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Brief Paper

$\mathscr{L}_\infty$ Optimal Control of SISO Continuous-time Systems*

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Key Words—$\mathscr{L}_\infty$ control; optimal control; optimization, disturbance rejection.

Abstract—In this paper we study the problem of designing a controller that minimizes the weighted amplitude of the time response due to a given, fixed input signal for SISO continuous-time systems. The main result of the paper shows that this problem admits a minimizing solution in $\mathscr{L}_\infty$ and that the optimal closed-loop system has a special structure: a sum of delayed step functions, all having the same amplitude. Thus the optimal controller has a non-rational transfer function. Although in the general case finding this controller entails solving an infinite-dimensional linear-programming problem, we show that in some special cases the optimal solutions have closed-form expressions and can be found by solving a set of algebraic equations. Finally, we address the issue of selecting time and frequency domain weighting functions. This paper together with our paper 'Rational $\mathscr{L}_\infty$-suboptimal controller for SISO continuous time systems' (*IEEE Trans. Autom. Control*, **AC-41**, 1358–1363 (1996)), which deals with the design of rational $\mathscr{L}_\infty$ suboptimal controllers for general systems, give a complete solution of the $\mathscr{L}_\infty$ control problem. © 1997 Elsevier Science Ltd. All rights reserved.

1. *Introduction*

In many cases the objective of a control system design can be stated simply as synthesizing an internally stabilizing controller that minimizes the response to some exogenous inputs. When the exogenous inputs are assumed arbitrary but with bounded energy and the outputs are also measured in terms of energy, this problem leads to the minimization of an $\mathscr{H}_\infty$ norm of the closed-loop system. The case where the exogenous inputs are bounded persistent signals and the outputs are measured in terms of the peak time-domain magnitude leads to the minimization of an $\mathscr{L}_1/l_1$ norm. $\mathscr{H}_\infty$ optimal control can now be solved by elegant state-space formulae (Doyle *et al.*, 1989) while $\mathscr{L}_1/l_1$ optimal control can be (approximately) solved by finite linear programming (Dahleh and Pearson 1987a, b, 1988a; Diaz-Bobillo and Dahleh 1993).

In some cases, following a common practice in engineering, the performance requirements are stated in terms of the response of the closed-loop system to a given, fixed test input (such as bounds on the rise time, settling time or maximum error to a step). In this case, if the output is measured in terms of its energy, the problem leads to the minimization of the closed-loop $\mathscr{H}_2$ norm, extensively studied in the 1960s and 1970s. On the other hand, if the outputs are measured in terms of the peak time-domain magnitude, it leads to the minimization of the $\mathscr{L}_\infty/l_\infty$ norm. l_∞ optimal control theory for SISO discrete-time systems was developed by Dahleh and Pearson (1988b) (for recent work in this context (see also Khammash, 1994; Elia *et al.*, 1994; and references therein). In this paper we address the $\mathscr{L}_\infty$ optimal control problem for SISO continuous-time systems. In addition to presenting a continuous-time counterpart to the results of Dahleh and Pearson (1988b), the contributions of this paper are as follows.

- Contrary to $\mathscr{H}_2$, $\mathscr{H}_\infty$ and $\mathscr{L}_1/l_1$ optimal control, where asymptotic stability of the closed-loop system is guaranteed, in this case the optimal closed loop is only guaranteed to be in $\mathscr{L}_\infty$. Thus, in general, it is neither exponentially nor bounded-input bounded-output stable. In Dahleh and Pearson (1988b) this problem was addressed by restricting the closed-loop system to a subspace $B \subset l_\infty$, algebraically equivalent to l_1. As a result, the optimal cost can be approached but not achieved. Moreover, it can be shown that the stability margin approaches zero as the closed-loop system approaches the optimal. In this paper we use a different approach. By observing that the stability of a weighted closed-loop system is sufficient (but not necessary) for BIBO stability of the actual closed-loop system, our optimization problem is still formulated in $\mathscr{L}_\infty$. This guarantees the existence of optimal solutions, while the BIBO stability requirement is addressed through appropriate weight selection.
- The structure of optimal solutions is identified: a sum, possibly infinite, of delayed step functions, all having the same amplitude. In general, finding these solutions entails solving an infinite-dimensional linear programming problem. However, we show that for some classes of systems the solution has a closed-form expression that can be found by solving a system of algebraic equations.

These results together with the method proposed in Wang and Sznaier (1994, 1996) for finding rational $\mathscr{L}_\infty$ suboptimal controllers give a complete solution of the $\mathscr{L}_\infty$ control problem.

2. *Preliminaries*

In this section we present the mathematical background required for solving the $\mathscr{L}_\infty$ optimal control problem. This material is standard in functional analysis and optimization textbooks (see e.g. Luenberger, 1969), and it is included here for ease of reference.

Let X be a normed linear space. The space of all bounded linear functionals on X is denoted by X^*. Consider $x \in X$, $r \in X^*$, then $\langle x, r\rangle$ denotes the value of the linear functional r at x. The induced norm on X^* is defined as

$$\|r\| = \sup_{x \in BX} |\langle x, r\rangle|.$$

where $BX \doteq \{x \in X : \|x\| \le 1\}$.

$\mathbb{R}_+$ and $\mathscr{L}_\infty(\mathbb{R}_+)$ denote respectively the set of nonnegative real numbers and the space of measurable functions $f(t)$ on $\mathbb{R}_+$ equipped with the norm:

$$\|f\|_\infty = \operatorname{ess}\sup_{\mathbb{R}_+} |f(t)| < \infty.$$

Similarly, $\mathscr{L}_1(\mathbb{R}_+)$ denotes the space of Lebesgue-integrable functions on $\mathbb{R}_+$ equipped with the norm

$$\|f\|_1 = \int_0^\infty |f(t)|\,\mathrm{d}t < \infty.$$

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Given $f \in \mathscr{L}_1(\mathbb{R}_+)$, we shall denote its Laplace transform by $F(s) = \mathscr{L}(f)$.

Definition 1. Let S be a subspace of X. The *annihilator subspace* of S, denoted by $S^\perp$ is defined as

$$S^\perp = \{r \in X^*: \langle x, r\rangle = 0\ \forall x \in S\}.$$

Next we recall a duality principle stating the equivalence of two optimization problems: one formulated in a normed space and the other in its dual. We shall exploit this result to recast the $\mathscr{L}_\infty$ control problem as an optimization problem in a finite-dimensional space.

Theorem 1. (Luenberger (1969, p. 121).) Let S be a subspace of a real normed linear space X. $x^* \in X^*$ be at a distance μ from $S^\perp$. Then

$$\mu = \min_{r^* \in S^\perp} \|x^* - r^*\| = \sup_{x \in BS} \langle x, x^*\rangle,$$

where the minimum is achieved for some $r_0^* \in S^\perp$. If the supremum on the right is achieved for some $x_0 \in BS$ then $\langle x^* - r_0^*, x_0\rangle = \|x^* - r_0^*\|\ \|x_0\|$ (i.e. $x^* - r_0^*$ is aligned with x_0).

A special case of the above theorem is the case when S is finite-dimensional. In this situation the supremum on the right will always exist, and hence both problems have solutions.

Now denote by A_∞ the space of all Laplace transforms of elements in $\mathscr{L}_\infty$. Elements of A_∞ are analytic in the open right half-plane $\mathrm{Re}\,(s) > 0$, and, if rational, have only simple poles on the imaginary axis.

Definition 2. A system $H(s)$ is said to be $\mathscr{L}_\infty$ *stable* if $H(s) \in A_\infty$.

Remark 1. Requiring $\mathscr{L}_\infty$ stability of the closed-loop system is usually not strong enough to guarantee acceptable performance, since it does not imply either BIBO or exponential stability. However, as we show in the sequel, this latter requirement can be enforced by requiring $\mathscr{L}_\infty$ stability of appropriately weighted closed-loop transfer functions.

3. *Problem formulation*

Consider the system represented by the block diagram in Fig. 1, where P represents the plant to be controlled, the scalar signals d and u represent a fixed exogenous disturbance and the control action respectively, z and y represent the regulated output subject to performance constraints and the measurements available to the controller respectively, δ is the impulse function, $W_I(s)$ is the Laplace transform of d, W_O is an output weighting function representing performance requirement, ζ is the weighted output, and S represents the generalized plant.

Our objective is to find an $\mathscr{L}_\infty$ internally stabilizing controller such that the maximum amplitude of the performance output $\zeta(t)$ is minimized. This is equivalent to minimizing the $\mathscr{L}_\infty$ norm of the impulse response $\phi(t)$ of the closed-loop system from δ to ζ:

$$\mu^* = \inf_{\mathscr{L}_\infty \text{ stabilizing } K} \|\zeta(t)\|_\infty = \inf_{\mathscr{L}_\infty \text{ stabilizing } K} \|\phi(t)\|_\infty \qquad (1)$$

Compared with the discrete-time counterpart of the problem (Dahleh and Pearson, 1988b), we relax asymptotic stability to $\mathscr{L}_\infty$ internal stability and drop the zero steady-state error requirement so that the infimum can be achieved. As we

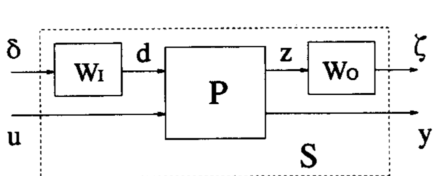

Fig. 1. The generalized plant.

mentioned before, asymptotic stability is certainly necessary for the actual (without input and output weights) closed-loop system from d to z, but not for the weighted one. We assume temporarily that we are satisfied with $\mathscr{L}_\infty$ internal stability of the weighted system, and we shall show later how to enforce internal asymptotic stability of the actual closed-loop system, through the use of appropriate weights. Additionally, these weights can be used to get a desired time response envelope for $z(t)$ or enforce a zero steady-state error requirement.

By a slight modification of the YJBK parametrization (Youla *et al.*, 1976), it can be shown that the set of all closed-loop transfer functions achievable with an $\mathscr{L}_\infty$ stabilizing controller can be parametrized as

$$\Phi(s) = H(s) - U(s)Q(s),$$

where H and U are rational stable transfer functions and Q is an arbitrary element in A_∞.

Assume now tht U has n distinct zeros z_j in the open right half-plane,† where $z_j = a_j + \mathrm{i}b_j$, and no zeros on the $i\omega$ axis. Let $\hat{T} = \{M \in A_\infty: (M(z_j) = 0,\ j = 1, \ldots, n\}$. Then we have the following lemma, which can be proved using arguments similar to those in Dahleh and Pearson (1987b).

Lemma 1. Let $M(s) = U(s)Q(s)$. Then $A \in A_\infty$ if and only if $M \in \hat{T}$.

Note that $M(z_j) = 0$ if and only if

$$\int_0^\infty m(t)\mathrm{e}^{-z_j t}\,\mathrm{d}t = 0.$$

Define $f_j = \mathrm{e}^{-a_j t}\cos b_j t$ and $g_j = \mathrm{e}^{-a_j t}\sin b_j t$. Then $M \doteq \mathscr{L}(m) \in \hat{T}$ if and only if $\langle m, f_j\rangle = 0$ and $\langle m, g_j\rangle = 0$ for $j = 1, \ldots, n$. Now consider the sets

$$T = \{m \in \mathscr{L}_\infty: \langle m, f_j\rangle = 0 \text{ and } \langle m, g_j\rangle = 0 \text{ for } j = 1, \ldots, n\},$$

$$S = \mathrm{span}\,\{f_j, g_j, j = 1, \ldots, n\}.$$

T will be viewed as a subspace of $\mathscr{L}_\infty$, and S as a subspace of $\mathscr{L}_1$. It follows that $T = S^\perp$, the annihilator subspace of S. The optimization problem (1) can now be written as

$$\mu^* = \min_{m \in S^\perp} \|h - m\|_\infty = \max_{r \in BS} \langle h, r\rangle. \qquad (2)$$

4. *Problem solution*

In this section we use the duality principle to reformulate the problem (1) in terms of another optimization problem. While this second problem is still infinite-dimensional, this reformulation allows for identifying the structure of the solutions. Moreover, as we show in the sequel, in some cases these solutions can be found exactly by solving a system of algebraic equations. Finding approximate solutions in the general case is briefly discussed in Section 4.2.

Theorem 2. (i) The solution to the problem (2) is given by

$$\mu^* = \min_{m \in S^\perp} \|h - m\|_\infty$$
$$= \max_{\alpha_j} \left[\sum_{j=1}^n \alpha_j \,\mathrm{Re}\, H(z_j) + \sum_{j=1}^n \alpha_{j+n} \,\mathrm{Im}\, H(z_j)\right] \qquad (3)$$

subject to

$$\int_0^\infty \left|\sum_{j=1}^n \alpha_j \mathrm{e}^{-a_j t}\cos b_j t + \sum_{j=1}^n \alpha_{j+n}\mathrm{e}^{-a_j t}\sin b_j t\right| \mathrm{d}t \le 1. \qquad (4)$$

(ii) An optimal solution $r^*(t) \in \mathscr{L}_1$ for the maximization problem always exists, where

$$r^*(t) = \sum_{j=1}^n \alpha_j^* \mathrm{e}^{-a_j t}\cos b_j t + \sum_{j=1}^n \alpha_{j+n}^* \mathrm{e}^{-a_j t}\sin b_j t.$$

(iii) The optimal solution $\phi = h - m$ always exists, and satisfies the following conditions:

† This assumption is made to simplify the developments in the sequel. In Section 4 we shall show that it can be easily removed.

(a) $|\phi(t)| \le \mu^*$, with $|\phi(t)| = \mu^*$ whenever $r^*(t) \ne 0$;

(b) $\phi(t) r^*(t) \ge 0$;

(c) $\phi(t)$ has the form

$$\phi(t) = \phi_0 \mathbf{1}(t) + \sum_{i=1}^{l} (-1)^i 2\phi_0 \mathbf{1}(t - t_i),$$

where $|\phi_0| = \mu^*$, $\mathbf{1}(t)$ is the unit step function, and the t_i are the points at which $r^*(t_i) = 0$, taken in increasing order and with appropriate multiplicity, $0 < t_1 \le t_2 \le \ldots \le t_l \ldots$; note that l may be infinite, in which case $t_l \to \infty$;

(d) $\Phi(z_j) = H(z_j)$ for $j = 1, \ldots, n$.

Proof. (i) Using Theorem 1, we have $\mu^* = \min_{m \in S^\perp} \|h - m\|_\infty = \max_{r \in BS} \langle r, h \rangle$. However, $r(t)$ has the representation

$$r(t) = \sum_{j=1}^{n} \alpha_j e^{-a_j t} \cos b_j t + \sum_{j=1}^{n} \alpha_{j+n} e^{-a_j t} \sin b_j t.$$

Hence

$$\langle r, h \rangle = \sum_{j=1}^{n} \alpha_j \operatorname{Re} H(z_j) + \sum_{j=1}^{n} \alpha_{j+n} \operatorname{Im}(z_j),$$

and $\|r\|_1 \le 1$ if and only if

$$\int_0^\infty \left| \sum_{j=1}^{n} \alpha_j e^{-a_j t} \cos b_j t + \sum_{j=1}^{n} \alpha_{j+n} e^{-a_j t} \sin b_j t \right| dt \le 1.$$

(ii) The existence of a solution to the maximization problem is guaranteed by the finite-dimensionality of S.

(iii) The existence of a solution to the primal problem follows from duality. Properties (a) and (b) follow from the alignment conditions. To prove (c), note that $r(t)$ is a continuous function and that $r^*(t) \not\equiv 0$ for any interval $[a, b]$ except in the trivial case $\mu^* = 0$. Hence $\phi(t)$ is constant between any two adjacent zero points. Moreover, we can assume without loss of generality that $r^*(t)$ changes sign at t_i (by considering points where $r^*(t_i) = 0$ but $r^*(t)$ does not change sign as zeros with a multiplicity of 2). The expression for $\phi(t)$ follows. Finally, property (d) is a restatement of the interpolation conditions. □

Consider now the case where the plant has a non-minimum-phase zero z_1 with multiplicity $l_1 > 1$. Then the conditions involving z_1 in Lemma 1 should be modified to $M^{(k)}(z_1) = 0$, $k = 0, 1, \ldots, l_1 - 1$, where $M^{(k)}$, denotes the kth derivative. These additional conditions can be accommodated by including the functionals $f_{1,k} \doteq t^k e^{-a_1 t} \cos b_1 t$ and $g_{1,k} \doteq t^k e^{-a_1 t} \sin b_1 t$, $k = 0, 1, \ldots, l - 1$, in S and modifying Theorem 2 accordingly.

It is interesting to compare optimal $\mathscr{L}_1$ and $\mathscr{L}_\infty$ closed-loop systems. The first observation is that both contain delay terms and hence have non-rational transfer functions even for rational plants. The optimal $\mathscr{L}_1$ closed-loop system is a finite sum of delayed pulse functions with different strengths. The optimal $\mathscr{L}_\infty$ closed loop is a (possibly infinite) sum of delayed step functions with the same amplitude. Thus a closed-form solution may not exist for an $\mathscr{L}_\infty$ optimal control problem.

4.1. *Exact solutions to two classes of systems*

(i) Systems with only real zeros. In this section we consider systems where all the unstable zeros z_j, $j = 1, \ldots, n$, of $U(s)$ are real. Then the optimization problem reduces to

$$\mu^* = \min_{m \in S^\perp} \|h - m\|_\infty = \max_{\alpha_j} \sum_{j=1}^{n} \alpha_j H(z_j)$$

subject to

$$\int_0^\infty \left| \sum_{j=1}^{n} \alpha_j e^{-z_j t} \right| dt \le 1.$$

The functional $r^*(t)$ has the form

$$r^*(t) = \sum_{j=1}^{n} \alpha_j e^{-z_j t}.$$

For this class of systems the results of Theorem 2 can be used to obtain a *closed form* of the solution by exploiting the following theorem.

Theorem 3. For the case where all the zeros z_j, $j = 1, \ldots, n$, are real the extremal functional $r^*(t)$ can equal zero at most at $n - 1$ points. The only exception is the trivial case $\mu^* = 0$.

Proof. This follows immediately from Gantmacher (1959, Example 1, p. 118).

Without loss of generality, we can always assume that $r^*(t) = 0$ at exactly $n - 1$ points, by adding additional zeros of $r^*(t)$ at $t = 0$ and changing the sign of ϕ_0 if necessary. The following corollary to Theorem 2 is now immediate.

Corollary 1. For the case where all the zeros z_j, $j = 1, \ldots, n$, are real the optimal solution $\phi = h - m$ has the form

$$\phi(t) = \phi_0 \mathbf{1}(t) + \sum_{i=1}^{n-1} (-1)^i 2\phi_0 \mathbf{1}(t - t_i),$$

where $|\phi_0| = \mu^*$ and $0 \le t_1 \le t_2 \le \ldots \le t_{n-1} < \infty$. The optimal closed-loop transfer function has the form

$$\Phi(s) = \frac{1}{s} \left[\phi_0 + \sum_{i=1}^{n-1} (-1)^i 2\phi_0 e^{-t_i s} \right], \tag{5}$$

and satisfies

$$\Phi(z_j) = H(z_j) \quad \text{for } j = 1, \ldots, n. \tag{6}$$

Remark 2. Since we have only n unknown variables t_i, $i = 1, \ldots, n - 1$, and ϕ_0, in this case the solution to (1) can be found by solving the set of n algebraic equations (6).

(ii) Systems with only one pair of complex zeros. In the case where $U(s)$ has only a pair of unstable complex zeros $z_1 = a + ib$ and $z_2 = a - ib$, the optimization problem (1) reduces to

$$\mu^* = \min_{m \in S^\perp} \|h - m\|_\infty = \max_{\alpha_j} [\alpha_1 \operatorname{Re} H(z_1) + \alpha_2 \operatorname{Im} H(z_1)]$$

subject to

$$\int_0^\infty |\alpha_1 e^{-at} \cos bt + \alpha_2 e^{-at} \sin bt| \, dt \le 1.$$

In this case the functional $r(t)$ has the form

$$r(t) = \alpha_1 e^{-at} \cos bt + \alpha_2 e^{-at} \sin bt = M_\alpha e^{-at} \sin(bt + \theta_\alpha) = M_\alpha e^{-at} \sin b(t + t_\alpha) \tag{7}$$

where M_α, θ_α and t_α depend on α, and $0 \le \theta_\alpha = bt_\alpha < \pi$.

It is easily seen from (7) that $r(t)$ will change sign periodically at points $t_k = k\pi/b - t_\alpha$, $k = 1, 2, \ldots$. So the optimal solution has the form

$$\phi(t) = \phi_0 \mathbf{1}(t) + \sum_{k=1}^{\infty} (-1)^i 2\phi_0 \mathbf{1}(t - t_k),$$

where $|\phi_0| = \mu^*$, and $t_k = k\pi/b - t_{\alpha^*}$, $k = 1, 2, \ldots$. Even though the optimal solution $\phi(t)$ is a sum of infinite terms, the optimal closed-loop transfer function still has a closed form:

$$\Phi(s) = \frac{\phi_0}{s} \left(1 - \frac{2e^{-t_1 s}}{1 + e^{-\pi s/b}} \right).$$

Note that $\Phi(s)$ has only two unknown variables, ϕ_0 and t_1. Hence it can be solved exactly again from the following interpolation conditions:

$$\Phi(z_j) = H(z_j) \quad \text{for } j = 1, 2. \tag{8}$$

4.2. *Approximation methods.* An exact solution to the $\mathscr{L}_\infty$ problem for general systems is not available at present. In Wang and Sznaier (1994, 1996) we did develop a method for finding a suboptimal controller yielding a closed-loop system with norm arbitrarily close to the optimal cost. An important feature of this method is that the resulting controllers are rational, and thus physically implementable. Because of space limitations, readers are referred to Wang and Sznaier (1994, 1996) for details about these approximations.

Note in passing that an alternative approach will be to mimic the approximation method used in Dahleh and Pearson (1987b) for the $\mathscr{L}_1$ case. By sampling the integral constraint (4), the continuous-time problem will be transformed into a discrete-time problem. Then an approximate solution can be found by finite linear programming (Dahleh and Pearson, 1988b). However, this method will result in a suboptimal controller that, like the optimal one, has an irrational transfer function, and thus an additional approximation is required in order to obtain a practically implementable controller.

5. *Stability and time response shaping*

We now return to the fundamental issue of stability. With reference to Fig. 1, thus far we have only imposed the requirement that $\Phi(s)$, the weighted closed-loop transfer function from $\delta(t)$ to $\zeta(t)$, must ge $\mathscr{L}_\infty$ stable. However since $\mathscr{L}_\infty$ stability does not imply asymptotic stability, additional steps are required in order to enforce asymptotic stability of the actual closed-loop system, that is, the closed-loop transfer function $\Phi_{zd}(s)$ between the *physical* input and output signals $d(t)$ and $z(t)$. Note that this is true even in the case where a suboptimal controller, obtained by using the finite-support approximation proposed in Wang and Sznaier (1994, 1996), is used. Though this approximation yields an asymptotically stable closed-loop system, the stability margin will approach zero as the approximation approaches the optimal controller.

In this section we show how to both enforce closed-loop asymptotic stability and achieve some desirable performance specifications through the use of appropriate weighting functions. We shall discuss both frequency- and time-domain weighting.

5.1. *Time-domain weighting.* One of the strengths of $\mathscr{L}_\infty$ optimal control is its ability to deal explicitly with time-domain specifications, such as overshoot and settling time. In general, these specifications can be described as

$$|z(t)| = |\phi_{z\delta}(t)| \le p(t) \quad \forall t \in \mathbb{R}^+$$

where $\phi_{z\delta}(t)$ is the closed-loop system from $\delta(t)$ to $z(t)$, and $p(t)$ is a bounded and nonnegative function. This confines the regulated output $z(t)$ within an envelope. It is shown in Dahleh and Pearson (1988b) that this can be achieved in l_∞ optimal control of discrete-time systems through time-domain weighting. Similar weighting can also be used for continuous-time systems. Let

$$\phi(t) = \zeta(t) = f(z(t)) = p^{-1}(t)z(t) \doteq w_O(t)\phi_{z\delta}(t),$$

and consider the following weighted minimization:

$$\mu^* = \inf_{\mathscr{L}_\infty \text{ stabilizing } K} \|\phi(t)\|_\infty = \inf_{\mathscr{L}_\infty \text{ stabilizing } K} \|w_O(t)\phi_{z\delta}(t)\|_\infty. \tag{9}$$

Suppose that a YJBK parametrization of all $\mathscr{L}_\infty$ stabilizing controllers for $\Phi_{z\delta}(s)$ is given by

$$\Phi_{z\delta}(s) = H(s) - U(s)Q(s).$$

Then the following result furnishes a solution to (9).

Theorem 4. Assume that $U(s)$ has n single right-half-plane zeros z_i and that $p(t) = e^{-\sigma t}$. Then the following hold.

(i)
$$\mu^* = \inf_{\mathscr{L}_\infty \text{ stabilizing } K} \|\phi(t)\|_\infty = \max_{\alpha_j}\left[\sum_{j=1}^{n} \alpha_j \operatorname{Re} H(z_j) + \sum_{j=1}^{n} \alpha_{j+n} \operatorname{Im} H(z_j)\right]$$

subject to

$$\int_o^\infty p(t)\left|\sum_{j=1}^{n} \alpha_j e^{-a_j t}\cos b_j t + \sum_{j=1}^{n} \alpha_{j+n} e^{-a_j t}\sin b_j t\right| dt \le 1.$$

(ii) An optimal solution $r^*(t)$ for the maximization problem always exists, where

$$r^*(t) = \sum_{j=1}^{n} \alpha_j^* e^{-a_j t}\cos b_j t + \sum_{j=1}^{n} \alpha_{j+n}^* e^{-a_j t}\sin b_j t.$$

(iii) The optimal solution $\phi(t)$ always exists, and satisfies the following conditions:

(a) $|\phi(t)| \le \mu^*$ and $|\phi(t)| = \mu^*$ whenever $r^*(t) \ne 0$;

(b) $\phi(t)r^*(t) \ge 0$;

(c) $\phi(t)$ has the form

$$\phi(t) = \phi_0 \mathbf{1}(t) + \sum_{i=1}^{l} (-1)^i 2\phi_0 \mathbf{1}(t - t_i),$$

where $|\phi_0| = \mu^*$, and the t_i are the points at which $r^*(t_i) = 0$ ordered in increasing order $0 \le t_1 \le t_2 \le \ldots \le t_l \ldots$; if l is infinite then $t_l \to \infty$;

(d) $\Phi_{z\delta}(z_j) = H(z_j)$ for $j = 1, \ldots, n$.

(iv) The optimal closed-loop system $\Phi_{z\delta}(s)$ and hence the optimal actual closed-loop system $\Phi_{zd}(s)$ are exponentially stable with a decay rate of σ for any $\sigma > 0$.

(v) If $\sigma > 0$ then the regulated output $z(t)$ has zero steady-state value.

Proof. Since the proof is similar to those before, rather than going into details, we shall just mention some key points. Following the same idea as used in the proof of Theorem 7 in Dahleh and Pearson (1988b), parts (i)–(iii) can be proved using duality, as in Theorem 2. In fact these results hold even for the most general form $p(t)$. Parts (iv) and (v) follow immediately from the fact that

$$|z(t)| = |\phi_{z\delta}(t)| = \mu^* p(t) = \mu^* e^{-\sigma t}. \qquad \square$$

Note in passing that when $p(t) = e^{-\sigma t}$, the EAS method (Wang and Sznaier 1994, 1996) can still be used to get rational suboptimal solutions arbitrarily close to the optimum.

5.2. *Frequency-domain weighting.* In this subsection we consider systems $U(s)$ having only real RHP zeros. In this case the $\mathscr{L}_\infty$ optimal weighted closed-loop systems $\Phi(s)$ has the form

$$\Phi(s) = \frac{1}{s}\left[\phi_0 + \sum_{i=1}^{l} (-1)^i 2\phi_0 e^{-t_i s}\right],$$

where $l = n - 1$ is finite. In this case $\Phi(s)$ has only one unstable pole at the origin. The situation involving complex RHP zeros is much more complicated, since l may be infinite and $\Phi(s)$ may have additional poles on the ω axis.

Suppose that the regulated output $z(t)$ is weighted in the frequency domain,

$$\zeta(s) = W_O(s)z(s),$$

and that the Laplace transform of $d(t)$ is $W_I(s)$. Then

$$\Phi(s) = W_O(s)\Phi_{zd}(s)W_I(s), \tag{10}$$

where $W_I(s)$ and $W_O(s)$ serve as input and output weights. $W_I(s)$ is completely determined by the problem (the dynamics of the disturbance), but $W_O(s)$ is a free parameter that can be used to achieve some desirable performance specifications.

Theorem 5. Assume that $U(s)$ has only real RHP zeros. Then we have the following.

(i) The $\mathscr{L}_\infty$ optimal actual closed-loop system $\Phi_{zd}(s)$ is asymptotically stable if and only if the input weight $W_I(s)$ or/and the output weight $W_O(s)$ have at least one pole at the origin.

(ii) The regulated output $z(t)$ will have zero steady-state error if and only if the output weight $W_O(s)$ has at least one pole at the origin.

Proof. (i) Since $\Phi(s)$ has only one unstable pole at the origin, (i) follows directly from the relationship (10) between $\Phi(s)$ and $\Phi_{zd}(s)$.

(ii) Note that

$$z(s) = W_O^{-1}(s)\zeta(s) = W_O^{-1}(s)\Phi(s). \quad (11)$$

It follows from (11) and (5) using the final-value theorem, that

$$\lim_{t\to\infty} z(t) = \lim_{s\to 0} W_O^{-1}(s)\left[\phi_0 + \sum_{i=1}^{n-1}(-1)^i 2\phi_0 e^{-t_i s}\right] = (-1)^{n-1}\phi_0 W_O^{-1}(0), \quad (12)$$

where $|\phi_0|$ equals the $\mathcal{L}_\infty$ optimal cost (different from zero except in trivial cases). Thus we have $\lim_{t\to\infty} z(t) = 0$ if and only if

$$W_O^{-1}(0) = 0.$$

The above equality implies that $W_O(s)$ has at least one pole at the origin. □

The above theorem states the following. For systems $U(s)$ with only real RHP zeros, asymptotic stability of the optimal $\mathcal{L}_\infty$ closed-loop system is automatically guaranteed if the dynamics of the disturbance $d(t)$ contains a mode at the origin, such as a step disturbance. If that is not the case then an output weight $W_O(s)$ containing a pole at the origin must be selected in order to guarantee asymptotic stability. To get zero steady-state error, just using disturbance dynamics as an input weight is enough for $\mathcal{H}_2$, $\mathcal{H}_\infty$ and $\mathcal{L}_1$ optimal control, but not for $\mathcal{L}_\infty$ optimal control. For the last, an output weight must be used as well.

Remark 3. When either the input weight $W_I(s)$ or the output weight $W_O(s)$ contains unstable modes, a precompensator containing these modes must be used before performing the YJBK parametrization.

6. *Examples*

Example 1. Consider the plant

$$P(s) = \frac{s-2}{s-1}$$

with a step disturbance d. We want to design a controller $C(s)$ to minimize the amplitude of the regulated output $z(t)$ $(z(s) = \Phi(s) = [1 + P(s)C(s)]^{-1}d(s))$.

Since $W_I(s) = d(s) = 1/s$, a precompensator containing this dynamic must be used. We choose a proper precompensator$(s+1)/s$. Then the problem becomes that of finding an $\mathcal{L}_\infty$ stabilizing compensator $\hat{C}(s)$ for the augmented plant

$$\hat{P}(s) = \frac{(s+1)(s-2)}{s(s-1)}$$

such that the amplitude of $z(t)(z(s) = \Phi(s) = [1 + \hat{P}(s)\hat{C}(s)]^{-1}d(s))$ is minimized. One YJBK parametrization is given by

$$\Phi(s) = H(s) - U(s)Q(s) = \frac{6(s-1)}{(s+1)(s+2)} - \frac{(s-1)(s-2)}{(s+1)(s+2)^2}Q(s).$$

Since $U(s)$ has only two real RHP zeros at 1 and 2, the optimal solution is of the form

$$\Phi(s) = \frac{\phi_0}{s}(1 - 2e^{-t_1 s}).$$

By solving the interpolation conditions

$$\Phi(z_i) = H(z_i), \quad i = 1, 2,$$

we get $\phi_0 = 2$ and $t_1 = \ln 2$. Hence the optimal cost $\mu^* = |\phi_0| = 2$ and the optimal solution is

$$\Phi(s) = \frac{2}{s}[1 - 2(s)^{-s}].$$

The actual closed-loop system is

$$\Phi_{zd}(s) = 2 - 4(2)^{-s},$$

which is stable, as expected, since the input weight contains a pole at origin. The optimal controller is

$$C(s) = \frac{(s-1)[4(2)^{-s} - 1]}{(s-2)[2 - 4(2)^{-s}]}.$$

It can be shown that 1 is not a zero nor is 2 a pole of the controller $C(s)$, so there are no unstable pole–zero cancellations. It can also be shown that $C(s)$ has no pole at the origin, even though a precompensator containing a pole at the origin has been used. This is not an unexpected result, since the regulated output $z(t)$ is not weighted in either the time or the frequency domain. The presence of a pole at the origin would lead to a zero steady-state value, contradicting the $\mathcal{L}_\infty$ optimal structure.

Example 2. Consider the same plant

$$P(s) = \frac{s-2}{s-1}$$

with a step disturbance d. We want to design a controller $C(s)$ to minimize the amplitude of the weighted output $\zeta(t) = e^t z(t)$ $(z(s) = \Phi_{z\delta}(s) = [1 + P(s)C(s)]^{-1}d(s))$.

This problem can again be solved exactly. It is easy to show that

$$\Phi(s) = \zeta(s) = z(s-1) = \Phi_{z\delta}(s-1).$$

From Example 1, we have that a YJBK parametrization is given by

$$\Phi_{z\delta}(s) = H(s) - U(s)Q(s) = \frac{6(s-1)}{(s+1)(s+2)} - \frac{(s-1)(s-2)}{(s+1)(s+2)^2}Q(s).$$

Since $U(s)$ has only two real RHP zeros at 1 and 2, the optimal solution is of the form

$$\Phi(s) = \frac{\phi_0}{s}(1 - 2e^{-t_1 s}).$$

By solving the interpolation conditions

$$\Phi(z_i + 1) = \Phi_{z\delta}(z_i) = H(z_i), \quad i = 1, 2,$$

we get $\phi_0 = \frac{3}{2}(2 + \sqrt{2})$ and $t_1 = \frac{1}{2}\ln 2$. Hence the optimal cost $\mu^* = |\phi_0| = \frac{3}{2}(2 + \sqrt{2})$ and

$$\Phi_{z\delta}(s) = \Phi(s+1) = \frac{\phi_0}{s+1}[1 - 2e^{-t_1(s+1)}].$$

The actual closed-loop system is

$$\Phi_{zd}(s) = \frac{\phi_0 s}{s+1}[1 - 2e^{-t_1(s+1)}].$$

The optimal controller is

$$C(s) = \frac{[1 + s - \phi_0 s(1 - 2e^{-t_1(s+1)})](s-1)}{\phi_0 s(1 - 2e^{-t_1(s+1)})(s-2)}.$$

It can be shown that 1 is not a zero nor is 2 a pole of the controller $C(s)$, so there are no unstable pole–zero cancellations. It can also be shown that $C(s)$ does indeed have a pole at origin this time.

7. *Conclusions*

In this paper we have formulated and studied the $\mathcal{L}_\infty$ optimal control problem for SISO continuous-time systems. We have shown that an optimal solution always exists, although, as in the $\mathcal{L}_1$ optimal control case, it has a nonrational transfer function. The resulting optimal closed-loop system is a (finite or infinite) sum of delayed step functions, and its magnitude is equal to the optimal cost almost everywhere.

In general, this optimal solution does not have a closed-form solution, and some approximation methods must be used to solve the problem (for details on obtaining *rational* suboptimal controllers with guaranteed error bounds see Wang and Sznaier 1994, 1996). However, we have shown that for two classes of systems the structure of the optimal solutions can be exploited to reduce the problem to that of solving a set of algebraic equations determined by the interpolation conditions. Finally, we have addressed the problem of enforcing additional performance requirements (such as stability degree, zero steady-state error or settling time bounds) through appropriate time- or/and frequency-domain weighting.

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References

Dahleh, M. A. and J. B. Pearson (1987a). l^1-optimal feedback controllers for MIMO discrete-time systems. *IEEE Trans. Autom. Control,* **AC-32,** 314–322.

Dahleh, M. A. and J. B. Pearson (1987b). $\mathscr{L}^1$-optimal compensators for continuous-time systems. *IEEE Trans. Autom. Control,* **AC-32,** 889–895.

Dahleh, M. A. and J. B. Pearson (1988a). Optimal rejection of persistent disturbances robust stability, and mixed sensitivity minimization. *IEEE Trans. Autom. Control,* **AC-33,** 722–731.

Dahleh, M. A. and J. B. Pearson (1988b). Minimization of a regulated response to a fixed input. *IEEE Trans. Autom. Control,* **AC-33,** 924–930.

Diaz-Bobillo, I. J. and M. A. Dahleh (1993). Minimization of the maximum peak-to-peak gain: the general multiblock problem. *IEEE Trans. Autom. Control,* **AC-38,** 1459–1482.

Doyle, J. K., Glover, P. Khargonekar and B. Francis (1989). State-space solutions to standard $\mathscr{H}_2$ and $\mathscr{H}_\infty$ control problems. *IEEE Trans. Autom. Control,* **AC-34,** 831–846.

Gantmacher, F. R. (1959). *Applications of the Theory of Matrices.* Interscience, New York.

Khammash, M. (1994). Robust steady-state tracking. In *Proc. American Control Conf.,* Baltimore, MD., pp. 791–795.

Elia, N., P. M. Young and M. A. Dahleh (1994). Robust performance for fixed inputs. In *Proc. 33rd IEEE Conf. on Decision and Control,* Lake Buena Vista, FL, pp. 2690–2695.

Luenberger, D. G. (1969). *Optimization by Vector Space Methods.* Wiley, New York.

Wang, Z.-Q. and M. Sznaier (1994). $\mathscr{L}_\infty$-optimal control of SISO continuous time systems and its rational approximations. In *Proc. 33rd IEEE Conf. on Decision and Control,* Lake Buena Vista, FL, pp. 34–39.

Wang, Z.-Q. and M. Sznaier (1996). Rational $\mathscr{L}_\infty$-suboptimal controller for SISO continuous time systems. *IEEE Trans. Autom. Control,* **AC-41,** 1358–1363.

Youla, D. C., H. A. Jabr and J. J. Bongiorno (1976). Modern Wiener–Hopf design of optimal controllers—Part 2: The multivariable case. *IEEE Trans. Autom. Control,* **AC-21,** 319–338.