Further Results on Rational Approximations of $\mathcal{L}^1$ Optimal Controllers

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***Abstract*—The continuous-time persistent disturbance rejection problem ($\mathcal{L}^1$ optimal control) leads to nonrational compensators, even for SISO systems [4], [7], [8]. As noted in [4], the difficulty of physically implementing these controllers suggest that the most significant applications of the continuous time $\mathcal{L}^1$ theory is to furnish bounds for the achievable performance of the plant. Recently, two different rational approximations of the optimal $\mathcal{L}^1$ controller were developed by Ohta *et al.* [6] and by Blanchini and Sznaier [1]. In this paper we explore the connections between these two approximations. The main result of the paper shows that both approximations belong to the same subset Ω_T of the set of rational approximations, and that the method proposed in [1] gives the best approximation, in the sense of providing the tightest upper bound of the approximation error, among the elements of this subset. Additionally, we exploit the structure of the dual to the $\mathcal{L}^1$ optimal control problem to obtain rational approximations with approximation error smaller than a prespecified bound.**

I. Introduction

A large number of control problems involve designing a controller capable of stabilizing a given linear time invariant system while minimizing the worst case response to some exogenous disturbances. This problem is relevant for instance for disturbance rejection, tracking and robustness to model uncertainty (see [2] and references therein). When the exogenous disturbances are modeled as bounded energy signals and performance is measured in terms of the energy of the output, this problem leads to the well known $\mathcal{H}_\infty$ theory. The case where the signals involved are persistent bounded signals leads to the $\mathcal{L}^1$ optimal control theory, formulated and further explored by Vidyasagar [7], [8] and solved by Dahleh and Pearson both in the discrete- [3], [5] and continuous-time [4] cases.

The $\mathcal{L}^1$ theory is appealing because it directly incorporates time-domain specifications. Moreover, it furnishes a complete solution to the robust performance problem (see [2] for a good tutorial and a list of relevant references). However, in contrast with the discrete time l^1 theory, the solution for the continuous-time $\mathcal{L}^1$ optimal control problem leads to nonrational compensators, even for SISO systems. As noted in [4], the difficulty of physically implementing these controllers suggests that the most significant application of the continuous time $\mathcal{L}^1$ theory is to provide performance bounds for the plant. Recently, two rational approximations to the optimal $\mathcal{L}^1$ controller were developed independently [6], [1]. Although these approximations are based upon different techniques ([6] follows an algebraic approach while [1] exploits the properties of the Euler approximating set), they seem to be strongly connected [1]. Noteworthy, they yield closed-loop plants with the same pole structure.

In this paper we explore the connection between these approaches. The main result of the paper shows that both belong to the same subset Ω_τ of the set of admissible rational approximations, and that the method proposed in [1] gives the best approximation (in the sense of providing the tightest upper bound of the error) among the elements of this set. Additionally, by exploiting the structure of the dual to the $\mathcal{L}^1$ optimal control problem we furnish a procedure to compute rational approximations with error smaller than a prespecified bound ϵ, and we show that the approximation error→0 as $O(\tau)$.

The paper is organized as follows. In Section II we introduce the notation to be used and we restate the main results concerning the $\mathcal{L}^1$ problem and its rational approximations. Section III contains the majority of the theoretical results. Here we compare the two approximation methods and we show that, in a sense, the method proposed in [1] yields the best rational approximation. In Section IV we present a simple design example and we compare the optimal $\mathcal{L}^1$ controller with its rational approximations. Finally, in Section V we summarize our results.

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II. Preliminaries

A. Notation and Definitions

R_+ denotes the set of nonnegative real numbers. $\mathcal{L}^\infty(R_+)$ denotes the space of measurable functions $f(t)$ equipped with the norm: $\|f\|_\infty = ess \cdot \sup_{R_+}|f(t)|$. $\mathcal{L}^1(R_+)$ denotes the space of Lebesgue integrable functions on R_+ equipped with the norm $\|f\|_1 \triangleq \int_0^\infty |f(t)|dt < \infty$. Similarly, l_1 denotes the space of absolutely summable sequences $h = \{h_i\}$ equipped with the norm $\|h\|_1 \triangleq \sum_{k=0}^\infty |h_i| < \infty$. $\mathcal{RL}^1$ denotes the subspace of $\mathcal{L}^1$ formed by matrices with real rational Laplace transform. A denotes the space whose elements have the form

$$h = h^L(t) + \sum_{k=0}^\infty h_i^l \delta(t - t_i)$$

where $h^L(t) \in L_1(R_+)$, $\{h_i^l\} \in l_1$ and $t_i \geq 0$, equipped with the norm $\|h\|_A \triangleq \|h^L\|_{L_1} + \|h^l\|_{l_1}$. Given a function $f(t) \in \mathcal{L}^1$ we denote its Laplace transform by $F(s) \in \mathcal{L}_\infty$; similarly, given $h \in A$, we denote its Laplace transform by $H(s)$. By a slight abuse of notation, we denote as $\|F(s)\|_1 \triangleq \|f(t)\|_1$ and $\|H(s)\|_A = \|h\|_A$. Throughout the paper we use packed notation to represent state-space realizations, i.e.,

$$G(s) = C(sI - A)^{-1}B + D \triangleq \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right).$$

Definition 1: Consider the continuous time system $G(s)$. Its Euler approximating system (EAS) is defined as the following discrete time system

$$G^E(z, \tau) = \left(\begin{array}{c|c} I + \tau A & \tau B \\ \hline C & D \end{array}\right). \tag{1}$$

From this definition it is easily seen that we can obtain the EAS of $G(s)$ by the simple variable transformation $s = (z - 1)/\tau$, i.e.,

$$G^E(z, \tau) = G\left(\frac{z-1}{\tau}\right).$$

On the other hand, for any given τ we can relate a discrete time system to a continuous system by the inverse transformation $z = 1 + \tau s$. It is obvious that the discrete time system is, in fact, the EAS of the continuous time system obtained in this form.

Definition 2: Consider a system of the form

$$\Phi(s) = T_1(s) + T_2(s)Q(s)$$

where $T_2(s)$ has all its zeros $\{z_1, z_2, \cdots, z_n\}$ in the open right-half plane and where, for simplicity, we assume that all the zeros are

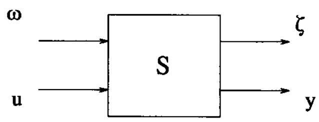

Fig. 1. The generalized plant.

distinct. $\Phi(s)$ is said to be admissible if it is stable and satisfies the interpolation conditions

$$\Phi(z_k) = T_1(z_k), \quad k = 1, \cdots, n.$$

B. The $\mathcal{L}^1$ Optimal Control Problem

Consider the system shown in Fig. 1, where S represents the system to be controlled; the scalar signals $\omega \in \mathcal{L}^\infty$ and u represent an exogenous disturbance and the control action respectively; and where ζ and y represent the output subject to performance constraints and the measurements available to the controller, respectively. As usual we will assume, without loss of generality, that any weights have been absorbed in the plant S. Then, the $\mathcal{L}^1$ optimal control problem can be stated as: Given the system (S) find an internally stabilizing controller $u(s) = K(s)y(s)$ such that the worst case (over the set of all $\omega(t) \in \mathcal{L}^\infty$, $\|\omega\|_\infty \le 1$) maximum amplitude of the performance output $\zeta(t)$ is minimized.

By using the YJBK parameterization of all stabilizing controllers [4], [9], the problem can be cast into the following model matching form

$$\mu_0 = \inf_{K \text{ stabilizing}} \|\Phi(s)\|_A = \inf_{Q \in A} \|T_1(s) + T_2(s)Q(s)\|_A \qquad (2)$$

where T_1, T_2 are rational stable transfer functions.

Next, we recall the main result of [4], showing that a solution to the $\mathcal{L}^1$ optimal control problem can be found by solving a semi-infinite linear programming problem.

Theorem 1 (Dahleh and Pearson [4]): Let $T_2(s)$ have n zeros z_i in the open right-half plane and no zeros on the jw axis. Then

$$\begin{aligned}\mu_0 &= \inf_{Q \in A} \|T_1(s) + T_2(s)Q(s)\|_A \\ &= \max_{\alpha_j} \left[\sum_{i=1}^{n} \alpha_i \operatorname{Re}\{T_1(z_i)\} + \sum_{i=1}^{n} \alpha_{i+n} \operatorname{Im}\{T_1(z_i)\}\right] \qquad (3)\end{aligned}$$

subject to

$$|r(t)| = \left|\sum_{i=1}^{n} \alpha_i \operatorname{Re}\{e^{-z_i t}\} + \sum_{i=1}^{n} \alpha_{i+n} \operatorname{Im}\{e^{-z_i t}\}\right| \le 1, \quad \forall t \in R_+. \qquad (4)$$

Furthermore, the following facts hold: i) the extremal functional $r^*(t)$ equals 1 at only finite points: $t_1, \cdots, t_m$; ii) an optimal solution $\Phi(s) = T_1(s) + T_2(s)Q(s)$ to the left side problem always exists; and iii) the optimal ϕ has the following form

$$\phi = \sum_{i=1}^{m} \phi_i \delta(t - t_i), \quad t_i \in R_+, \quad m \text{ finite} \qquad (5)$$

and satisfies the following conditions

a) $\phi_i r^*(t_i) \ge 0$;

b) $\sum_{i=1}^{m} |\phi_i| = \mu_0$;

c) $\sum_{i=1}^{m} \phi_i e^{-z_k t_i} = T_1(z_k)$, $k = 1, \cdots, n$.

Remark 1: It was shown in [4] that we need to satisfy constraints (4) only for all $t \le t_{\max}$ where $t_{\max}$ is finite and can be determined a priori. Even so, there are still infinite constraints, and therefore the dual problem is a semi-infinite linear programming problem.

C. Rational Approximations to the Optimal $\mathcal{L}^1$ Controller

From (5) it follows that, unlike in the discrete-time case, the $\mathcal{L}^1$ optimal controller is irrational even if the plant is rational. Prompted by the difficulty in physically implementing a controller with a nonrational transfer function, two rational approximation methods have been recently developed by Ohta *et al.* [6] and by Blanchini and Sznaier [1]. In the sequel we briefly review these results. For brevity, we refer to the former as the OMK method and to the latter as the EAS method.

Theorem 2 (Ohta et al., [6]): Let

$$T_1(s) = \left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right) \quad \text{and} \quad T_2(s) = \left(\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right)$$

be minimal realizations. Define

$$\begin{aligned} L &= B_2 D_2^{-1} \\ \hat{A} &= A_2 - LC_2 \\ M &= LD_1 + R_1 B_1 \end{aligned}$$

where R_1 is the unique solution of the linear matrix equation

$$\hat{A} R_1 - R_1 A_1 = LC_1.$$

Then, there exist finite sets $\{t_1\, t_2 \cdots t_m\}$ and $\{\phi_1\, \phi_2 \cdots \phi_m\}$ such that

$$M = \sum_{i=1}^{m} \phi_i \exp(-\hat{A} t_i) L \qquad (6)$$

and $\mu_0 = \sum_{i=1}^{m} |\phi_i|$. For $\tau > 0$, define $N(t_i, \tau) \triangleq$ the smallest integer larger than or equal to t_i/τ, $i = 1, 2, \cdots, m$, and $N(\tau) \triangleq N(t_m, \tau)$. Finally, denote by $\phi(\tau)$ the minimizer of $\|\phi(\tau) - \phi\|_2$ subject to

$$M = \sum_{i=1}^{m} \phi_i(\tau)(I + \tau \hat{A})^{-N(t_i, \tau)} L \qquad (7)$$

where $\phi(\tau) = [\phi_1(\tau), \cdots, \phi_m(\tau)]$, and $\phi = [\phi_1, \cdots, \phi_m]$. Consider the rational system $\Phi(s, \tau)$ with the following state-space realization

$$\Phi(s, \tau) = \left(\begin{array}{c|c} A(\tau) & B_4(\tau) \\ \hline C_3(\tau) & D_{34}(\tau) \end{array}\right)$$

where the matrices A, B_4, C_3, D_{34} are defined as follows

$$A(\tau) = \tau^{-1} \begin{pmatrix} -1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & -1 & 1 \\ 0 & \cdots & \cdots & 0 & 0 & -1 \end{pmatrix}, \quad N(\tau) \text{ by } N(\tau)$$

$$B_4(\tau)\text{: its } k\text{th element} \begin{cases} 0, & \text{if } k \notin \{N(t_i, \tau),\ i = 1, 2, \cdots, m\} \\ \phi_i(\tau), & \text{if } k = N(t_i, \tau) \end{cases}$$

$$C_3(\tau) = (\tau^{-1} \quad 0 \quad \cdots \quad 0), \qquad 1 \text{ by } N(\tau)$$

$$D_{34}(\tau) = \begin{cases} 0, & \text{if } N(t_1, \tau) \ne 0 \\ \phi_1(\tau), & \text{if } N(t_1, \tau) = 0. \end{cases}$$

Then, as $\tau_i \to 0$, we have that $\Phi(s, \tau) \to \Phi_{\mathrm{OPT}}(s)$ uniformly in the wide sense in the open half plane $\operatorname{Re}(s) > -\sigma$ for some $\sigma > 0$; and $\|\Phi\|_A$, as well as its upper bound $\gamma = \sum_{i=1}^{q} |\phi_i(\tau)|$, converge to μ_0.

Remark 2: As we will show later, (6) and (7) are just another version of the interpolation condition.

Next, we recall the main result of [1] showing that the $\mathcal{L}^1$ norm of a stable transfer function is bounded above by the l^1 norm of its EAS. Moreover, this bound can be made arbitrarily tight by taking the parameter τ in (1) small enough. This result is the basis for the approximation procedure proposed in [1].

Theorem 3 (Blanchini and Sznaier [1]): Consider a continuous time system with rational Laplace transform $\Phi(s)$ and its EAS, $\Phi^E(z, \tau)$. If $\Phi^E(z, \tau)$ is asymptotically stable, then $\Phi(s)$ is also asymptotically stable and such that

$$\|\Phi(s)\|_1 \leq \|\Phi^E(z, \tau)\|_1.$$

Conversely, if $\Phi(s)$ is asymptotically stable and such that $\|\Phi(s)\|_1 \triangleq \mu_c$, then for all $\mu > \mu_c$ there exists $\tau^* > 0$ such that for all $0 < \tau \leq \tau^*$, $\Phi^E(z, \tau)$ is asymptotically stable and such that $\|\Phi^E(z, \tau)\|_1 \leq \mu$.

Theorem 4 [1]: Consider a strictly decreasing sequence $\tau_i \to 0$, and define

$$\mu_i \triangleq \inf_{\text{stabilizing K}} \|\Phi_{cl}^E(z, \tau_i)\|_1$$

where $\Phi_{cl}^E(z, \tau_i)$ denotes the closed-loop transfer function. Then the sequence μ_i is nonincreasing and such that $\mu_i \to \mu_0$, the optimal $\mathcal{L}^1$ cost.

Corollary: A suboptimal rational solution to the $\mathcal{L}^1$ optimal control problem for continuous time systems, with cost arbitrarily close to the optimal cost, can be obtained by solving a discrete-time l_1 optimal control problem for the corresponding EAS. Moreover, if $K(z)$ denotes the optimal l_1 compensator for the EAS, the suboptimal $\mathcal{L}^1$ compensator is given by $K(\tau s + 1)$.

III. Analysis of the Different Rational Approximations

In this section we analyze the rational approximations generated by the OMK and EAS methods. The main result shows that both approximations belong to a certain subset Ω_τ of the set of rational approximations, and that the EAS method generates the best approximation among the elements of this subset. We begin by showing that the two expressions for matrix M in Theorem 2 are just another version of the interpolation conditions.

A. Characterization of All Rational Approximations

Lemma 1: For a closed-loop system of the form

$$\Phi(s) = \sum_{i=1}^{q} \phi_i e^{-t_i s}, \qquad t_i \in \mathcal{R}_+$$

the following two conditions are equivalent

a) $\Phi(z_k) = \sum_{i=1}^{q} \phi_i e^{-z_k t_i} = T_1(z_k),\ k = 1, \cdots, n.$

b) $M = \sum_{i=1}^{q} \phi_i \exp(-\hat{A} t_i) L.$

Moreover, we have $\|\Phi(s)\|_A = \gamma \triangleq \sum_{i=1}^{q} |\phi_i|$.

Proof: b)⇒ a) can be proved following the proof procedure of [6, lemma 2] by simply replacing Φ_{OPT} with Φ. Similarly, the fact that b) is necessary for a) to hold can also be concluded from the proof. The expression for $\|\Phi(s)\|_A$ follows from direct calculations.

In the next lemma we give a complete characterization of all rational approximations.

Lemma 2: For any rational closed-loop system

$$\Phi(s) = \sum_{j=1}^{q_i} \sum_{i=1}^{q_i} \phi_{ij} (1 + \lambda_j s)^{-N_{ij}}$$

where $\text{Re}(\lambda_j) > 0$ and N_{ij} integers, the following two conditions are equivalent

a) $\Phi(z_k) = \sum_{j=1}^{q_j} \sum_{i=1}^{q_i} \phi_{ij} (1 + \lambda_j z_k)^{-N_{ij}} = T_1(z_k),\ k = 1, \cdots, n.$

b) $M = \sum_{j=1}^{q_j} \sum_{i=1}^{q_i} \phi_{ij} (I + \lambda_j \hat{A})^{-N_{ij}} L.$

Moreover, we have $\|\Phi(s)\|_A \leq \gamma \triangleq \sum_{j=1}^{q_j} \sum_{i=1}^{q_i} |\phi_{ij}|$.

Proof: a)⇔ b) can be proved using the same idea. The calculations, through straightforward, are tedious, and are omitted here for space reasons. By direct calculation we have

$$\begin{aligned} \|\Phi(s)\|_A &= \int_0^\infty \left| \sum_{j=1}^{q_j} \sum_{i=1}^{q_i} \phi_{ij} \frac{1}{(N_{ij} - 1)! \lambda_j^{N_{ij}}} t^{N_{ij} - 1} e^{-t/\lambda_j} \right| dt \\ &\leq \sum_{j=1}^{q_j} \sum_{i=1}^{q_i} |\phi_{ij}| \int_0^\infty \frac{1}{(N_{ij} - 1)! \lambda_j^{N_{ij}}} t^{N_{ij} - 1} e^{-t/\lambda_j}\, dt \\ &= \gamma \triangleq \sum_{j=1}^{q_j} \sum_{i=1}^{q_i} |\phi_{ij}|. \end{aligned}$$ □

B. Comparison of the OMK and the EAS Rational Approximations

Lemma 2 gives a characterization of all rational admissible closed-loop systems. All these closed-loop systems can be thought as candidate rational approximations of the $\mathcal{L}^1$-optimal closed-loop system. In the sequel we concentrate on a specific subset Ω_τ and we show that both the OMK and the EAS methods generate approximations that belong to this subset. For $\tau > 0$, define

$$\Omega_\tau = \left\{ \Phi(s) = \sum_{i=1}^{q} \phi_i(\tau)(1 + \tau s)^{-N_i} : M = \sum_{i=1}^{q} \phi_i(\tau)(I + \tau \hat{A})^{-N_i} L \right\}. \tag{8}$$

By direct calculation, the closed-loop system obtained by OMK methods is

$$\Phi(s) = \sum_{i=1}^{m} \phi_i(\tau)(1 + \tau s)^{-N(t_i, \tau)}.$$

Suppose that the l_1-optimal closed-loop system for the EAS is given by

$$\Phi^E(z) = \sum_{i=1}^{q} \phi_i^E(\tau) z^{-N_i^E}$$

then the closed-loop system obtained using the EAS method is

$$\Phi(s) = \sum_{i=1}^{q} \phi_i^E(\tau)(1 + \tau s)^{-N_i^E}.$$

It follows that the approximations generated by both methods belong to the set Ω_τ, with a specific $\{N_i\}$ determined by each method.

Remark 3: In the OMK method, $\{N(t_i, \tau)\}$ depend directly on $\{t_i\}$, and hence on the $\mathcal{L}^1$ optimal closed-loop system. Hence, obtaining a rational approximation requires solving the $\mathcal{L}^1$ optimal control problem first. However, as pointed out in Remark 1, solving exactly this problem entails solving a semi-infinite linear programming problem. The EAS method requires only solving a discrete l_1 optimal control problem, which is considerable easier, since only finite-dimensional linear programming is involved.

Remark 4: Note that additional OMK-like rational approximations can be chosen among the elements of Ω_τ by simply modifying the rule for selecting $N(t_i, \tau)$. For instance, N could be selected as the largest integer smaller than or equal to t_i/τ or $t_i/\tau + 0.5$. Clearly, the convergence property also holds for these approximations.

As it is shown in Theorem 6, the EAS method can be interpreted as approximating the original optimization problem, as opposed to directly approximating its irrational solution. This makes it quite unique. Besides the computational advantages, we show in the sequel that the EAS method has two other important merits.

Theorem 5: The rational approximation of the $\mathcal{L}^1$ optimal controller given by the EAS method is the best one in the set Ω_τ in the sense that it leads to the smallest upper bound γ.

To prove Theorem 5, we need to prove first the following results.

Lemma 3: Consider the following discrete time systems

$$T_1^E = \left(\begin{array}{c|c} A_{1E} & B_{1E} \\ \hline C_{1E} & D_{1E} \end{array}\right) \triangleq \left(\begin{array}{c|c} I + \tau A_1 & \tau B_1 \\ \hline C_1 & D_1 \end{array}\right)$$

and

$$T_2^E = \left(\begin{array}{c|c} A_{2E} & B_{2E} \\ \hline C_{2E} & D_{2E} \end{array}\right) \triangleq \left(\begin{array}{c|c} I + \tau A_2 & \tau B_2 \\ \hline C_2 & D_2 \end{array}\right).$$

Define

$$L_E = B_{2E} D_{2E}^{-1}$$
$$\hat{A}_E = A_{2E} - L_E C_{2E}$$
$$M_E = L_E D_{1E} + \hat{A}_E^{-1} R_{1E} B_{1E}$$

where R_{1E} is the unique solution of the linear matrix equation

$$\hat{A}_E^{-1} R_{1E} A_{1E} - R_{1E} + L_E C_{1E} = 0.$$

Then we have

$$M_E = \tau M$$

where M is defined in Theorem 2.

Proof: First we show that $\hat{A}_E$ is always invertible. Note that

$$(T_2^E)^{-1} = \left(\begin{array}{c|c} \hat{A}_E & -L_E \\ \hline D_{2E}^{-1} C_{2E} & D_{2E}^{-1} \end{array}\right).$$

Since it is assumed that $T_2(s)$ has only unstable zeros, so does $T_2^E(z)$. This means all the zeros (poles) of $T_2^E(z)$ $((T_2^E)^{-1})$ have magnitudes larger than 1. Invertibility of $\hat{A}_E$ follows immediately. Recall now that R_1 is the unique solution of the following linear matrix equation

$$\hat{A} R_1 - R_1 A_1 - L C_1 = 0.$$

We can verify that $\hat{A}_E^{-1} R_{1E}$ also satisfies the above equation

$$\begin{aligned} &\hat{A}\hat{A}_E^{-1} R_{1E} - \hat{A}_B^{-1} R_{1E} A_1 - L C_1 \\ &= \tau^{-1}(R_{1E} - \hat{A}_E^{-1} R_{1E} A_{1E} - L_E C_{1E}) = 0. \end{aligned}$$

Hence $R_1 = \hat{A}_E^{-1} R_{1E}$ and $M_E = \tau M$.

Lemma 4: Consider the discrete time l_1 optimal control problem for the EAS system

$$\mu_E = \inf_{Q \text{ stable}} \|T_1^E + T_2^E Q\|_1.$$

A closed-loop system $\Phi^E(z) = \sum_{i=1}^q \phi_i z^{-N_i}$ is admissible, i.e., satisfies the interpolation conditions

$$\Phi^E(z_k^E) = T_1^E(z_k^E), \qquad k = 1, \cdots, n \tag{9}$$

where z_k^E are the zeros of T_2^E, if and only if

$$M_E = \sum_{i=1}^q \phi_i \hat{A}_E^{-N_i} L_E. \tag{10}$$

Proof: From the definition of EAS, (9) is equivalent to

$$\Phi(z_k) = \sum_{i=1}^q \phi_i (1 + \tau z_k)^{-N_i} = T_1(z_k), \qquad k = 1, \cdots, n.$$

From Lemma 3, (10) is equivalent to

$$\begin{aligned} M &= \sum_{i=1}^q \phi_i (I + \tau \hat{A})^{-N_i} L \Leftrightarrow \\ \tau M &= \sum_{i=1}^q \phi_i (I + \tau A_2 - \tau L C_2)^{-N_i} \tau \acute{L} \\ M_E &= \sum_{i=1}^q \phi_i \hat{A}_E^{-N_i} L_E. \end{aligned} \tag{11}$$

Equivalence of (9) and (10) follows now from Lemma 2.

Note that Lemma 4 is true for any discrete time systems since a discrete time systems can always be thought as an EAS of some continuous time systems.

Proof of Theorem 6: Consider a set of admissible closed-loop systems

$$\Omega_E = \left\{ \Phi^E(z) = \sum_{i=1}^q \phi_i z^{-N_i} : M_E = \sum_{i=1}^q \phi_i \hat{A}_E^{-N_i} L_E \right\}. \tag{12}$$

From Lemma 3 it follows that conditions (8) and (12) are identical. Therefore, for every $\Phi^E(z)$ in Ω_E there exists a corresponding $\Phi(s) = \Phi^E(1 + \tau s)$ in Ω_τ, and vice versa. Since

$$\gamma = \sum_{i=1}^q |\phi_i| = \|\Phi^E\|_1$$

it follows that the closed-loop system Φ^E obtained by solving the optimal l^1 control problem for the EAS yields the smallest γ among the elements of the set Ω_E. Hence the rational closed-loop system obtained by EAS methods also has the smallest upper bound γ among the set Ω_τ.

C. The EAS Method Revisited

Although the results of [6] and [1] show that the optimal $\mathcal{L}^1$ controllers can be approximated arbitrarily close with a rational controller, these results did not provide a way of obtaining an approximation with error smaller than a prespecified bound; rather, they required solving a sequence of problems and checking the approximation error until the desired precision was achieved. In this section we indicate how to select the parameter τ for the EAS method in such a way that the error of the resulting approximation is smaller than a prespecified bound. Moreover, we show that this approximation error converges to 0 as fast as τ.

Theorem 6: Given any $\epsilon > 0$, we can find a τ a priori for the EAS method such that

$$\mu_E \leq \mu_0(1 + \epsilon).$$

Moreover, the approximation error converges to zero as $O(\tau)$.

Proof: Consider the optimal l_1 control problem for the EAS

$$\mu_E = \inf_{Q \text{ stable}} \|T_1^E(z) + T_2^E(z) Q(z)\|_1 \tag{13}$$

and its dual

$$\mu_E = \max_{\alpha_j} \left[\sum_{i=1}^n \alpha_i \operatorname{Re}\{T_1^E(z_i^E)\} + \sum_{i=1}^n \alpha_{i+n} \operatorname{Im}\{T_1^E(z_i^E)\} \right] \tag{14}$$

subject to

$$\begin{aligned} |r^E(k, \alpha)| &\triangleq \left| \sum_{i=1}^n \alpha_i \operatorname{Re}\{(z_i^E)^{-k}\} + \sum_{i=1}^n \alpha_{i+n} \operatorname{Im}\{(z_i^E)^{-k}\} \right| \\ &\leq 1, \qquad 0, 1, 2, \cdots \end{aligned} \tag{15}$$

where T_1^E and T_2^E are the EAS of T_1 and T_2, respectively, and where z_k^E denotes the zeros of T_2^E. From the relationship between the EAS and its corresponding continuous system, the above dual problem is equivalent to

$$\mu_E = \max_{\alpha_j}\left[\sum_{i=1}^{n}\alpha_i \operatorname{Re}\{T_1(z_1)\} + \sum_{i=1}^{n}\alpha_{i+n}\operatorname{Im}\{T_1(z_i)\}\right] \quad (16)$$

subject to

$$\left|\sum_{i=1}^{n}\alpha_i \operatorname{Re}\{(1+\tau z_i)^{-k}\} + \sum_{i=1}^{n}\alpha_{i+n}\operatorname{Im}\{(1+\tau z_i)^{-k}\}\right| \le 1, \quad k = 0, 1, 2, \cdots \quad (17)$$

which can further be thought as an approximation of the dual problem of $\mathcal{L}^1$-optimal control problem

$$\mu_0 = \max_{\alpha_j}\left[\sum_{i=1}^{n}\alpha_i \operatorname{Re}\{T_1(z_i)\} + \sum_{i=1}^{n}\alpha_{i+n}\operatorname{Im}\{T_1(z_i)\}\right] \quad (18)$$

subject to

$$|r(t, \alpha)| \triangleq \left|\sum_{i=1}^{n}\alpha_i \operatorname{Re}\{e^{-z_i t}\} + \sum_{i=1}^{n}\alpha_{i+n}\operatorname{Im}\{e^{-z_i t}\}\right| \le 1, \quad \forall t \in R_+ \quad (19)$$

in the sense that the constraints (19) are firstly sampled at the time interval $t_k = k\tau$ and then the irrational terms $e^{-z_i t_k}$ are approximated by rational terms $(1+\tau z_i)^{-k}$.

For simplicity, in the sequel we will assume that all the zeros z_i are real as in [4], although the proofs can be easily extended to encompass complex zeros as well.

An upper bound on $\|\alpha\|_1$ for all α satisfying the constraints (15) can be derived as follows. Define the following sets

$$S_c = \left\{\alpha: \left|\sum_{i=1}^{n}\alpha_i e^{-z_i t}\right| \le 1\, \forall t \ge 0\right\}$$
$$S(\tau) = \left\{\alpha: \left|\sum_{i=1}^{n}\alpha_i (1+\tau z_i)^{-k}\right| \le 1\, \forall k \ge 0\right\}$$
$$R(\tau) = \left\{\alpha: \left|\sum_{i=1}^{n}\alpha_i (1+\tau z_i)^{-k}\right| \le 1\; k = 0, 1, \cdots, n-1\right\}. \quad (20)$$

From [1] it can be shown that, if $\tau \le \bar{\tau}$, then $S_c \subseteq S(\tau) \subseteq S(\bar{\tau})$ and $S(\tau) \subseteq R(\tau)$. Hence

$$\sup_{\alpha\in S_c}\|\alpha\|_1 \le \sup_{\alpha\in S(\tau)}\|\alpha\|_1 \le \sup_{\alpha\in S(\bar{\tau})}\|\alpha\|_1 \le \sup_{\alpha\in R(\bar{\tau})}\|\alpha\|_1 \le \|F^{-1}(\bar{\tau})\|_{\infty,1} \quad (21)$$

where

$$F(\tau) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ (1+\tau z_1)^{-1} & (1+\tau z_2)^{-1} & \cdots & (1+\tau z_n)^{-1} \\ \vdots & \vdots & \cdots & \vdots \\ (1+\tau z_1)^{-n+1} & (1+\tau z_2)^{-n+1} & \cdots & (1+\tau z_n)^{-n+1} \end{pmatrix}.$$

$\|F^{-1}\|_{\infty,1}$ denotes the induced norm of F^{-1} from l_n^∞ to l_n^1, and where $\bar{\tau}$ is fixed.

Note also that constraints (19) will be automatically satisfied for all $t > t_{\max}$ where $t_{\max}$ is finite and can be determined a priori [4]. Given any $t \le t_{\max}$, assume that the constraints (17) are satisfied and consider any $t \le t_{\max}$. Selecting k such that $t_k \le t < t_k + \tau$ we have that

$$\begin{aligned}
&\left|\sum_{i=1}^{n}\alpha_i (1+\tau z_i)^{-1} - \sum_{i=1}^{n}\alpha_i e^{-z_i t}\right| \\
&\le \|\alpha\|_1 \max_i\{|(1+\tau z_i)^{-k} - e^{-z_i t}|\} \\
&\le \|\alpha\|_1 \Big(\max_i\{|(1+\tau z_i)^{-k} - e^{-z_i t_k}|\} \\
&\quad + \max_i\{|e^{-z_i t_k} - e^{-z_i t}|\}\Big) \\
&\le \|\alpha\|_1 \Big(\max_i\{t_{\max}\tau^{-1}|(1+\tau z_i)^{-1} - e^{-z_i \tau}|\} \\
&\quad + \max_i\{1 - e^{-z_i\tau}\}\Big) \triangleq \epsilon(\tau). \qquad (22)
\end{aligned}$$

The first inequality is immediate. The second one follows from the triangle inequality. The last one can be proved as follows: If $|a| \le 1$ and $|b| \le 1$, then

$$|a^k - b^k| = |a-b|\,|a^{k-1} + a^{k-2}b + \cdots + b^{k-1}| \le k|a-b|$$

so we have

$$\begin{aligned}|(1+\tau z_i)^{-k} - e^{-z_i t_k}| &\le k|(1+\tau z_i)^{-1} - e^{-z_i\tau}| \\ &\le t_{\max}\tau^{-1}|(1+\tau z_i)^{-1} - e^{-z_i\tau}|.\end{aligned}$$

Note that both terms in the parenthesis can be made as small as one desires. So given any $\epsilon > 0$, we can choose a τ such that the right-hand side of the inequality is less than or equal to ϵ. For this value of τ we have

$$|r(t, \alpha)| = \left|\sum_{i=1}^{n}\alpha_i e^{-z_i t}\right| \le 1 + \epsilon, \qquad \forall t \in R_+.$$

In particular the above inequality holds for α^* which solves the dual problem of EAS. Since

$$\langle T_1, r(t, \alpha^*)\rangle = \sum_{i=1}^{n}\alpha_i^* T_1(z_i) = \mu_E$$

we have that

$$\mu_0 = \max_{r\ne 0}\frac{\langle T_1, r\rangle}{\|r\|_\infty} = \max_{\|r\|_\infty \le 1+\epsilon}\frac{\langle T_1, r\rangle}{1+\epsilon} \ge \frac{\mu_E}{1+\epsilon}.$$

Finally, the fact that $\epsilon(\tau) = O(\tau)$ follows from considering the Taylor expansion of equation (22).

IV. An Example

Consider the example introduced in [4] and further studied in [1] and [6]. The plant is

$$P(s) = \frac{s-1}{s-2}.$$

The control objective is to minimize $\|\Phi\|_1 = \|PC(1+PC)^{-1}\|_1$. The optimal closed-loop system is [4]

$$\Phi_{\rm OPT}(s) = 1.7071 - 4.1213e^{-0.8814s}$$

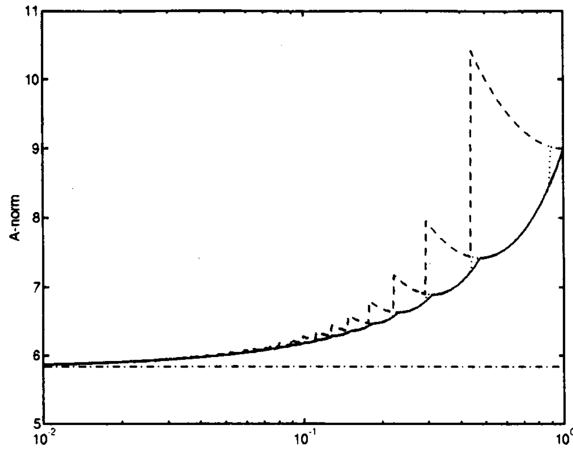

Fig. 2. Upper bound γ versus τ: EAS method—solid line; OMK method—dotted line, and OMK-like method—dashed line.

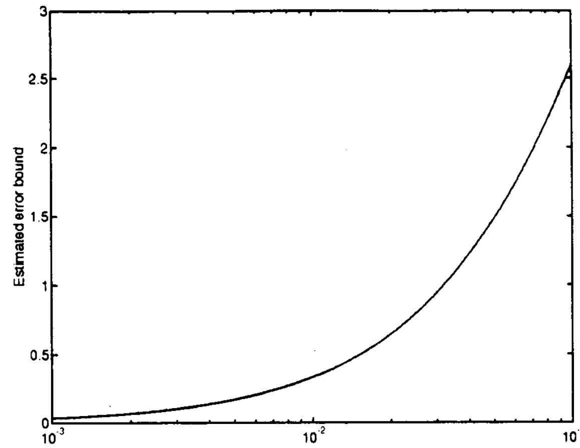

Fig. 3. An estimated error bound.

with an optimal cost $\mu_0 = 5.8284$. For $\tau = 0.45$, the rational closed-loop system obtained using the EAS method is

$$\Phi(s) = 1.8001 - 5.4878(1 + 0.45s)^{-3}$$

with $\gamma = \|\Phi(s)\|_A = 7.2879$. The OMK method yields

$$\Phi(s) = 2.3947 - 5.0348(1 + 0.45s)^{-2}$$

with $\gamma = \|\Phi(s)\|_A = 7.4295$. Finally, if we consider the OMK-like approximation obtained by selecting $N(t_i, \tau) \triangleq$ largest integer smaller than or equal to t_i/τ, $i = 1, \cdots, m$, we obtain

$$\Phi(s) = 4.222 - 6.122(1 + 0.45s)^{-1}$$

with $\gamma = \|\Phi(s)\|_A = 10.344$. Fig. 2 shows the upper bounds γ corresponding to different approximation methods versus τ. For this example the A norm of the closed-loop system coincides with its upper bound in all cases (since there are only 2 interpolation constraints). It is interesting to note that while the bound obtained using the EAS method decreases monotonously with τ (theoretically proved in Theorem 4), those corresponding to the OMK and OMK-like methods do not. An estimated error bound $\epsilon(\tau)$ curves for this example is shown in Fig. 3. It is very close to a straight line if a linear scale is used for the τ axis.

V. Conclusions

A recent research effort [3]–[5], [7], [8], has led to techniques for designing optimal compensators that minimize the worst case output amplitude with respect to all inputs of bounded amplitude. In the discrete-time SISO case, minimizing the l^1 norm of the closed-loop impulse response yields a rational compensator. Unfortunately, the solution to the continuous-time version of the problem is nonrational. Prompted by the difficulty of physically implementing a system with a nonrational transfer function, rational approximations were recently developed [6], [1].

In this paper we compare these approximations and we show that they are strongly connected. Indeed, both approximations can be considered as elements of the same subset Ω_τ of the set of rational approximations.

In Section III-B we show that the EAS method proposed in [1] yields the best approximation (in the sense of providing the tightest upper bound of the error) among the elements of this set.

Finally, in Section III-C, we exploit the structure of the dual problem to provide a procedure that allows for selecting the parameter τ for the EAS method to guarantee that the approximation error is smaller than a prespecified bound ϵ. Moreover, we also show that this approximation error→0 as $O(\tau)$.

We believe that these results, combined with the features of the EAS method mentioned in [1], namely, the facts that i) it removes the ill-posedness due to the presence of zeros on the imaginary axis; ii) it leads to computationally simple problems; iii) it furnishes a monotonically nonincreasing bound; and iv) it is easily extendable to the MIMO case, make this method an attractive tool for the design of controllers for continuous time systems.

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