[13] K. S. Narendra and L. S. Valavani, "Stable adaptive controller design-direct control," *IEEE Trans. Automat. Contr.*, vol. 23, pp. 570–583, Aug. 1978.

[14] P. Ioannou, "Reply to comments on the robust stability analysis of adaptive controllers using normalizations," *IEEE Trans. Automat. Contr.*, vol. 35, p. 1184, 1990.

[15] P. A. Ioannou and G. Tao, "Frequency domain conditions for strictly positive real functions," *IEEE Trans. Automat. Contr.*, vol. 32, pp. 53–54, 1987.

Rational $\mathcal{L}_\infty$-Suboptimal Controllers for SISO Continuous-Time Systems

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***Abstract*— In this paper we study the problem of minimizing the weighted amplitude of the time response due to a given fixed input signal for single-input/single-output (SISO) continuous-time systems and focus on obtaining rational suboptimal solutions. An EAS (Euler approximating system)-based method is proposed for designing a *rational* $\mathcal{L}_\infty$-suboptimal controller for SISO systems. It is shown that this rational approximation is the best one among a set of rational approximations, in the sense of providing the tightest upper bound and that it can approximate the optimal cost arbitrarily close.**

I. Introduction

In many cases the objective of a control system design can be stated simply as finding a controller which stabilizes the feedback system and minimizes some output responses to some exogenous inputs. Depending on how the exogenous inputs are modeled, this leads to different mathematical optimization problems. For example, when the exogenous inputs are assumed arbitrary but with bounded energy, and the outputs are also measured in terms of energy, this problem leads to the minimization of the $\mathcal{H}_\infty$ norm of the closed-loop system; when the exogenous inputs are bounded persistent signals and the outputs are measured in terms of the peak time-domain magnitude, this problem leads to an $\mathcal{L}_1/l_1$ norm minimization. $\mathcal{H}_\infty$-optimal control can now be solved by elegant state-space formulas [8], while $\mathcal{L}_1/l_1$-optimal control can be (approximately) solved by finite linear programming [3]–[5], [7]. In some cases performance specifications are given in terms of the response to fixed exogenous inputs (such as the step response). The case where the input is fixed and the output is measured in terms of its energy leads to the minimization of the closed-loop $\mathcal{H}_2$ norm extensively studied in the 1960's and 1970's.

The case where the exogenous inputs are assumed fixed and the outputs are measured in terms of the peak magnitude leads to the minimization of an $\mathcal{L}_\infty/l_\infty$ norm. l_∞-optimal control theory for single-input/single-output (SISO) discrete-time systems was developed by Dahleh and Pearson in [6]. In this paper we address the $\mathcal{L}_\infty$-optimal control problem for SISO continuous-time systems and focus on designing rational suboptimal controllers.

In [11] we have identified the structure of $\mathcal{L}_\infty$-optimal solutions. While exact solutions are not yet available except in some special cases, we have shown that as in $\mathcal{L}_1$-optimal control, $\mathcal{L}_\infty$-optimal solutions contain delay terms. Thus, both the closed-loop system and the controller have irrational transfer functions, even if the plant and all the weights are rational. Since it is difficult to implement an irrational controller, this raises the additional question of existence of rational approximations. This issue is not trivial since a delay generally cannot be uniformly approximated by a rational system either in $\mathcal{L}_1$ or $\mathcal{L}_\infty$ spaces. References [2], [9], and [12] answered the question in the $\mathcal{L}_1$ case in an affirmative way and presented two methods to get rational $\mathcal{L}_1$-suboptimal controllers. In this paper we will show that the same idea used in [2] [using the Euler approximating system (EAS)] can still be employed. However, using this technique requires imposing additional interpolation constraints. A modified EAS-based method is proposed and is shown to yield the best approximation, in the sense of providing the tightest upper bound, among a set of rational approximations. Moreover, we indicate how to select *a priori* the approximation to meet a given approximation error bound.

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II. Preliminaries

A. Notation

Let R_+ denote the set of nonnegative real numbers. $\mathcal{L}_\infty(R_+)$ denotes the space of measurable functions $f(t)$ equipped with the norm: $\|f\|_{\mathcal{L}_\infty} = \text{ess.}\sup_{R_+} |f(t)| < \infty$, and $\mathcal{L}_1(R_+)$ denote the space of Lebesgue integrable functions on R_+ equipped with the norm $\|f\|_{\mathcal{L}_1} = \int_0^\infty |f(t)|dt < \infty$. Similarly, l_1 denotes the space of sequences $h = \{h_i\}$ such that $\|h\|_{l_1} = \sum_{i=0}^\infty |h_i| < \infty$, and l_∞ denotes the space of sequences $h = \{h_i\}$ such that $\|h\|_{l_\infty} = \sup_i |h_i| < \infty$. Throughout the paper we will use the capital letter $F(s)$ [or $H(z)$] to denote the Laplace transform (or $\mathcal{Z}$-transform) of $f(t)$ (or $\{h_i\}$) and packed notation to represent state-space realizations, i.e.,

$$G(s) = C(sI - A)^{-1}B + D \triangleq \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

B. The $\mathcal{L}_\infty$-Control Problem

Definition 1: A system $F(s)(H(z))$ is $\mathcal{L}_\infty$ stable (l_∞ stable) if $f(t) \in \mathcal{L}_\infty(R_+)(h = \{h_i\} \in l_\infty)$. A controller $C(s)(C(z))$ is an $\mathcal{L}_\infty$-stabilizing (l_∞-stabilizing) controller if it renders the overall closed-loop system $\Phi(s)(\Phi(z))\mathcal{L}_\infty$ stable (l_∞ stable).

By using this concept we can precisely state the $\mathcal{L}_\infty$-control problem as follows. Consider the system shown in Fig. 1, where P represents the plant to be controlled; the scalar signals d and u represent a fixed exogenous disturbance and the control action, respectively; z and y represent the regulated output subject to performance constraints and the measurements available to the controller, respectively; δ is the impulse function; $W_I(s)$ is the Laplace transform of d; W_O is an output weighting function representing performance requirement; ζ is the weighted output; and S represents the generalized plant. Then the $\mathcal{L}_\infty$-control problem can be stated as follows.

Problem 1 ($\mathcal{L}_\infty$): Find an $\mathcal{L}_\infty$-internally stabilizing controller such that the $\mathcal{L}_\infty$ norm of the impulse response $\phi(t)$ of the closed-loop system from δ to ζ is minimized, i.e.,

$$\mu^* = \inf_{\mathcal{L}_\infty-\text{stabilizing } K} \|\zeta(t)\|_\infty = \inf_{\mathcal{L}_\infty-\text{stabilizing } K} \|\phi(t)\|_\infty. \quad (1)$$

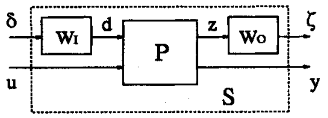

Fig. 1. The generalized plant.

Remark 1: Note that $\mathcal{L}_\infty$ stability does not imply either exponential or bounded-input/bounded-output stability. Thus, compared to the discrete-time counterpart of the problem [6], we relax asymptotic stability to $\mathcal{L}_\infty$-internal stability. Clearly, asymptotic stability is necessary for the actual (without input and output weights) closed-loop system from d to z but not for the weighted one. Relaxing the stability requirement allows for finding an optimal solution for the $\mathcal{L}_\infty$ problem (see [11] for details). Internal asymptotic stability of the actual closed-loop system can be enforced through the use of appropriate weights [11].

By using a slight generalization of the Youla parameterization, the $\mathcal{L}_\infty$-optimal control problem can be recast as [11]

$$\begin{aligned}\mu^* &= \inf_{\mathcal{L}_\infty-\text{stabilizing } K} \|\phi(t)\|_{\mathcal{L}_\infty} \\ &= \inf_{q \in \mathcal{L}_\infty} \|h(t) - u(t) * q(t)\|_{\mathcal{L}_\infty}\end{aligned} \tag{2}$$

where $H(s)$ and $U(s)$ are both stable functions. In the sequel we will assume that $U(s)$ has no zeros on the imaginary axis. Thus (by absorbing its stable zeros in Q if necessary), we can assume that all the zeros of $U(s)$ are unstable. Under this assumption, the solution to (1) is given by [11]

$$\mu^* = \max_{\alpha_j} \left[\sum_{j=1}^{n} \alpha_j \operatorname{Re} H(z_j) + \sum_{j=1}^{n} \alpha_{j+n} \operatorname{Im} H(z_j) \right] \tag{3}$$

subject to

$$\int_0^\infty \left| \sum_{j=1}^{n} \alpha_j \operatorname{Re}\{e^{-z_j t}\} + \sum_{j=1}^{n} \alpha_{j+n} \operatorname{Im}\{e^{-z_j t}\} \right| dt \le 1 \tag{4}$$

where z_j denotes the zeros of $U(s)$.

In [11] we analyzed the structure of the solutions to this infinite-dimensional optimization problem and we showed that, in general, they contain delay terms thus leading to nonrational transfer functions. Given the difficulty of implementing these controllers, and motivated by the results of [2], we will search for rational approximations to the optimal solution. To this effect, proceeding as in [2], we introduce the EAS.

C. The EAS and Its Properties

Definition 2: Consider the continuous-time system

$$G(s) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{5}$$

Its EAS is defined as the following discrete-time system:

$$G^E(z, \tau) = \begin{pmatrix} I + \tau A & \tau B \\ C & D \end{pmatrix} \tag{6}$$

where $\tau > 0$.

From this definition it is easily seen that we can obtain the EAS of $G(s)$ by the simple variable transformation $s = \frac{z-1}{\tau}$, i.e., $G^E(z,\tau) = G(\frac{z-1}{\tau})$. Moreover, by a slight generalization of [2, Th. 2] it can be easily shown that if $G^E(z,\tau)$ is asymptotically stable (or l_∞ stable), then (5) is also asymptotically stable (or $\mathcal{L}_\infty$ stable). Conversely, if $G(s)$ is asymptotically stable, there exists $\tau_{\max} > 0$ such that for all $0 < \tau \le \tau_{\max} G^E(z,\tau)$ is asymptotically stable.

Definition 3: Consider the system

$$\dot{x}(t) = Ax(t). \tag{7}$$

A set $\Sigma \subset R^n$ is a *positively invariant set* of (7) if for any initial condition $x_o \in \Sigma$, the corresponding trajectory $x(t, x_o) \in \Sigma$ for all t. A similar definition holds for the case of discrete-time systems.

We now introduce a key property of the EAS, the fact that *for strictly proper systems*, the l_∞ norm of the impulse response of the EAS scaled by τ^{-1} is an upper bound of the $\mathcal{L}_\infty$ norm of the impulse response of the corresponding continuous-time system.

Theorem 1: Consider the strictly proper continuous-time system

$$\begin{aligned}\dot{x} &= Ax + Bv \\ \zeta &= Cx\end{aligned} \tag{8}$$

and its corresponding EAS

$$\begin{aligned}x(k+1) &= (I + \tau A)x(k) + \tau B v(k) \\ \zeta(k) &= Cx(k).\end{aligned} \tag{9}$$

Assume that $G^E(z,\tau)$ is l_∞ stable. Then we have that (8) is $\mathcal{L}_\infty$ stable and such that $\|g(t)\|_{\mathcal{L}_\infty} \le \frac{1}{\tau}\|g^E(k,\tau)\|_{l_\infty}$, where $g(t)$ and $g^E(k,\tau)$ are the impulse responses of $G(s)$ and $G^E(z,\tau)$, respectively.

Proof: $\mathcal{L}_\infty$ stability of (8) follows from l_∞ stability of $G^E(z,\tau)$ and [2, Th. 2]. The second claim will be established by extending the proof of [10, Th. 1] to l_∞-stable systems. To simplify the expressions, take $k = -1$ as initial time for the EAS so that $\frac{1}{\tau}g^E(k,\tau) = Cx(k,\tau)$ and $g(t) = Cx(t)$, where $x(t)$ and $x(k,\tau)$ are the free-state responses of (8) and (9), respectively, taking the vector B as the initial condition. Denote by n the dimension of A. We assume (without loss of generality) that (A, B) is a reachable pair. This is both a necessary and sufficient condition for the EAS to be reachable for all $\tau \ge 0$. The reachability of $(I + \tau A, \tau B)$ implies that the sequence $x(k,\tau)$ spans R^n. Denote by $S(\tau)$ the convex hull of the set of points $\{\pm x(i,\tau),\ i = 0, 1, \cdots\}$. By definition $S(\tau)$ is a positively invariant set for $x(k+1) = (I + \tau A)x(k)$. Since $x(i,\tau)$, $i = 0, 1, \cdots$, span R^n, the set $S(\tau)$ is convex and contains the origin in its interior.

Denote by $\bar{S}(\tau)$ the closure of $S(\tau)$. Since $G^E(z,\tau)$ is l_∞ stable, $\bar{S}(\tau)$ is compact. We now prove that it is a positively invariant set for $x(k+1) = (I + \tau A)x(k)$. For any point x_0 on the boundary $\partial\overline{S(\tau)}$, we can find a sequence $\{x_i,\ i = 1, 2, \cdots\}$ in $S(\tau)$ approaching x_0. The sequence $\{(I + \tau A)x_i,\ i = 1, 2, \cdots\}$ is also inside $S(\tau)$ and approaches $(I + \tau A)x_0$. So we must have $(I + \tau A)x_0 \in \bar{S}(\tau)$.

By generalizing the proof of [1, Th. 2.2] from polytopes to general convex, compact sets, it can be shown that $\bar{S}(\tau)$ is also a positively invariant set for $\dot{x}(t) = Ax(t)$. Since $x(0) = B = x(0,\tau) \in S(\tau)$, it follows that the state impulse response $x(t)$ of $(A, B) \in \bar{S}(\tau)$. Define the set

$$P(\rho) = \{x \in R^n : |Cx| \le \rho, \rho > 0\} \tag{10}$$

then

$$\begin{aligned}\frac{1}{\tau}\|g^E(k,\tau)\|_{l_\infty} &= \sup_{k \ge 0} |Cx(k,\tau)| \\ &= \inf\{\rho \ge 0 : x(k,\tau) \in P(\rho), \text{ for all } k \ge 0\}.\end{aligned} \tag{11}$$

Therefore the points $\pm x(i,\tau)$, $i \ge 0$, are in the set $P(\frac{1}{\tau}\|g^E(k,\tau)\|_{l_\infty})$. Since this set is convex and closed, both $S(\tau)$ and $\bar{S}(\tau)$ are its subsets. As $x(t) \in \bar{S}(\tau)$, we have $\|g(t)\|_{\mathcal{L}_\infty} = \sup_{t>0} |Cx(t)| \le \frac{1}{\tau}\|g^E(k,\tau)\|_{l_\infty}$. □

Remark 2: It is important to note that Theorem 1 only holds for strictly proper systems. When D does not equal zero, the impulse response $g(t)$ of the continuous-time system will have an impulse function and hence it will no longer belong to $\mathcal{L}_\infty$. It follows that an $\mathcal{L}_\infty$-optimal solution will always render the closed-loop system strictly proper. Since discrete-time l_∞-optimal solutions do not share this property, additional precautions must be used when attempting to use the EAS approximation.

III. Main Results

Motivated by [2] and Theorem 1, we may want to consider the l_∞-optimal control problem for the corresponding EAS system

$$\inf_{q \in l_\infty} \|h^E - u^E * q\|_{l_\infty} \tag{12}$$

where $h^E(t)$ and $u^E(t)$ are the EAS of $h(t)$ and $u(t)$, respectively. However the l_∞-optimal closed-loop system is not strictly proper in general. To apply Theorem 1 we must add to the optimization problem (12) an additional constraint, namely that $\Phi^E(z) = H^E(z) - U^E(z)Q(z)$ must be strictly proper, or equivalently $\phi_o^E = \Phi(\infty) = 0$, resulting in the following nonstandard l_∞-optimization problem:

$$\mu_E = \inf_{q \in l_\infty} \|h^E - u^E * q\|_{l_\infty} \text{ subject to } \phi_o^E = 0. \tag{13}$$

Consider first the simpler case where both H^E and U^E are strictly proper. In this case the additional condition is automatically satisfied, and it is easily seen that (13) is equivalent to[1]

$$\mu_E = \inf_{q \in l_\infty} \|S_L * (h^E - u^E * q)\|_{l_\infty} \tag{14}$$

where S_L denotes the left-shift operator. Consider now the case where $H^E(z)$ and $U^E(z)$ are proper, but not strictly proper. Clearly, for the $\mathcal{L}_\infty$ problem (2) to have a finite solution, if $h_o^E = H^E(\infty) \neq 0$, then we must have $u_o^E \doteq U^E(\infty) \neq 0$ and must select $q_o \doteq Q(\infty) = h_o^E/u_o^E$. Define $\tilde{H}(z) \doteq H^E(z) - U^E(z) * q_o$ (note that $\tilde{H}(z)$ is *strictly proper* and such that $\tilde{H}(z_i^E) = H^E(z_i^E)$). The l_∞ problem can be rewritten as

$$\begin{aligned}\mu_E &= \inf_{q \in l_\infty} \|H^E(z) - U^E(z)Q(z)\|_{l_\infty} \text{ subject to } \phi_o^E = 0\\
&= \inf_{q \in l_\infty} \left\| H^E(z) - U^E(z)q_o - \frac{U^E(z)}{z}\sum_{i=1}^{\infty} q_i z^{-i+1}\right\|_{l_\infty}\\
&\qquad\qquad \text{subject to } q_o = \frac{h_o^E}{u_o^E}\\
&= \inf_{\tilde{q} \in l_\infty} \|\tilde{H}(z) - \tilde{U}(z)\tilde{Q}(z)\|_{l_\infty}\\
&= \inf_{\tilde{q} \in l_\infty} \|z\tilde{H}(z) - U^E(z)\tilde{Q}(z)\|_{l_\infty}\end{aligned} \tag{15}$$

where $\tilde{U}(z) = \frac{U^E(z)}{z}$ and $\tilde{Q}(z) = z(Q(z) - q_o)$, and where that last equality follows from the fact that both $\tilde{H}(z)$ and $\tilde{U}(z)$ are strictly proper. Since $\tilde{H}(z_i^E) = H^E(z_i^E)$ for all the unstable zeros of $U^E(z)$, it is straightforward to verify that both (14) and (15) have the same dual problem. These results are summarized in the following lemma.

Lemma 1: The nonstandard l_∞ problem (13) can be solved by transforming it to a standard l_∞ problem in either form (14) or form (15). Furthermore, both (14) and (15) lead to the same dual problem

$$\mu_E = \max_{\beta_j} \left[\sum_{i=1}^{n} \beta_i \operatorname{Re}\{z_i^E H^E(z_i^E)\} + \sum_{i=1}^{n} \beta_{i+n} \operatorname{Im}\{z_i^E H^E(z_i^E)\}\right]$$

$$\text{subject to } \sum_{k=0}^{\infty}\left|\sum_{i=1}^{n} \beta_i \operatorname{Re}\{(z_i^E)^{-k}\} + \sum_{i=1}^{n} \beta_{i+n} \operatorname{Im}\{(z_i^E)^{-k}\}\right| \leq 1 \tag{16}$$

where z_i^E denote the unstable zeros of $U^E(z)$.

After solving this dual problem, we get a strictly proper closed-loop system $\Phi^E(z)$ and an optimal controller $K^E(z)$. Transforming back to their continuous counterparts $\Phi_{\mathrm{EAS}}(s) = \Phi^E(1 + \tau s)$ and $K_{\mathrm{EAS}}(s) = K^E(1 + \tau s)$, we have that $\Phi_{\mathrm{EAS}}(s)$ is $\mathcal{L}_\infty$ stable and, from Theorem 1, $\|\phi_{\mathrm{EAS}}(t)\|_{\mathcal{L}_\infty} \leq \frac{\mu_E}{\tau}$. It can be shown that $\Phi_{\mathrm{EAS}}(s)$ satisfies the interpolation conditions. Hence we can use $\Phi_{\mathrm{EAS}}(s)$ and $K_{\mathrm{EAS}}(s)$ as approximations to the optimal closed-loop system $\Phi(s)$ and optimal controller $K(s)$ of the $\mathcal{L}_\infty$ problem, respectively.

Next we show that the error of the resulting approximation goes to zero when $\tau \to 0$. Furthermore, we indicate how to select τ *a priori* to meet any prespecified error bound.

Theorem 2: Given any $\epsilon > 0$, we can find a τ *a priori* for the EAS method such that

$$\mu^* \leq \|\phi^{\mathrm{EAS}}\|_{\mathcal{L}_\infty} \leq \frac{\mu_E}{\tau} \leq \mu^*(1 + \epsilon)$$

where ϕ^{EAS} represents the impulse response of the closed-loop system obtained using the EAS method. Moreover, the approximation error converges to zero as fast as $\mathcal{O}(\tau)$.

Proof: Consider the optimal l_∞-control problem for the EAS (14) or (15) and their dual (16). By [2, Th. 2] there exists $\tau_{\max}$ such that U^E is stable for all $0 < \tau < \tau_{\max}$. Moreover, U^E has the same number of unstable zeros as $U(s)$ since $U(s)$ has only unstable zeros. Let

$$\begin{aligned}\alpha_i &= \beta_i \operatorname{Re}\{z_i^E\} + \beta_{i+n} \operatorname{Im}\{z_i^E\}\\
\alpha_{i+n} &= -\beta_i \operatorname{Im}\{z_i^E\} + \beta_{i+n} \operatorname{Re}\{z_i^E\}.\end{aligned}$$

By direct calculations it can be easily shown that an alternative form of (16) is

$$\mu_E = \max_{\alpha_j} \left[\sum_{i=1}^{n} \alpha_i \operatorname{Re}\{H^E(z_i^E)\} + \sum_{i=1}^{n} \alpha_{i+n} \operatorname{Im}\{H^E(z_i^E)\}\right] \tag{17}$$

subject to

$$\|r_E(k,\alpha)\|_{l_1} = \sum_{k=0}^{\infty}\left|\sum_{i=1}^{n} \alpha_i \operatorname{Re}\{(z_i^E)^{-k-1}\} + \sum_{i=1}^{n} \alpha_{i+n} \operatorname{Im}\{(z_i^E)^{-k-1}\}\right| \leq 1. \tag{18}$$

From the relationship between the EAS and its corresponding continuous system, the above dual problem is equivalent to

$$\mu_E = \max_{\alpha_j} \left[\sum_{i=1}^{n} \alpha_i \operatorname{Re}\{H(z_i)\} + \sum_{i=1}^{n} \alpha_{i+n} \operatorname{Im}\{H(z_i)\}\right] \tag{19}$$

subject to

$$\|r_E(k,\alpha)\|_{l_1} = \sum_{k=0}^{\infty}\left|\sum_{i=1}^{n} \alpha_i \operatorname{Re}\{(1 + \tau z_i)^{-k-1}\} + \sum_{i=1}^{n} \alpha_{i+n} \operatorname{Im}\{(1 + \tau z_i)^{-k-1}\}\right| \leq 1. \tag{20}$$

Let us see how this problem can be related to the $\mathcal{L}_\infty$ problem. The dual problem of $\mathcal{L}_\infty$-optimal control is

$$\mu^* = \max_{\alpha_j} \left[\sum_{i=1}^{n} \alpha_i \operatorname{Re}\{H(z_i)\} + \sum_{i=1}^{n} \alpha_{i+n} \operatorname{Im}\{H(z_i)\}\right] \tag{21}$$

[1] Alternatively, the original l_∞ problem can be solved, adding the additional interpolation constraint $\Phi(\infty) = H(\infty) = 0$. A simple computation shows that this is equivalent to (14).

subject to

$$\|r(t,\alpha)\|_{L_1} = \int_0^\infty \left| \sum_{i=1}^n \alpha_i \operatorname{Re}\{e^{-z_i t}\} + \sum_{i=1}^n \alpha_{i+n} \operatorname{Im}\{e^{-z_i t}\} \right| dt \le 1. \quad (22)$$

The difference between the two problems is in the constraints. Let us first sample $r(t,\alpha)$ at the time points $t_k = k\tau$ and approximate the integral $\int_0^\infty r(t,\alpha)dt$ with an infinite sum $\sum_{k=1}^\infty \tau r(t_k,\alpha)$. Second, approximate the irrational terms $e^{-z_i t_k}$ in $r(t_k,\alpha)$ by rational terms $(1+\tau z_i)^{-k}$, obtaining a constraint identical to (20), up to a scaling factor τ. This scaling leads to the same scaling in the optimal cost. So the EAS method can be thought of as a two-step approximation of the original $\mathcal{L}_\infty$-optimal control problem.

We now derive an upper bound on any α satisfying the constraint (18). Moreover, this bound is linear in τ. Assume, without loss of generality, that $1/\tau$ is an integer. Define

$$r_E^j(k,\alpha) = \sum_{i=1}^n \alpha_i \operatorname{Re}\left\{ \left(z_i^E\right)^{-k/\tau - j} \right\} + \sum_{i=1}^n \alpha_{i+n} \operatorname{Im}\left\{ \left(z_i^E\right)^{-k/\tau - j} \right\} \quad j = 1, 2, \cdots, 1/\tau.$$

Then $\|r_E(k,\alpha)\|_{l_1} = \sum_{j=1}^{1/\tau} \|r_E^j(k,\alpha)\|_{l_1}$. A bound on $\|\alpha\|_1$ for all α satisfying the constraint $\|r_E(k,\alpha)\|_{l_1} \le 1$ can be obtained as follows. Consider the following sets:

$$S_1(\tau) \doteq \left\{\alpha \colon \left\| r_E^{\frac{1}{\tau}}(k,\alpha) \right\|_{l_1} \le 1 \right\}$$
$$R(\tau) \doteq \left\{\alpha \colon \left| r_E^{\frac{1}{\tau}}(k,\alpha) \right| \le 1 \quad \text{for} \quad k = 0, 1, \cdots, p-1 \right\}$$

where $p \ge n$. Clearly $S_1(\tau) \subset R(\tau)$. Hence $\sup_{\alpha \in S_1} \|\alpha\|_1 \le \sup_{\alpha \in R} \|\alpha\|_1$. Moreover, from Theorem 1 it can be shown that if $\tau_1 < \tau_2$, then $S_1(\tau_1) \subset S_1(\tau_2)$. It follows that given $\bar{\tau}$, a bound on $\|\alpha(\tau)\|_1$ for all $\tau \le \bar{\tau}$ is given by

$$\sup_{\alpha \in S_1(\tau)} \|\alpha(\tau)\|_1 \le \sup_{\alpha \in R(\bar{\tau})} \|\alpha\|_1 \le \|F^{-1}(\bar{\tau})\|_{1,1} \quad (23)$$

where $F(\cdot)$ is a $p \times n$ matrix defined by

$$F(\tau) = \left(\operatorname{Re}\left\{ \left(z_j^E\right)^{-i/\tau} \right\} \middle| \operatorname{Im}\left\{ \left(z_j^E\right)^{-i/\tau} \right\} \right) \quad j = 1, \cdots, n \quad \text{and} \quad i = 1, \cdots, p$$

and where $\|\cdot\|_{1,1}$ indicates the matrix norm induced from $(R^p, \|\cdot\|_1)$ to $(R^n, \|\cdot\|_1)$. Note that since $p \ge n$, $F(\tau)$ has full column rank. Hence its left inverse $F^{-1}(\tau)$ is well defined. Similarly, it can be shown easily that an upper bound on any α satisfying $\|r_E^j(k,\alpha)\|_{l_1} \le 1$ is given by

$$\|\alpha\|_1 \le \|F^{-1}(\bar{\tau})\|_{1,1} \max_i \left\{ \left| \left(z_i^E\right)^{-\left(\frac{1}{\tau} - j\right)} \right| \right\} \le \|F^{-1}(\bar{\tau})\|_{1,1}. \quad (24)$$

Therefore, (24) provides a common upper bound independent of j on $\|\alpha\|_1$ satisfying $\|r_E^j(k,\alpha)\|_{l_1} \le 1, \quad j = 1, \cdots, 1/\tau$. It follows that an upper bound on any α satisfying $\|r_E(k,\alpha)\|_{l_1} = \sum_{i=1}^{\frac{1}{\tau}} \|r_E^j(k,\alpha)\|_1 \le 1$ is given by

$$\|\alpha\|_1 \le \tau \|F^{-1}(\bar{\tau})\|_{1,1} \quad (25)$$

where $\bar{\tau}$ is fixed. Now, given any $t \in R_+$, assume that the constraint (18) is satisfied. Selecting k such that $t_k < t \le t_{k+1}$ we have that

$$\begin{aligned}
&\left| \left[\sum_{i=1}^n \alpha_i \operatorname{Re}\{(1+\tau z_i)^{-k-1}\} + \sum_{i=1}^n \alpha_{i+n} \operatorname{Im}\{(1+\tau z_i)^{-k-1}\} \right] - \left[\sum_{i=1}^n \alpha_i \operatorname{Re}\{e^{-z_i t}\} + \sum_{i=1}^n \alpha_{i+n} \operatorname{Im}\{e^{-z_i t}\} \right] \right| \\
&\le \|\alpha\|_1 \max_i \{ |(1+\tau z_i)^{-k-1} - e^{-z_i t}| \} \\
&\le \|\alpha\|_1 \left(\max_i \{ |(1+\tau z_i)^{-k-1} - e^{-z_i t_{k+1}}| \} + \max_i \{ |e^{-z_i t_{k+1}} - e^{-z_i t}| \} \right) \\
&\le \|\alpha\|_1 \left(\max_i \left\{ |e^{-z_i t_{k+1}}| \left| \left(\frac{e^{z_i \tau}}{1+\tau z_i} \right)^{k+1} - 1 \right| \right\} + \max_i \{ |e^{-z_i t_k}| |1 - e^{-z_i \tau}| \} \right) \\
&\le \|\alpha\|_1 \left(\max_i \left\{ (k+1) \left| \left(\frac{1}{1+\tau z_i} \right)^k \right| \left| \left(\frac{e^{z_i \tau}}{1+\tau z_i} \right) - 1 \right| \right\} + \max_i \{ |e^{-z_i t_k}| |1 - e^{-z_i \tau}| \} \right) \\
&\triangleq \hat{\epsilon}(k,\tau).
\end{aligned} \quad (26)$$

From (26) it follows that $|r(t,\alpha)| \le |r_E(k,\alpha)| + \hat{\epsilon}(k,\tau)$ and

$$\begin{aligned}
\|r(t,\alpha)\|_{\mathcal{L}_1} &= \int_0^\infty |r(t,\alpha)| dt = \sum_{k=0}^\infty \int_{t_k}^{t_{k+1}} |r(t,\alpha)| dt \\
&\le \tau \sum_{k=0}^\infty [|r_E(k,\alpha)| + \hat{\epsilon}(k,\tau)] \\
&\le \tau \left[1 + \sum_{k=0}^\infty \hat{\epsilon}(k,\tau) \right] = \tau(1+\epsilon(\tau))
\end{aligned} \quad (27)$$

where $\epsilon(\tau) \triangleq \sum_{k=0}^\infty \hat{\epsilon}(k,\tau)$. Each of the terms in $\sum_{k=0}^\infty \hat{\epsilon}$ can be bounded as follows:

$$\begin{aligned}
\epsilon_1(\tau) &\triangleq \sum_{k=0}^\infty \max_i \left\{ (k+1) \left| \left(\frac{1}{1+\tau z_i} \right)^k \right| \left| \frac{e^{z_i \tau}}{1+\tau z_i} - 1 \right| \right\} \\
&\le n \max_i \left\{ \frac{|1+\tau z_i|}{(|1+\tau z_i| - 1)^2} \frac{|\tau z_i|^2}{2(1 - |\tau z_i|)} \right\} \\
&\le n \max_i \left\{ \frac{|z_i|^2}{2(\operatorname{Re}\{z_i\})^2} \frac{1 + |\tau z_i|}{1 - |\tau z_i|} \right\} \\
&\qquad \left(\to n \max_i \left\{ \frac{|z_i|^2}{2(\operatorname{Re}\{z_i\})^2} \right\} \right)
\end{aligned} \quad (28)$$

$$\begin{aligned}
\epsilon_2(\tau) &\triangleq \sum_{k=0}^\infty \max_i \{ |e^{-z_i t_k}| |1 - e^{-z_i \tau}| \} \\
&\le n \max_i \left\{ \frac{|1 - e^{-z_i \tau}|}{1 - |e^{-z_i \tau}|} \right\} \qquad \left(\to n \max_i \left\{ \frac{|z_i|}{\operatorname{Re}\{z_i\}} \right\} \right).
\end{aligned} \quad (29)$$

Thus

$$\begin{aligned}
\epsilon(\tau) &\triangleq \sum_{k=0}^\infty \hat{\epsilon}(k,\tau) \\
&\le \|\alpha\|_1 n \left(\max_i \left\{ \frac{1 + |\tau z_i|}{2(1 - |\tau z_i|)} \right\} + \max_i \left\{ \frac{|e^{z_i \tau} - 1|}{|e^{z_i \tau}| - 1} \right\} \right) \\
&\le \tau \|F^{-1}(\bar{\tau})\|_{1,1} \\
&\quad \times n \left(\max_i \left\{ \frac{1 + |\tau z_i|}{2(1 - |\tau z_i|)} \right\} + \max_i \left\{ \frac{|e^{z_i \tau} - 1|}{|e^{z_i \tau}| - 1} \right\} \right).
\end{aligned} \quad (30)$$

Given any $\epsilon > 0$, we can always choose a τ such that $\epsilon(\tau) \le \epsilon$. Note that (28) also holds for α^* which solves the dual problem of EAS and that

$$\langle h, r(t,\alpha^*) \rangle = \sum_{i=1}^n \alpha_i^* H(z_i) = \mu_E$$

so we have

$$\mu^* = \max_{r \ne 0} \frac{\langle h, r \rangle}{\|r\|_{\mathcal{L}_1}} = \max_{\|r\|_{\mathcal{L}_1} \le \tau(1+\epsilon)} \frac{\langle h, r \rangle}{\tau(1+\epsilon)} \ge \frac{\mu_E}{\tau(1+\epsilon)}.$$

Finally, to prove the last part of the theorem consider the Taylor expansion of (30). Note that

$$\begin{aligned}\frac{|e^{z_i\tau}-1|}{|e^{z_i\tau}|-1} &= \frac{|z_i\tau+\frac{1}{2!}(z_i\tau)^2+\frac{1}{3!}(z_i\tau)^3+\cdots|}{e^{\mathrm{Re}\{z_i\}\tau}-1}\\ &\le \frac{|z_i\tau|+\frac{1}{2!}|z_i\tau|^2+\frac{1}{3!}|z_i\tau|^3+\cdots}{\mathrm{Re}\{z_i\}\tau+\frac{1}{2!}(\mathrm{Re}\{z_i\}\tau)^2+\frac{1}{3!}(\mathrm{Re}\{z_i\}\tau)^3+\cdots}.\end{aligned} \tag{31}$$

Thus we have that $\frac{\mu^E}{\tau}-\mu^*\le\epsilon\mu^*\to 0$ as $\mathcal{O}(\tau)$ [since $\epsilon\to 0$ as $\mathcal{O}(\tau)$]. □

The following lemma is a straightforward application of [6, Th. 5] to the EAS.

Lemma 2: Given any $\epsilon>0$, an l_∞-suboptimal control for the EAS problem with cost μ_E such that

$$\mu_E\le\mu\le\mu_E(1+\epsilon)$$

can be found by solving a system of linear equations. Moreover, the suboptimal closed-loop system has finite support. A lower bound for the number of support N can be obtained from

$$\frac{(z_m^E)^{-(N+1)}}{\left(1-(z_m^E)^{-1}\right)}\|F^{-1}(0)\|_{1,1}\le\epsilon \tag{32}$$

where $F(0)\triangleq(\mathrm{Re}\{e^{-iz_j}\}\mid\mathrm{Im}\{e^{-iz_j}\})\ j=1,\cdots,n$ and $i=1,\cdots,p$, and where $z_m^E=\min_i\{|z_i^E|\}=\min_i\{|1+\tau z_i|\}$.

Combining the results of Theorem 2 and Lemma 3, it follows that a rational suboptimal solution to the $\mathcal{L}_\infty$-control problem can always be found, and its cost can be made arbitrarily close to the optimal cost μ^*. Denoting the $\mathcal{L}_\infty$ norm of the rational suboptimal solution as μ_R, then by Theorem 1 we have $\mu_R\le\mu/\tau$. Given any $\epsilon>0$, we can easily find $\epsilon_1>0$ and $\epsilon_2>0$ such that

$$\epsilon_1+\epsilon_2+\epsilon_1\epsilon_2\le\epsilon.$$

If we select τ according to Theorem 2 such that the error bound ϵ_1 is satisfied and select N according to Lemma 2 such that error bound ϵ_2 is satisfied, we have that

$$\begin{aligned}\mu_R-\mu^* &\le\frac{\mu}{\tau}-\mu^*\\ &=\left(\frac{\mu}{\tau}-\frac{\mu_E}{\tau}\right)+\left(\frac{\mu_E}{\tau}-\mu^*\right)\\ &\le\frac{\mu_E}{\tau}\epsilon_2+\mu^*\epsilon_1\\ &\le\mu^*(\epsilon_1+\epsilon_2+\epsilon_1\epsilon_2)\le\mu^*\epsilon.\end{aligned}$$

Therefore the given error bound ϵ for the rational suboptimal solution is satisfied. Moreover, since the finite support l_∞-suboptimal solution for the EAS problem is internally asymptotically stable, so is the corresponding suboptimal solution to the $\mathcal{L}_\infty$-control problem.

We have shown above that the EAS method is an effective way to obtain a rational approximation to the $\mathcal{L}_\infty$-control problem. There exist, of course, many ways to develop rational approximations. Next we will show that the EAS approximation is the best one among a certain subset $\Omega(\tau,N)$ of the set of rational approximations, in the sense that it yields the tightest upper bound. For a given $0<\tau<\tau_{\max}$ and a given $N>0$, the closed-loop system obtained using the EAS method is

$$\Phi(s)=\sum_{i=1}^N\phi_i^E(\tau)(1+\tau s)^{-i}.$$

Define

$$\Omega(\tau,N)=\left\{\Phi(s)=\sum_{i=1}^N\phi_i(\tau)(1+\tau s)^{-i}:\Phi(z_k)=H(z_k),k=1,\cdots,n\right\}. \tag{33}$$

Clearly, all elements in $\Omega(\tau,N)$ (including the EAS approximation) can be thought of as rational approximations of the $\mathcal{L}_\infty$-optimal closed-loop system. Let $\gamma\triangleq\max_i\{|\phi_i(\tau)|,\ i=1,\cdots,N\}$. Since γ is the l_∞ norm of the EAS of $\Phi(s)=\sum_{i=1}^N\phi_i(\tau)(1+\tau s)^{-i}$, it follows from Theorem 1 that

$$\|\phi(t)\|_{\mathcal{L}_\infty}\le\frac{\gamma}{\tau}$$

and

$$\lim_{\tau\to 0}\frac{\gamma}{\tau}=\|\phi(t)\|_{\mathcal{L}_\infty}.$$

The following theorem presents a result based on this upper bound of $\|\phi(t)\|_{\mathcal{L}_\infty}$.

Theorem 3: The rational approximation of the $\mathcal{L}_\infty$-optimal controller given by the EAS method leads to the smallest upper bound γ/τ among the elements of the set $\Omega(\tau,N)$.

Proof: Consider the following set:

$$\Omega_E(\tau,N)=\left\{\Phi^E(z)=\sum_{i=1}^N\phi_iz^{-i}:\Phi^E(z_k^E)=H^E(z_k^E),k=1,2,\cdots,n\right\}. \tag{34}$$

For every $\Phi^E(z)$ in $\Omega_E(\tau,N)$ there is a $\Phi(s)=\Phi^E(1+\tau s)$ in $\Omega(\tau,N)$, and *vice versa*. Since $\gamma=\|\phi^E\|_{l_\infty}$, the closed-loop system Φ^E obtained by solving the optimal l_∞-control problem for the EAS certainly has the smallest γ among the elements of the set $\Omega_E(\tau,N)$. It follows that the rational closed-loop system obtained using the EAS methods also has the smallest upper bound γ/τ among the elements of the set $\Omega(\tau,N)$. □

Remark 3: The theorem only states that the rational approximation obtained using the EAS method is the best one in the sense that it leads to the smallest upper bound γ/τ. However, following a procedure similar to [10], it can be shown that this bound converges monotonically to the optimal cost. Hence the gap between the upper bound and the actual $\mathcal{L}_\infty$ norm vanishes as $\tau\to 0$.

IV. An Example

Consider the plant

$$P(s)=\frac{s-2}{s-1}$$

with a step disturbance d. We want to design a controller $C(s)$ to minimize the amplitude of the regulated output $z(t)$ $(z(s)=\Phi(s)=(1+P(s)C(s))^{-1}d(s))$.

Since the output is not weighted, we have only an input weight $W_I(s)=d(s)=\frac{1}{s}$. To get a Youla parameterization, a precompensator containing this dynamics must be used. We choose a proper precompensator $\frac{s+1}{s}$. Then the problem becomes that of finding an $\mathcal{L}_\infty$-stabilizing compensator $\hat{C}(s)$ for the augmented plant

$$\hat{P}(s)=\frac{(s+1)(s-2)}{s(s-1)}$$

such that the amplitude of $z(t)$ is minimized. One Youla parameterization is given by

$$\begin{aligned}\Phi(s)&=H(s)-U(s)Q(s)\\ &=\frac{6(s-1)}{(s+1)(s+2)}-\frac{(s-1)(s-2)}{(s+1)(s+2)^2}Q(s).\end{aligned}$$

For this plant the exact $\mathcal{L}_\infty$-optimal solution can be obtained and is given below (see [11])

$$\Phi(s) = \frac{2}{s}[1 - 2(2)^{-s}]$$

with optimal cost $\mu^* = 2$ and corresponding $\mathcal{L}_\infty$-optimal controller given by

$$C(s) = \frac{(s-1)(4(2)^{-s} - 1)}{(s-2)(2 - 4(2)^{-s})}.$$

We proceed now to find a rational controller using the EAS method. Let $\epsilon = 0.1$. With $\tau = 0.001$, according to (30) with $\bar{\tau} = 1$ we have $\epsilon(\tau) = 0.0901 \leq 0.1$. The Youla parameterization of the EAS problem is

$$\Phi(z) = z(H^E(z) - U^E(z)Q(z))$$

where

$$H^E(z) = H\left(\frac{z-1}{\tau}\right) = \frac{0.006(z - 1.001)}{(z - 0.999)(z - 0.998)}$$
$$U^E(z) = U\left(\frac{z-1}{\tau}\right) = \frac{0.001(z - 1.001)(z - 1.002)}{(z - 0.999)(z - 0.998)^2}.$$

Since $zU^E(z)$ has only two unstable real zeros, the l_∞-optimal control for the EAS problem can also be solved exactly. The optimal cost is

$$\mu_E = 0.002\,001\,4$$

and the optimal solution is of the form

$$\Phi(z) = \phi_0 \sum_{i=0}^{q-1} z^{-i} + \phi_q z^{-q} - \phi_0 \sum_{i=q+1}^{\infty} z^{-i}$$

with $q = 693$, $\phi_0 = 0.002\,001\,4$, and $\phi_q = -0.000\,024\,7$. So

$$\begin{aligned}\Phi^E(z) &= z^{-1}\Phi(z)\\ &= 0.002\,001\,4 \frac{1}{z-1}(1 - 2z^{-693}) + .001\,976\,7 z^{-694}\end{aligned}$$

and the rational closed-loop system obtained using the EAS method is

$$\begin{aligned}\Phi_{\mathrm{EAS}}(s) = \Phi^E(1 + \tau s) &= 2.0014 \frac{1}{s}(1 - 2(1 + .001s)^{-693})\\ &\quad + .001\,976\,7(1 + .001s)^{-694}\end{aligned}$$

with $\|\phi_{\mathrm{EAS}}(t)\|_{\mathcal{L}_\infty} \leq \mu_E/\tau = 2.0014$.

Note in passing that τ could be much larger than the value estimated from (30) to satisfy the given approximation error ϵ. For this example, even with $\tau = 0.1$, we can get an optimal cost $\mu_E = 0.213\,64$ which still satisfies the ten percent error requirement. This larger value of τ results in a lower-order approximation

$$\Phi_{\mathrm{EAS}}(s) = 2.1364 \frac{1}{s}(1 - 2(1 + .1s)^{-7}) + 0.120\,52(1 + .1s)^{-8}.$$

V. Conclusion

We have shown in [11] that the $\mathcal{L}_\infty$-optimal control problem leads to nonrational closed-loop systems, even when the augmented plant is rational. In this paper we present an EAS-based method to obtain *rational* $\mathcal{L}_\infty$-suboptimal solution for general SISO systems. The suboptimal cost can be made arbitrarily close to the optimal cost by selecting a sufficiently small value of the parameter τ. Moreover, τ can be selected *a priori* to satisfy a given approximation error bound. It is also shown that the rational approximation obtained using the EAS method is the best one among certain sets of rational approximations, in the sense of yielding the tightest upper bound.

Even though rational approximations are available, further work toward obtaining exact solutions for general systems is worth pursuing. This is not only of theoretical interests but will also enhance our understanding of the $\mathcal{L}_\infty$-optimal control problem and the structure of the optimal solutions. A model reduction technique in the context of $\mathcal{L}_\infty$-optimization would be of significant practical value since rational $\mathcal{L}_\infty$-suboptimal controllers may have very high order.

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References

[1] F. Blanchini, "Feedback control for linear time-invariant systems with state and control bounds in the presence of disturbances," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 1231–1234, 1990.

[2] F. Blanchini and M. Sznaier, "Rational $\mathcal{L}^1$ suboptimal compensators for continuous-time systems," *IEEE Trans. Automat. Contr.*, vol. 39, no. 7, pp. 1487–1492, July 1994.

[3] M. A. Dahleh and J. B. Pearson, "l^1-optimal feedback controllers for MIMO discrete-time systems," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 314–322, Apr. 1987.

[4] ——, "$\mathcal{L}^1$-optimal compensators for continuous-time systems," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 889–895, Oct. 1987.

[5] ——, "Optimal rejection of persistent disturbances, robust stability, and mixed sensitivity minimization," *IEEE Trans. Automat. Contr.*, vol. AC-33, pp. 722–731, Aug. 1988.

[6] ——, "Minimization of a regulated response to a fixed input," *IEEE Trans. Automat. Contr.*, vol. AC-33, pp. 924–930, Oct. 1988.

[7] I. J. Diaz-Bobillo and M. A. Dahleh, "Minimization of the maximum peak-to-peak gain: The general multiblock problem," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 1459–1482, Oct. 1993.

[8] J. Doyle, K. Glover, P. Khargonekar, and B. Francis, "State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 831–846, Aug. 1989.

[9] Y. Ohta, H. Maeda, and S. Kodama, "Rational approximation of $\mathcal{L}_1$ optimal controllers for SISO systems," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1683–1691, Nov. 1992.

[10] M. Sznaier and F. Blanchini, "Mixed $\mathcal{L}_\infty/\mathcal{H}_\infty$ suboptimal controllers for continuous-time systems," *IEEE Trans. Automat. Contr.*, vol. 40, no. 11, pp. 1831–1840, Nov. 1995.

[11] Z.-Q. Wang and M. Sznaier, "$\mathcal{L}_\infty$-optimal control of SISO continuous time systems and its rational approximations," in *Proc. 33rd Conf. Decision Contr.*, Lake Buena Vista, FL, Dec. 1994, pp. 34–39.

[12] Z.-Q. Wang, M. Sznaier, and F. Blanchini, "Further results on rational approximations of $\mathcal{L}^1$-optimal controllers," *IEEE Trans. Automat. Contr.*, vol. 40, no. 3, pp. 552–557, 1995.