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Further Results on Rational Approximations of $\mathcal{L}^1$-Optimal Controllers

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1. Introduction

When disturbances and outputs are persistent bounded signals, the worst case disturbance rejection problem leads to the $\mathcal{L}^1$ optimal control theory, which was formulated by Vidyasagar [6] and solved by Dahleh and Pearson both in the discrete [2][4] and continuous time [3] cases. In contrast with the discrete time l^1 theory, the solution to the continuous–time $\mathcal{L}^1$ optimal control problem leads to irrational compensators. As noted in [3], the difficulty of physically implementing these controllers suggests that the most significant application of the continuous time $\mathcal{L}^1$ theory is to provide performance bounds for the plant. Recently, two rational approximations to the optimal $\mathcal{L}^1$ controller were developed independently [5] [1]. Although these approximations are based upon different techniques ([5] follows an algebraic approach while [1] exploits the properties of the Euler Approximating System), they seem to be strongly connected [1].

In this paper we explore the connection between these approaches. The main results of the paper show that both belong to the same subset Ω_τ of the set of admissible rational approximations, and that the method proposed in [1] gives the best approximation among the elements of this set. Additionally by exploiting the structure of the dual to the $\mathcal{L}^1$ optimal control problem we furnish a procedure to compute rational approximations with error smaller than a prespecified bound ϵ, and we show that the approximation error $\rightarrow 0$ as $\mathcal{O}(\tau)$.

2. Preliminaries

2.1. Notation and Definitions

R_+ denotes the set of nonnegative real numbers. $\mathcal{L}^\infty(R_+)$ denotes the space of measurable functions $f(t)$ equipped with the norm: $\|f\|_\infty = ess.\sup_{R_+} |f(t)|$. $\mathcal{L}^1(R_+)$ denotes the space of Lebesgue integrable functions on R_+ equipped with the norm $\|f\|_1 \triangleq \int_0^\infty |f(t)|dt < \infty$. Similarly, l_1 denotes the space of absolutely summable sequences $h = \{h_i\}$ equipped with the norm $\|h\|_1 \triangleq \sum_{k=0}^\infty |h_i| < \infty$. $\mathcal{RL}^1$ denotes the subspace of $\mathcal{L}^1$ formed by matrices with real rational Laplace transform. A denotes the space whose elements have the form:

$$h = h^L(t) + \sum_{k=0}^\infty h_i^l \delta(t - t_i)$$

where $h^L(t) \in L_1(R_+)$, $\{h_i^l\} \in l_1$ and $t_i \geq 0$, equipped with the norm $\|h\|_A \triangleq \|h^L\|_{L_1} + \|h^l\|_{l_1}$. Given a function $f(t) \in \mathcal{L}^1$ we will denote its Laplace transform by $F(s) \in \mathcal{L}_\infty$; similarly, given $h \in A$, we will denote its Laplace transform by $H(s)$. By a slight abuse of notation, we will denote as $\|F(s)\|_1 \triangleq \|f(t)\|_1$ and $\|H(s)\|_A = \|h\|_A$. Throughout the paper we will use packed notation to represent state–space realizations, i.e.

$$G(s) = C(sI - A)^{-1}B + D \triangleq \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

Definition 1 *Consider the continuous time system $G(s)$. Its Euler Approximating System (EAS) is defined as the following discrete time system:*

$$G^E(z,\tau) = \left(\begin{array}{c|c} I + \tau A & \tau B \\ \hline C & D \end{array}\right) \qquad (1)$$

From this definition it is easily seen we can obtain the EAS of $G(s)$ by the simple variable transformation $s = \frac{z-1}{\tau}$, i.e.

$$G^E(z,\tau) = G(\frac{z-1}{\tau})$$

On the other hand, for any given τ we can relate a discrete time system to a continuous system by the inverse transformation $z = 1 + \tau s$. It is obvious that the discrete time system is, in fact, the EAS of the continuous time system obtained in this form.

Definition 2 *Consider a system of the form:*

$$\Phi(s) = T_1(s) + T_2(s)Q(s)$$

where $T_2(s)$ has all its zeros $\{z_1, z_2, \ldots, z_n\}$ in the open right–half plane. $\Phi(s)$ is said to be admissible if it is stable and satisfies the interpolation conditions

$$\Phi(z_k) = T_1(z_k),\ k = 1, \ldots, n$$

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2.2. The $\mathcal{L}^1$ Optimal Control Problem

Consider the system represented by the block diagram in Figure 1, where S represents the system to be controlled; the scalar signals $w \in \mathcal{L}^\infty$ and u represent an exogenous disturbance and the control action respectively; and where ζ and y represent the output subject to performance constraints and the measurements available to the controller respectively. As usual we will assume, without loss of generality, that any weights have been absorbed in the plant S. Then, the $\mathcal{L}^1$ optimal control problem can be stated as: Given the system (S) find an internally stabilizing controller $u(s) = K(s)y(s)$ such that the worst case (over the set of all $w(t) \in \mathcal{L}^\infty$, $\|w\|_\infty \leq 1$) maximum amplitude of the performance output $\zeta(t)$ is minimized.

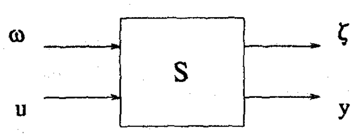

Figure 1: The Generalized Plant

By using the YJBK parametrization of all stabilizing controllers [3], the problem can be cast into the following model matching form:

$$\mu_0 = \inf_{K \text{ stabilizing}} \|\Phi(s)\|_A = \inf_{Q \in A} \|T_1(s) - T_2(s)Q(s)\|_A$$

where T_1, T_2 are rational stable transfer functions, and where, without loss of generality, we can assume that T_2 has all its zeros in the open–right half plane.

Next, we recall the main result of [3], showing that a solution to the $\mathcal{L}^1$–optimal control problem can be found by solving a semi–infinite linear programming problem.

Theorem 1 *(Dahleh and Pearson, [3]) Let $T_2(s)$ have n zeros z_i in the open right-half plane and no zeros on the jw–axis. Then:*

$$\begin{aligned}\mu_0 &= \inf_{Q \in A} \|T_1(s) + T_2(s)Q(s)\|_A \\ &= \max_{\alpha_j} \left[\sum_{i=1}^{n} \alpha_i Re\{T_1(z_i)\} + \sum_{i=1}^{n} \alpha_{i+n} Im\{T_1(z_i)\} \right] \quad (2)\end{aligned}$$

subject to:

$$\|r(t)\| = \left| \sum_{i=1}^{n} \alpha_i Re\{e^{-z_i t}\} + \sum_{i=1}^{n} \alpha_{i+n} Im\{e^{-z_i t}\} \right| \leq 1, \ \forall t \in R_+ \quad (3)$$

Furthermore, the following facts hold: i) the extremal functional $r^*(t)$ equals 1 at only finite points: $t_1, \dots, t_m$; ii) an optimal solution $\Phi(s) = T_1(s) - T_2(s)Q(s)$ to the leftside problem always exists; and iii) the optimal ϕ has the following form:

$$\phi = \sum_{i=1}^{m} \phi_i \delta(t - t_i), \ t_i \in \mathcal{R}_+, \ \text{m finite} \quad (4)$$

and satisfies the following conditions:

a). $\phi_i r^*(t_i) \geq 0$;

b). $\sum_{i=1}^{m} |\phi_i| = \mu_0$;

c). $\sum_{i=1}^{m} \phi_i e^{-z_k t_i} = T_1(z_k), \ k = 1, \dots, n.$

Remark 1 *It was shown in [3] that we need to satisfy constraints (3) only for all $t \leq t_{max}$ where t_{max} is finite and can be determined* a priori. *Even so, there are still infinite constraints, and therefore the dual problem is a semi-infinite linear programming problem.*

2.3. Rational Approximations to the Optimal $\mathcal{L}^1$ Controller

From equation (4) it follows that, unlike in the discrete–time case, the $\mathcal{L}^1$-optimal controller is irrational even if the plant is rational. Prompted by the difficulty in physically implementing a controller with an irrational transfer function, two rational approximation methods have been recently developed independently by Ohta *et. al.* [5] and by Blanchini and Sznaier [1]. For brevity, we will refer to the former as the OMK method and to the latter as the EAS method.

Theorem 2 *(Ohta et al., [5]) Let $T_1(s) = \left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right)$ and $T_2(s) = \left(\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right)$ be minimal realizations. Define:*

$$L = B_2 D_2^{-1}$$
$$\hat{A} = A_2 - LC_2$$
$$M = LD_1 + R_1 B_1$$

where R_1 is the unique solution of the matrix linear equation

$$\hat{A}R_1 - R_1 A_1 = LC_1$$

Then, there exist finite sets $\{t_1 \ t_2 \ \dots \ t_m\}$ and $\{\phi_1 \ \phi_2 \ \dots \ \phi_m\}$ such that:

$$M = \sum_{i=1}^{m} \phi_i exp(-\hat{A}t_i)L \quad (5)$$

and

$$\mu_0 = \sum_{i=1}^{m} |\phi_i|$$

For $\tau > 0$, define $N(t_i, \tau) \triangleq$ the smallest integer larger than or equal to t_i/τ, $i = 1, 2, \dots, m$, and $N(\tau) \triangleq N(t_m, \tau)$. Finally, denote by $\phi(\tau)$ the minimizer of $\|\phi(\tau) - \phi\|_2$ subject to

$$M = \sum_{i=1}^{m} \phi_i(\tau)(I + \tau\hat{A})^{-N(t_i,\tau)} L \quad (6)$$

where $\phi(\tau) = [\phi_1(\tau), \dots, \phi_m(\tau)]$, and $\phi = [\phi_1, \dots, \phi_m]$. Consider the rational system $\Phi(s, \tau)$ with the following state-space realization:

$$\Phi(s, \tau) = \left(\begin{array}{c|c} A(\tau) & B_4(\tau) \\ \hline C_3(\tau) & D_{34}(\tau) \end{array}\right)$$

Where A, B_4, C_3, D_{34} are $N(\tau) \times N(\tau)$, $N(\tau) \times 1$, $1 \times N(\tau)$ and 1×1 matrices, respectively; and are defined as follows:

$$A(\tau) = \tau^{-1} \begin{pmatrix} -1 & 1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ 0 & 0 & -1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & -1 & 1 \\ 0 & \dots & \dots & 0 & 0 & -1 \end{pmatrix},$$

$$B_4(\tau)[k] = \begin{cases} 0, & \text{if } k \notin \{N(t_i,\tau), i = 1,2,...,m\} \\ \phi_i(\tau), & \text{if } k = N(t_i,\tau) \end{cases}$$

$$C_3(\tau) = (\tau^{-1} \quad 0 \quad \dots \quad 0),$$

$$D_{34}(\tau) = \begin{cases} 0, & \text{if } N(t_1,\tau) \neq 0 \\ \phi_1(\tau), & \text{if } N(t_1,\tau) = 0. \end{cases}$$

Then, as $\tau \to 0$, we have that $\Phi(s,\tau) \to \Phi_{OPT}(s)$ uniformly in the wide sense in the open half plane $Re(s) > -\sigma$ for some $\sigma > 0$; and $\|\Phi\|_A$, as well as its upper bound $\gamma = \sum_{i=1}^{q} |\phi_i(\tau)|$, converge to μ_0, the optimal cost.

Next, we recall the main result of [1] showing that the $\mathcal{L}^1$ norm of a stable transfer function is bounded above by the l^1 norm of its Euler Approximating System (EAS). Moreover, this bound can be made arbitrarily tight by taking the parameter τ in (1) small enough. This result is the basis for the approximation procedure proposed in [1].

Theorem 3 *(Blanchini and Sznaier, [1]) Consider a continuous time system with rational Laplace transform $\Phi(s)$ and its EAS, $\Phi^E(z,\tau)$. If $\Phi^E(z,\tau)$ is asymptotically stable, then $\Phi(s)$ is also asymptotically stable and such that:*

$$\|\Phi(s)\|_1 \le \|\Phi^E(z,\tau)\|_1$$

Conversely, if $\Phi(s)$ is asymptotically stable and such that $\|\Phi(s)\|_1 \stackrel{\Delta}{=} \mu_c$, then for all $\mu > \mu_c$ there exists $\tau^* > 0$ such that for all $0 < \tau \le \tau^*$, $\Phi^E(z,\tau)$ is asymptotically stable and such that $\|\Phi^E(z,\tau)\|_1 \le \mu$.

Theorem 4 *[1] Consider a strictly decreasing sequence $\tau_i \to 0$, and define:*

$$\mu_i \stackrel{\Delta}{=} \inf_{stabilizing K} \|\Phi^E_{cl}(z,\tau_i)\|_1$$

where $\Phi^E_{cl}(z,\tau_i)$ denotes the closed-loop transfer function. Then the sequence μ_i is non-increasing and such that $\mu_i \to \mu_0$, the optimal $\mathcal{L}^1$ cost.

Corollary: A suboptimal rational solution to the $\mathcal{L}^1$ Optimal Control Problem for continuous time–systems, with cost arbitrarily close to the optimal cost, can be obtained by solving a discrete–time l_1 optimal control problem for the corresponding EAS. Moreover, if $K(z)$ denotes the optimal l_1 compensator for the EAS, the suboptimal $\mathcal{L}^1$ compensator is given by $K(\tau s + 1)$.

3. Analysis of the Different Rational Approximations

In this section we analyze the rational approximations generated by the OMK and EAS methods. The main result shows that both approximations belong to a certain subset Ω_τ of the set of rational approximations, and that the EAS method generates the best approximation among the elements of this subset. We begin by showing that the two expressions (5) and (6) for matrix M in Theorem 2 are just another version of the interpolation conditions.

3.1. Characterization of all Rational Approximations

Lemma 1 *For a closed loop system of the form*

$$\Phi(s) = \sum_{i=1}^{q} \phi_i e^{-t_i s}, \; t_i \in \mathcal{R}_+,$$

the following two conditions are equivalent

a) $\Phi(z_k) = \sum_{i=1}^{q} \phi_i e^{-z_k t_i} = T_1(z_k),\ k = 1,\dots,n.$

b) $M = \sum_{i=1}^{q} \phi_i exp(-\hat{A} t_i) L.$

Moreover, we have

$$\|\Phi(s)\|_A = \gamma \stackrel{\Delta}{=} \sum_{i=1}^{q} |\phi_i|.$$

Proof: b) $\Rightarrow$ a) can be proved following the proof procedure of Lemma 2 in [5] by simply replacing Φ_{OPT} with Φ. Similarly, the fact that b) is necessary for a) to hold can also be concluded from the proof. The expression for $\|\Phi(s)\|_A$ follows from direct calculations.

□

In the next lemma we give a complete characterization of all rational approximations.

Lemma 2 *For any rational closed loop system*

$$\Phi(s) = \sum_{j=1}^{q_j} \sum_{i=1}^{q_i} \phi_{ij} (1 + \lambda_j s)^{-N_{ij}}$$

where $Re(\lambda) > 0$ and N_{ij} integers, the following two conditions are equivalent

a) $\Phi(z_k) = \sum_{j=1}^{q_j} \sum_{i=1}^{q_i} \phi_{ij} (1 + \lambda_j z_k)^{-N_{ij}} = T_1(z_k),\ k = 1,\dots,n.$

b) $M = \sum_{j=1}^{q_j} \sum_{i=1}^{q_i} \phi_{ij} (I + \lambda_j \hat{A})^{-N_{ij}} L.$

Moreover, we have

$$\|\Phi(s)\|_A \le \gamma \stackrel{\Delta}{=} \sum_{j=1}^{q_j} \sum_{i=1}^{q_i} |\phi_{ij}|$$

Proof: a) $\Leftrightarrow$ b) can be proved using the same idea. The calculations, though straightforward, are tedious, and are omitted here for space reason. By direct calculation we have

$$\|\Phi(s)\|_A = \int_0^\infty \left| \sum_{j=1}^{q_j} \sum_{i=1}^{q_i} \phi_{ij} \frac{1}{(N_{ij}-1)!\lambda_j^{N_{ij}}} t^{N_{ij}-1} e^{-t/\lambda_j} \right| dt$$
$$\le \sum_{j=1}^{q_j} \sum_{i=1}^{q_i} |\phi_{ij}| \int_0^\infty \frac{1}{(N_{ij}-1)!\lambda_j^{N_{ij}}} t^{N_{ij}-1} e^{-t/\lambda_j} dt$$
$$= \gamma \stackrel{\Delta}{=} \sum_{j=1}^{q_j} \sum_{i=1}^{q_i} |\phi_{ij}|$$

□

3.2. Comparison of the OMK and the EAS Rational Approximations

Lemma 2 gives a characterization of all rational admissible closed-loop systems. All these closed loop systems can be thought as candidate rational approximations of the $\mathcal{L}^1$-optimal closed loop system. In the sequel we will concentrate on a specific subset Ω_τ and we will show that both the OMK and the EAS methods generate approximations that belong to this subset. For $\tau > 0$, define:

$$\Omega_\tau = \Big\{ \Phi(s) = \sum_{i=1}^{q} \phi_i(\tau)(1 + \tau s)^{-N_i} : \quad M = \sum_{i=1}^{q} \phi_i(\tau)(I + \tau \hat{A})^{-N_i} L \Big\} \qquad (7)$$

By direct calculation, the closed loop system obtained by OMK method is:

$$\Phi(s) = \sum_{i=1}^{m} \phi_i(\tau)(1+\tau s)^{-N(t_i,\tau)}$$

Suppose that the l_1-optimal closed loop system for the EAS is given by:

$$\Phi^E(z) = \sum_{i=1}^{q} \phi_i^E(\tau) z^{-N_i^E}$$

then the closed loop system obtained using the EAS method is:

$$\Phi(s) = \sum_{i=1}^{q} \phi_i^E(\tau)(1+\tau s)^{-N_i^E}$$

It follows that the approximations generated by both methods belong to the set Ω_τ, with a specific $\{N_i\}$ determined by each method.

Remark 2 *In the OMK method, $\{N(t_i,\tau)\}$ depend directly on $\{t_i\}$, and hence on the $\mathcal{L}^1$-optimal closed loop system. Hence, obtaining a rational approximation requires solving the $\mathcal{L}^1$-optimal control problem first. However as pointed out in Remark 1, solving exactly this problem entails solving a semi-infinite linear programming problem. The EAS method requires only solving a discrete l_1 optimal control problem, which is considerable easier, since only finite–dimensional linear programming is involved.*

Remark 3 *Note that additional OMK–like rational approximations can be chosen among the elements of Ω_τ by simply modifying the rule for selecting $N(t_i,\tau)$. For instance, N could be selected as the largest integer smaller than or equal to t_i/τ or $t_i/\tau + 0.5$. Clearly, the convergence property also holds for these approximations.*

As it will be shown in Theorem 6, the EAS method can be interpreted as approximating the original optimization problem, as opposed to directly approximating its irrational solution. This makes it quite unique. Besides the computational advantages, we will show in the sequel that the EAS method has two other important merits.

Theorem 5 *The rational approximation of the $\mathcal{L}^1$-optimal controller given by the EAS Method is the best one in the set Ω_τ in the sense that it leads to the smallest upper bound γ.*

In order to prove Theorem 5, we need to prove first the following results:

Lemma 3 *Consider the following discrete time systems:*

$$T_1^E = \left(\begin{array}{c|c} A_{1E} & B_{1E} \\ \hline C_{1E} & D_{1E}\end{array}\right) \triangleq \left(\begin{array}{c|c} I+\tau A_1 & \tau B_1 \\ \hline C_1 & D_1\end{array}\right)$$

and

$$T_2^E = \left(\begin{array}{c|c} A_{2E} & B_{2E} \\ \hline C_{2E} & D_{2E}\end{array}\right) \triangleq \left(\begin{array}{c|c} I+\tau A_2 & \tau B_2 \\ \hline C_2 & D_2\end{array}\right)$$

Define:

$$L_E = B_{2E} D_{2E}^{-1}$$
$$\hat{A}_E = A_{2E} - L_E C_{2E}$$
$$M_E = L_E D_{1E} + \hat{A}_E^{-1} R_{1E} B_{1E}$$

where R_{1E} is the unique solution of the matrix linear equation

$$\hat{A}_E^{-1} R_{1E} A_{1E} - R_{1E} + L_E C_{1E} = 0$$

Then we have

$$M_E = \tau M.$$

Proof: Firstly we show that $\hat{A}_E$ is always invertible. Note that

$$(T_2^E)^{-1} = \left(\begin{array}{c|c} \hat{A}_E & -L_E \\ \hline D_{2E}^{-1} C_{2E} & D_{2E}^{-1}\end{array}\right).$$

Since it is assumed that $T_2(s)$ has only unstable zeros, so does $T_2^E(z)$. This means all the zeros (poles) of $T_2^E(z)$ ($(T_2^E)^{-1}$) have magnitudes larger than 1. Invertibility of $\hat{A}_E$ follows immediately. Recall now that R_1 is the unique solution of the following matrix linear equation

$$\hat{A} R_1 - R_1 A_1 - L C_1 = 0$$

We can verify that $\hat{A}_E^{-1} R_{1E}$ also satisfies the above equation

$$\begin{aligned} & \hat{A}\hat{A}_E^{-1} R_{1E} - \hat{A}_E^{-1} R_{1E} A_1 - L C_1 \\ = \; & \tau^{-1}(R_{1E} - \hat{A}_E^{-1} R_{1E} A_{1E} - L_E C_{1E}) \\ = \; & 0 \end{aligned}$$

So $R_1 = \hat{A}_E^{-1} R_{1E}$ and $M_E = \tau M$ follows.

□

Lemma 4 *Consider the discrete time l_1 optimal control problem for the EAS system:*

$$\mu_E = \inf_{Q stable} \|T_1^E + T_2^E Q\|_1$$

A closed loop system $\Phi^E(z) = \sum_{i=1}^{q} \phi_i z^{-N_i}$ is admissible, i.e. satisfies the interpolation conditions

$$\Phi^E(z_k^E) = T_1^E(z_k^E), \; k = 1, \ldots, n \quad (8)$$

where z_k^E are the zeros of T_2^E, if and only if

$$M_E = \sum_{i=1}^{q} \phi_i \hat{A}_E^{-N_i} L_E. \quad (9)$$

Proof: From the definition of EAS, (8) is equivalent to

$$\Phi(z_k) = \sum_{i=1}^{q} \phi_i (1+\tau z_k)^{-N_i} = T_1(z_k), \; k = 1, \ldots, n.$$

From Lemma 3, (9) is equivalent to

$$\begin{aligned} M &= \sum_{i=1}^{q} \phi_i (I+\tau \hat{A})^{-N_i} L \iff \\ \tau M &= \sum_{i=1}^{q} \phi_i (I+\tau A_2 - \tau L C_2)^{-N_i} \tau L \iff \quad (10) \\ M_E &= \sum_{i=1}^{q} \phi_i \hat{A}_E^{-N_i} L_E \end{aligned}$$

Equivalence of (8) and (9) follows now from Lemma 2.

□

Note that Lemma 4 is true for any discrete time system since a discrete time system can always be thought as an EAS of some continuous time system.

Proof of Theorem 5: Consider a set of admissible closed loop systems

$$\Omega_E = \left\{ \Phi^E(z) = \sum_{i=1}^{q} \phi_i z^{-N_i} \quad : M_E = \sum_{i=1}^{q} \phi_i \hat{A}_E^{-N_i} L_E \right\} \quad (11)$$

From Lemma 3 it follows that conditions (7) and (11) are identical. Therefore, for every $\Phi^E(z)$ in Ω_E there exists a corresponding $\Phi(s) = \Phi^E(1+\tau s)$ in Ω_τ, and *vice versa.* Since

$$\gamma = \sum_{i=1}^{q} |\phi_i| = \|\Phi^E\|_1$$

it follows that the closed loop system Φ^E obtained by solving the optimal l^1 control problem for the EAS yields the smallest γ among the elements of the set Ω_E. Hence the rational closed loop system obtained by EAS methods also has the smallest upper bound γ among the set Ω_τ.

□

3.3. The EAS Method Revisited

Although the results of [5] and [1] show that the optimal $\mathcal{L}^1$ controllers can be approximated arbitrarily close with a rational controller, these results did not provide a way of obtaining an approximation with error smaller than a prespecified bound; rather, they required solving a sequence of problems and checking the approximation error until the desired precision was achieved. In this section we indicate how to select the parameter τ for the EAS method in such a way that the error of the resulting approximation is smaller than a prespecified bound. Moreover, we show that this approximation error converges to 0 as fast as τ. Finally, we provide an alternative, simpler proof to EAS method that does not use the properties of Positively Invariant Sets.

Theorem 6 *Given any $\epsilon > 0$, we can find a τ* a priori *for the EAS method such that*

$$\mu_E \le \mu_0(1+\epsilon)$$

Moreover, the approximation error converges to zero as $\mathcal{O}(\tau)$.

Proof: Consider the optimal l_1 control problem for the EAS:

$$\mu_E = \inf_{Q\ stable} \|T_1^E(z) + T_2^E(z)Q(z)\|_1 \quad (12)$$

and its dual:

$$\mu_E = \max_{\alpha_j} \left[\sum_{i=1}^{n} \alpha_i Re\{T_1^E(z_i^E)\} + \sum_{i=1}^{n} \alpha_{i+n} Im\{T_1^E(z_i^E)\} \right] \quad (13)$$

subject to:

$$\left| \sum_{i=1}^{n} \alpha_i Re\{(z_i^E)^{-k}\} + \sum_{i=1}^{n} \alpha_{i+n} Im\{(z_i^E)^{-k}\} \right| \le 1,\ k = 0,1,2,\dots \quad (14)$$

where T_1^E and T_2^E are the EAS of T_1 and T_2, respectively, and where z_k^E denotes the zeros of T_2^E. From the relationship between the EAS and its corresponding continuous system, the above dual problem is equivalent to

$$\mu_E = \max_{\alpha_j} \left[\sum_{i=1}^{n} \alpha_i Re\{T_1(z_i)\} + \sum_{i=1}^{n} \alpha_{i+n} Im\{T_1(z_i)\} \right] \quad (15)$$

subject to:

$$\left| \sum_{i=1}^{n} \alpha_i Re\{(1+\tau z_i)^{-k}\} + \sum_{i=1}^{n} \alpha_{i+n} Im\{(1+\tau z_i)^{-k}\} \right| \le 1,$$
$$k = 0,1,2,\dots \quad (16)$$

which can further be thought as an approximation of the dual problem of $\mathcal{L}^1$-optimal control problem

$$\mu_0 = \max_{\alpha_j} \left[\sum_{i=1}^{n} \alpha_i Re\{T_1(z_i)\} + \sum_{i=1}^{n} \alpha_{i+n} Im\{T_1(z_i)\} \right] \quad (17)$$

subject to:

$$\left| \sum_{i=1}^{n} \alpha_i Re\{e^{-z_i t}\} + \sum_{i=1}^{n} \alpha_{i+n} Im\{e^{-z_i t}\} \right| \le 1 \quad \forall t \in R_+ \quad (18)$$

in the sense that the constraints (18) are firstly sampled at the time interval $t_k = k\tau$ and then the irrational terms $e^{-z_i t_k}$ are approximated by rational terms $(1+\tau z_i)^{-k}$.

For simplicity, in the sequel we will assume that all the zeros z_i are real as in [3], although the proofs can be easily extended to encompass complex zeros as well.

An upper bound on $\|\alpha\|_1$ for all α satisfying the constraints (13) can be derived by using a procedure similar to the proof of Theorem 5 in [2]. Define a $p \times n$ matrix

$$F = \begin{pmatrix} 1 & 1 & \dots & 1 \\ (z_1^E)^{-1} & (z_2^E)^{-1} & \dots & (z_n^E)^{-1} \\ \vdots & \vdots & \dots & \vdots \\ (z_1^E)^{-p+1} & (z_2^E)^{-p+1} & \dots & (z_n^E)^{-p+1} \end{pmatrix}$$

where p is any integer not smaller than n. It can be easily shown that F has full column rank and satisfies:

$$F_i \alpha = r^E(i-1, \alpha)$$

where F_i denotes the i^{th} row of F. Hence:

$$\|\alpha\|_1 = \|F^{-1}F\alpha\|_1 \le \|F^{-1}\|_{\infty,1} \|F\alpha\|_\infty \le \|F^{-1}\|_{\infty,1}$$

where F^{-1} denotes the left inverse of F and where $\|F^{-1}\|_{\infty,1}$ denotes the induced norm of F^{-1} from l_n^∞ to l_p^1.

Note also that constraints (18) will be automatically satisfied for all $t > t_{max}$ where t_{max} is finite and can be determined *a priori* [3]. Given any $t \le t_{max}$, assume that the constraints (16) are satisfied and consider any $t \le t_{max}$. Selecting k such that $t_k \le t < t_k + \tau$ we have that:

$$\begin{aligned} & \left| \sum_{i=1}^{n} \alpha_i (1+\tau z_i)^{-k} - \sum_{i=1}^{n} \alpha_i e^{-z_i t} \right| \\ \le\ & \|\alpha\|_1 \max_i \{ |(1+\tau z_i)^{-k} - e^{-z_i t}| \} \\ \le\ & \|\alpha\|_1 (\max_i \{ |(1+\tau z_i)^{-k} - e^{-z_i t_k}| \} \\ & + \max_i \{ |e^{-z_i t_k} - e^{-z_i t}| \}) \\ \le\ & \|\alpha\|_1 (\max_i \{ t_{max}\tau^{-1} |(1+\tau z_i)^{-1} - e^{-z_i \tau}| \} \\ & + \max_i \{ 1 - e^{-z_i \tau} \}) \\ \triangleq\ & \epsilon(\tau) \end{aligned} \quad (19)$$

The first inequality is immediate. The second one follows from the triangle inequality. The last one can be proved as follows: If $|a| \leq 1$ and $|b| \leq 1$ then

$$|a^k - b^k| = |a-b||a^{k-1} + a^{k-2}b + \ldots + b^{k-1}| \leq k|a-b|$$

so we have

$$\begin{aligned} & |(1+\tau z_i)^{-k} - e^{-z_i t_k}| \\ \leq\ & k|(1+\tau z_i)^{-1} - e^{-z_i \tau}| \\ \leq\ & t_{max}\tau^{-1}|(1+\tau z_i)^{-1} - e^{-z_i\tau}| \end{aligned} \qquad (20)$$

Note that both terms in the parenthesis can be made as small as one desires. So given any $\epsilon > 0$, we can choose a τ such that the right-hand side of the inequality is less than or equal to ϵ. For this value of τ we have:

$$|r(t,\alpha)| = \left|\sum_{i=1}^{n} \alpha_i e^{-z_i t}\right| \leq 1+\epsilon \quad \forall t \in R_+$$

In particular the above inequality holds for α^* which solves the dual problem of EAS. Since

$$\langle T_1, r(t,\alpha^*)\rangle = \sum_{i=1}^{n} \alpha_i^* T_1(z_i) = \mu_E$$

we have that

$$\mu_0 = \max_{r \neq 0} \frac{\langle T_1, r\rangle}{\|r\|_\infty} = \max_{\|r\|_\infty \leq 1+\epsilon} \frac{\langle T_1, r\rangle}{1+\epsilon} \geq \frac{\mu_E}{1+\epsilon}$$

Finally, the fact that $\epsilon(\tau) = \mathcal{O}(\tau)$ follows from considering the Taylor expansion of equation (19).

□

Remark 4 *By using the results of Lemma 2 and Theorems 2 and 5, the main result of [1] can be proved without using the concept of positively invariant sets. The fact that the A-norm of the resulting closed-loop system is less than $\mu^E(\tau)$ follows from Lemma 2. The convergence property is a direct result of Theorem 2 and Theorem 5.*

4. An Example

Consider the example introduced in [3] and further studied in [1] and [5]. The plant is

$$P(s) = \frac{s-1}{s-2}$$

The control objective is to minimize $\|\Phi\|_1 = \|PC(1+PC)^{-1}\|_1$. The optimal closed loop system is [3]:

$$\Phi_{OPT}(s) = 1.7071 - 4.1213e^{-0.8814s}$$

with an optimal cost $\mu_0 = 5.8284$. For $\tau = 0.45$, the rational closed loop system obtained using the EAS method is:

$$\Phi(s) = 1.8001 - 5.4878(1+0.45s)^{-3}$$

with $\gamma = \|\Phi(s)\|_A = 7.2879$. The OMK method yields:

$$\Phi(s) = 2.3947 - 5.0348(1+0.45s)^{-2}$$

with $\gamma = \|\Phi(s)\|_A = 7.4295$. Finally, if we consider the OMK-like approximation obtained by selecting $N(t_i,\tau) \triangleq$ *largest integer smaller than or equal to* t_i/τ, $i = 1,\ldots,m$, we obtain:

$$\Phi(s) = 4.222 - 6.122(1+0.45s)^{-1}$$

with $\gamma = \|\Phi(s)\|_A = 10.344$.

5. Conclusions

A recent research effort [2],[3],[4],[6],[7], has lead to techniques for designing optimal compensators that minimize the worst case output amplitude with respect to all inputs of bounded amplitude. In the discrete-time SISO case, minimizing the l^1 norm of the closed-loop impulse response yields a rational compensator. Unfortunately, the solution to the continuous-time version of the problem is irrational. Prompted by the difficulty of physically implementing a system with an irrational transfer function, rational approximations were recently developed [5],[1].

In this paper we compare these approximations and we show that they are strongly connected. Indeed, both approximations can be considered as elements of the same subset Ω_τ of the set of rational approximations.

In section 3.2 we show that the EAS method proposed in [1] yields the best approximation (in the sense of providing the tightest upper bound of the error) among the elements of this set.

Finally, in section 3.3, we exploit the structure of the dual problem to provide a procedure that allows for selecting the parameter τ for the EAS method to guarantee that the approximation error is smaller than a prespecified bound ϵ. Moreover, we also show that this approximation error $\rightarrow 0$ as $\mathcal{O}(\tau)$.

We believe that these results, combined with the features of the EAS method mentioned in [1], namely the facts that i) it removes the ill-posedness due to the presence of zeros on the imaginary axis; ii) it leads to computationally simple problems; and iii) it is easily extensible to the MIMO case, make this method an attractive tool for the design of controllers for continuous time systems.

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