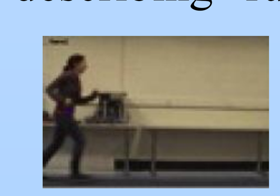
and the measurement equation $y_t = \xi_t + \eta_t$

Actual	G_1, G_2, G_3	G_1, G_2, G_3	G_1, G_2, G_3	G_2, G_3
A Priori Information	G_1, G_2, G_3	G_1, G_2, G_3	G_1, G_2	G_1, G_2
Results using sparsification	feasible, $s_{i,t}^* \in \{0,1\}$	feasible, $s_{i,t}^* \notin \{0,1\}$	infeasible	infeasible
Interpretation	Not invalidated	no decision	invalidated	invalidated
Time (sec.)	3.6808	4.4611	0.1949	0.1980
Results using Moments	-3.0399e-07	-2.4735e-07	7.3123	14.2226
Interpretation	not invalidated	not invalidated	invalidated	invalidated
Times (sec.)	6.8146	6.4891	2.7642	2.4306

Contextually Abnormal Activity Recognition

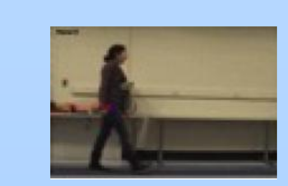
Given submodels describing "run" and "walk"

run



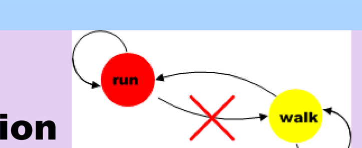
$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 0.6058 & 0.0003 \\ 0.2597 & 0.8589 \end{pmatrix}\begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} 0.3608 & 0.1853 \\ -0.2381 & 0.1006 \end{pmatrix}\begin{pmatrix} x_{t-2} \\ y_{t-2} \end{pmatrix}$$

walk

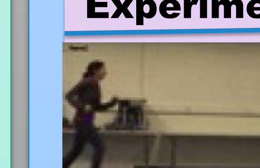


$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 0.4747 & 0.0628 \\ -0.3424 & 1.2250 \end{pmatrix}\begin{pmatrix} x_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} 0.5230 & -0.1144 \\ 0.3574 & -0.2513 \end{pmatrix}\begin{pmatrix} x_{t-2} \\ y_{t-2} \end{pmatrix}$$

A Priori information

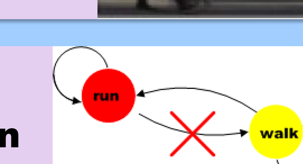


Experimental Data *time*

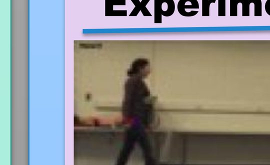


Algorithms	results
Results using Sparsification **with** Constraints	Infeasible
Interpretation	**Invalidated**
Time (sec.)	2.1836
Results using Moments **with** Constraints	0.1751
Interpretation	**Invalidated**
Times (sec.)	1.2968e03

A Priori information



Experimental Data *time*



Algorithms	results
Results using Sparsification **with** Constraints	Feasible, $s_{i,t} \in \{0,1\}$
Interpretation	**Not Invalidated**
Time (sec.)	1.0577
Results using Moments **with** Constraints	-3.3581e-08
Interpretation	**Not Invalidated**
Times (sec.)	0.8407e03

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