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A Convex Optimization Approach to Design of Information Structured Linear Constrained Controllers

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***Abstract*— In this paper we consider the synthesis of sparse stabilizing static state feedback controllers subject to state and control constraints. The presence of information constraints renders this problem generically NP-hard. However, as we show in the paper, a convergent sequence of tractable convex relaxations with optimality certificates can be obtained by transforming the problem into a polynomial optimization form. The effectiveness of the proposed technique is illustrated with a numerical example.**

I. Introduction

Most realistic control problems involve both physical and information flow constraints. The later typically arise from the fact that using a centralized controller may not be feasible, due to communications constraints. In such cases using a decentralized controller can be crucial. Indeed decentralized controllers are widely used in many different application domains such as multi machine power systems [27], cooperative robotic vehicles [8] and micro electromechanical systems (MEMS)[7].

Given its importance, the problem of constructing decentralized controllers has been the object of considerable recent attention. While it is known that this problem is generically NP-hard, and hence intractable [5], it has been shown that, if the so co-called quadratic invariance property holds, then it reduces to a tractable convex optimization via the Youla parameterization [20], [22], [13]. Alternative approaches that do not rely on the quadratic invariance property include LMI based methods [30], [18], [21], [19]) and non-convex optimization [28], [15], [14], [15], While successful in many practical cases, LMI based approaches can only handle a subset of information structures and plants. On the other hand, optimization based approaches can handle generic structures, but, due to the non-convexity of the problem may result in sub-optimal solutions. Finally, [26] recently proposed an approach based on polynomial optimization. However, no control or state constraints were considered.

Similarly, the problem of synthesizing constrained controllers has been extensively studied. A line of research has approached this problem from the standpoint of invariant sets, with earlier work concentrating on polyhedral ones [24], [2], [25], [6], [16], while later work also considered ellipsoidal sets characterized in term of LMIs [10], [9], [1]. A potential difficulty with these approaches is that the use of polyhedral sets generally leads to bilinear optimization problems, while the use of ellipsoids typically leads to conservative results. Alternatively, constraints can be addressed in a convex framework by using the Youla parameterization [11], [23]. However, the resulting problem is infinite-dimensional, necessitating some form of approximation, and the resulting controllers can have arbitrarily high order.

When addressing problems exhibiting both physical and information flow constraints, if quadratic invariance holds, in principle one could pursue a modified approach based on [11] or [23]. However, as noted before, this leads to potentially very high order controllers. Alternative, the LMI based approaches that handle information and physical constraints could be combined, but this approach will inherit the restrictions of the former and the conservatism of the later.

Motivated by these difficulties, in this paper we propose a novel approach to solve the Information Structured Linear Constrained Regulation problem (ISLCRP), that is, the problem of synthesizing stabilizing controllers subject to information and physical constraints. The key idea is to recast the problem into a polynomial optimization form and use recent results in the field to solve the latter via a convergent sequence of convex optimization problems. Further, when combined with rank-minimization ideas, this approach leads to computationally efficient algorithms with optimality certificates. The paper is organized as follows. Section II provides background information about convex relaxations of polynomial optimization problems and polyhedral Lyapunov functions and a formal definition

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of the problem under consideration. Section III presents the proposed polynomial optimization based approach, including computational complexity considerations and briefly discusses how to extend this approach to the continuous time case. These results are illustrated in Section IV with a numerical example. Lastly, we present some concluding remarks in Section V.

II. PRELIMINARIES

In this section we present, for ease of reference, the notation used throughout the paper, some background results needed to transform the ISLCRP into a tractable convex optimization and formally define the problem under consideration.

A. Notation

$\mathbb{R}$, $\mathbb{N}$	set of real number and non-negative integers
$\boldsymbol{M}, \boldsymbol{x}$	a matrix, vector of suitable size
$\boldsymbol{M} \geq 0$	each element of $\boldsymbol{M}$ is non-negative
$\boldsymbol{M} \succeq 0$	matrix $\boldsymbol{M}$ is positive semi-definite
$\sigma_i(\boldsymbol{M})$	the i-th largest singular value of $\boldsymbol{M}$
$\\|\boldsymbol{u}\\|_\infty$	ℓ^∞ norm of $\boldsymbol{u}$: $\\|\boldsymbol{u}\\|_\infty \doteq \max_i \|u_i\|$
$\\|\boldsymbol{H}\\|_\infty$	ℓ^∞ to ℓ^∞ induced norm of $\boldsymbol{H}$: $\\|\boldsymbol{H}\\|_\infty \doteq \max_i \sum_j \|h_{ij}\|$

B. Convex relaxations of polynomial optimization problems

Consider the problem of minimizing a multivariate polynomial over a semi-algebraic set $\mathcal{K}$,

$$p^* = \min_{\boldsymbol{x} \in \mathcal{K}} \quad p(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \tag{1}$$

where $\mathcal{K} = \{\boldsymbol{x} \in \mathbb{R}^n : g_k(\boldsymbol{x}) \geq 0, \; k = 1, \cdots, N\}$. Here the constraints $g_k(\boldsymbol{x})$ are polynomials of the form $g_k(\boldsymbol{x}) = \sum_\alpha g_{k,\alpha} \boldsymbol{x}^\alpha$, where $\boldsymbol{\alpha} \in \mathbb{N}^n$, $\boldsymbol{x}^{\boldsymbol{\alpha}} = \prod_{i=1}^n x_i^{\alpha_i}$, and where $p_{\boldsymbol{\alpha}}$ and $g_{k,\alpha}$ denote the coefficients corresponding to the monomials $\boldsymbol{x}^{\boldsymbol{\alpha}}$. As shown in [12], this problem is equivalent to the following convex, albeit infinite dimensional, one

$$p^* = \min_{\mu \in \mathcal{P}(\mathcal{K})} \int p(\boldsymbol{x}) \mu(d\boldsymbol{x}) = \min_{\mu} \sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} \boldsymbol{m}_{\boldsymbol{\alpha}} \tag{2}$$

where $\mathcal{P}(\mathcal{K})$ denotes the set of probability measure supported on $\mathcal{K}$ and $\boldsymbol{m}_{\boldsymbol{\alpha}} \doteq \int_K \boldsymbol{x}^{\boldsymbol{\alpha}} \mu(d\boldsymbol{x})$ is the $\boldsymbol{\alpha}^{\text{th}}$ moment with respect to μ. Notice that the objective function in (2) is an affine function of $\boldsymbol{m}_{\boldsymbol{\alpha}}$, subject to the (convex) constraint that $\boldsymbol{m}_{\boldsymbol{\alpha}}$ must be the moments of some probability measure μ. As shown in [12], this constraint can be enforced by introducing several (infinite-dimensional) positive semi-definite constraints of the form $\boldsymbol{M}(\boldsymbol{m}_{\boldsymbol{\alpha}}) \succeq 0$ and $\boldsymbol{L}(g_k \boldsymbol{m}_{\boldsymbol{\alpha}}) \succeq 0, \; k = 1 \dots N$, where $\boldsymbol{M}$ (the moment matrix) and $\boldsymbol{L}$ (the localizing matrices) are affine in $\boldsymbol{m}$. Finally, finite dimensional relaxations of (2) with cost $p^d_{\boldsymbol{m}}$ can be obtained by replacing these matrices with truncated versions, containing moments of order up to $2d$, of the form

$$\begin{aligned} \boldsymbol{M}_d(\boldsymbol{m})_{i,j} &= E_\mu(\boldsymbol{x}^{\boldsymbol{\alpha}(i)} \boldsymbol{x}^{\boldsymbol{\alpha}(j)}), \; \forall i, j \leq S_d \\ \boldsymbol{L}_d(g_k \boldsymbol{m})_{i,j} &= E_\mu(\boldsymbol{x}^{\boldsymbol{\alpha}(i)} \boldsymbol{x}^{\boldsymbol{\alpha}(j)} g_k(\boldsymbol{x})), \\ & \qquad \forall i, j \leq S_{d - \lceil \frac{\deg(g_k(\boldsymbol{x}))}{2} \rceil} \end{aligned} \tag{3}$$

where $S_d \doteq \binom{d+n}{n}$, $\boldsymbol{x}^{\boldsymbol{\alpha}(i)}$ denotes the i-th element in the lexicographical ordered polynomial ring and $E_\mu(.)$ denotes expectation w.r.t. μ. As shown in [12], $p^d_{\boldsymbol{m}} \uparrow p^*$ monotonically. Further, if for some d

$$\text{rank}\{\boldsymbol{M}_d(\boldsymbol{m})\} = \text{rank}\{\boldsymbol{M}_{d-1}(\boldsymbol{m})\} \tag{4}$$

then the relaxation of order d is exact, that is $p^d_{\boldsymbol{m}} = p^*$.

C. Polyhedral Lyapunov Functions

Consider a discrete time linear system:

$$\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{x}(t) \tag{5}$$

and the function

$$V(\boldsymbol{x}) = \|\boldsymbol{G}\boldsymbol{x}\|_\infty \tag{6}$$

where, $\boldsymbol{G}$ has full column rank.

Definition 1: $V(\boldsymbol{x})$ is a polyhedral Lyapunov function for (5) iff along its trajectories

$$V(\boldsymbol{x}(t+1)) - V(\boldsymbol{x}(t)) < 0 \; \forall \boldsymbol{x} \neq 0$$

Remark 1: As shown in [3], existence of a polyhedral Lyapunov function is a necessary and sufficient condition for asymptotic stability of (5).

D. Problem Statement

Consider a discrete-time system of the form

$$\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{7}$$

where $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n \times m}$. Our goal is to design a stabilizing static state feedback law $\boldsymbol{u}(t) = \boldsymbol{F}\boldsymbol{x}(t)$ subject to the constraints:

$$-\boldsymbol{d} \leq \boldsymbol{u} \leq \boldsymbol{d} \tag{8}$$

for all initial conditions $\boldsymbol{x}(0)$ in some initial condition set $\mathcal{P}_o$, with an information structure of the form,

$$\boldsymbol{F} \in \boldsymbol{S} \tag{9}$$

where $\boldsymbol{d}$ is a real vector with positive components and $\boldsymbol{S}$ is a given matrix with $1/0$ entries that specifies the desired sparsity pattern for the controller. In the sequel, we will refer to this problem as *the Information Structured Linear Constrained Regulation problem (ISLCRP).*

III. Main Result

A. Semi-Algebraic Characterization of All the Admissible State Feedback Gains

According to the Proposition 1 in [24], the existence of a control law to the traditional linear constrained regulation problem is equivalent to the existence of a positively invariant set $\Omega \subseteq \mathbb{R}^n$ of the resulting closed-loop system such that

$$\mathcal{P}_o \subseteq \Omega \subseteq \mathcal{P}(\boldsymbol{F}, \boldsymbol{d}) \tag{10}$$

where, $\mathcal{P}_o$ denotes the set of initial states, and $\mathcal{P}(\boldsymbol{F}, \boldsymbol{d}) = \{\boldsymbol{x} \in \mathbb{R}^n : -\boldsymbol{d} \leq \boldsymbol{F}\boldsymbol{x} \leq \boldsymbol{d}\}$ denotes the set in which the control inputs satisfy the constraint (8). Furthermore, as shown by Proposition 3 in [24], a set of the form $\mathcal{P}(\boldsymbol{G}, \boldsymbol{w}) = \{\boldsymbol{x} \in \mathbb{R}^n : -\boldsymbol{w} \leq \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{w}\}$ where $\boldsymbol{G} \in \mathbb{R}^{q \times n}$ is positively invariant for the closed loop system $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}$ if and only if there exists a matrix $\boldsymbol{H} \in \mathbb{R}^{q \times q}$ such that

$$\boldsymbol{G}(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}) = \boldsymbol{H}\boldsymbol{G}, \; \|\boldsymbol{H}\|_\infty \leq 1. \tag{11}$$

Remark 2: Note that the condition above does not necessarily imply asymptotic stability of $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}$. Indeed, when $\text{rank}(\boldsymbol{G}) < n$ the equation above holds even for unstable systems as long as the unstable modes of $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}$ are unobservable from $\boldsymbol{G}$. As shown in [24], guaranteeing closed loop stability requires either explicitly enforcing that the closed loop poles lie strictly inside the unit disk, or alternatively, requiring that $\boldsymbol{G}$ has full column rank and slightly modifying (11) to enforce a strict inequality. In this case, $V(\boldsymbol{x}) \doteq \|\boldsymbol{G}\boldsymbol{x}\|_\infty$ is a polyhedral Lyapunov function for the closed loop system.

Based on the observation above, we will look for a full column rank matrix $\boldsymbol{G}$ and matrices $\boldsymbol{F}, \boldsymbol{H}$ such that the strict inequality in (11) holds. The next result will be used to enforce the full column rank condition.

Lemma 1: Assume that (11) holds for some full column rank matrix $\boldsymbol{G}$ and some $\|\boldsymbol{H}\|_\infty \leq \sigma < 1$. Then, there exist some ϵ such that $\boldsymbol{G}_\epsilon \doteq \begin{bmatrix} \epsilon \boldsymbol{I} \\ \boldsymbol{G} \end{bmatrix}$ also satisfies (11) for a suitable chosen $\boldsymbol{H}_\epsilon$ with $\|\boldsymbol{H}_\epsilon\|_\infty < 1$.

Proof: Follows by selecting

$$\boldsymbol{H}_\epsilon \doteq \begin{bmatrix} 0 & \boldsymbol{H}_{12} \\ 0 & \boldsymbol{H} \end{bmatrix}$$

with $\boldsymbol{H}_{12} = \epsilon(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F})(\boldsymbol{G}^T\boldsymbol{G})^{-1}\boldsymbol{G}^T$ and selecting ϵ small enough so that $\|\boldsymbol{H}_{12}\|_\infty < 1$. ■

Note that since $\boldsymbol{G}$ has full column rank, the set $\mathcal{P}(\boldsymbol{G}, \boldsymbol{d})$ is a compact polyhedron and thus, one can always select ϵ small enough so that $\mathcal{P}(\boldsymbol{G}, \boldsymbol{w}) \subset \{\boldsymbol{x} \colon \epsilon|x_i| \leq w_i\}$. Thus $\mathcal{P}(\boldsymbol{G}, \boldsymbol{w}) = \mathcal{P}(\boldsymbol{G}_\epsilon, \boldsymbol{w})$.

Consider now the constraint that the control action must satisfy $\|\boldsymbol{u}\|_\infty \leq \boldsymbol{d}$ for all points $\boldsymbol{x} \in \mathcal{P}(\boldsymbol{G}, \boldsymbol{w})$, that is

$$-\boldsymbol{d} \leq \boldsymbol{F}\boldsymbol{x} \leq \boldsymbol{d}, \; \forall \boldsymbol{x} \in \mathcal{P}(\boldsymbol{G}, \boldsymbol{w}) \tag{12}$$

By using Lemma 1 above, combined with the *Extended Farkas Lemma* [11], it follows that (12) is equivalent to the existence of a matrix $\boldsymbol{Y}$ such that

$$\begin{array}{c} \boldsymbol{Y} \begin{bmatrix} \boldsymbol{G}_\epsilon \\ -\boldsymbol{G}_\epsilon \end{bmatrix} = \begin{bmatrix} \boldsymbol{F} \\ -\boldsymbol{F} \end{bmatrix} \\ \boldsymbol{Y} \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{w} \end{bmatrix} \leq \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{d} \end{bmatrix} \\ \boldsymbol{Y} \geq 0 \end{array} \tag{13}$$

Finally, the information structure constraints can be embedded easily as $\boldsymbol{F}_{[i,j]} = 0$ for all (i,j) such that $\boldsymbol{S}_{[i,j]} = 0$. Combining all the observations above leads to the following result, providing a semi-algebraic characterization of all admissible state feedback gains

Theorem 1: The ISLCRP is solvable if and only if there exist matrices $\boldsymbol{G}, \boldsymbol{F}, \boldsymbol{H}, \boldsymbol{Y}$, a vector $\boldsymbol{w}$ and a scalar ϵ such that the following conditions hold:

$$\begin{array}{c} \boldsymbol{G}_\epsilon(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}) = \boldsymbol{H}\boldsymbol{G}_\epsilon, \; \|\boldsymbol{H}\|_\infty \leq \sigma \\ \boldsymbol{Y} \begin{bmatrix} \boldsymbol{G}_\epsilon \\ -\boldsymbol{G}_\epsilon \end{bmatrix} = \begin{bmatrix} \boldsymbol{F} \\ -\boldsymbol{F} \end{bmatrix} \\ \boldsymbol{Y} \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{w} \end{bmatrix} \leq \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{d} \end{bmatrix} \\ \boldsymbol{Y} \geq 0 \\ \boldsymbol{F}_{[i,j]} = 0, \; \forall (i,j) \in \boldsymbol{O} \end{array} \tag{14}$$

where, $\boldsymbol{O} = \{(i,j) : \boldsymbol{S}_{[i,j]} = 0\}$ denotes the locations of non-available information channels.

B. Convex Optimization based Synthesis Approach

Theorem 1 above allows for reformulating the ISLCRP as a (non-convex) polynomial feasibility problem. As we show next, a convex relaxation can be obtained by exploring the moment-based relaxation introduced in Section II. Note that it is often desirable to maximize the region where the system can be stabilized subject to the control constraints. In the sequel, we will achieve this by maximizing a scaling factor δ^{-1}, such that

$$\mathcal{P}(\boldsymbol{F}, \boldsymbol{d}) \supseteq \delta^{-1}\mathcal{P}(\boldsymbol{G}_\epsilon, \boldsymbol{w}) = \{\boldsymbol{x} : \begin{bmatrix} \boldsymbol{G}_\epsilon \\ -\boldsymbol{G}_\epsilon \end{bmatrix} \boldsymbol{x} \leq \delta^{-1} \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{w} \end{bmatrix}\} \tag{15}$$

which leads to the following optimization problem

$$
\begin{array}{c}
\min \delta \quad \text{subject to} \\
\boldsymbol{G}_\epsilon(\boldsymbol{A}+\boldsymbol{BF}) = \boldsymbol{HG}_\epsilon,\ \|\boldsymbol{H}\|_\infty \le \sigma \\
\boldsymbol{Y}\begin{bmatrix}\boldsymbol{G}_\epsilon \\ -\boldsymbol{G}_\epsilon\end{bmatrix} = \begin{bmatrix}\boldsymbol{F} \\ -\boldsymbol{F}\end{bmatrix} \\
\boldsymbol{Y}\begin{bmatrix}\boldsymbol{w} \\ \boldsymbol{w}\end{bmatrix} \le \delta \begin{bmatrix}\boldsymbol{d} \\ \boldsymbol{d}\end{bmatrix} \\
\boldsymbol{Y} \ge 0 \\
\boldsymbol{F}_{[i,j]} = 0,\ \forall (i,j) \in \boldsymbol{O}
\end{array} \tag{16}
$$

or, equivalently, the feasibility problem

$$
\begin{array}{c}
\boldsymbol{G}_\epsilon(\boldsymbol{A}+\boldsymbol{BF}) = \boldsymbol{HG}_\epsilon,\ \|\boldsymbol{H}\|_\infty \le \sigma \\
\boldsymbol{Y}\begin{bmatrix}\boldsymbol{G}_\epsilon \\ -\boldsymbol{G}_\epsilon\end{bmatrix} = \begin{bmatrix}\boldsymbol{F} \\ -\boldsymbol{F}\end{bmatrix} \\
\boldsymbol{Y}\begin{bmatrix}\boldsymbol{w} \\ \boldsymbol{w}\end{bmatrix} \le \bar{\delta} \begin{bmatrix}\boldsymbol{d} \\ \boldsymbol{d}\end{bmatrix} \\
\boldsymbol{Y} \ge 0 \\
\boldsymbol{F}_{[i,j]} = 0,\ \forall (i,j) \in \boldsymbol{O}
\end{array} \tag{17}
$$

Note that these are polynomial optimization problems of the form of (1) and thus can be reduced to a sequence of SDPs of the form[1]

$$
\begin{array}{ll}
\min\limits_{\boldsymbol{m}} & f(\boldsymbol{m}) \\
\text{s.t.} & \boldsymbol{M}_d(\boldsymbol{m}) \succeq 0 \\
& \boldsymbol{L}_d(g_k(\cdot) \ge 0) \succeq 0 \\
& \boldsymbol{L}_d(h_l(\cdot) = 0) = \boldsymbol{0}
\end{array} \tag{18}
$$

Here $f(\boldsymbol{m}) = m_{1,\delta}$ (the first order moment of δ) for the optimization problem, and $f(\boldsymbol{m}) = 1$ for the feasibility problem; and $\boldsymbol{L}_d(g_k(\cdot)), \boldsymbol{L}_d(h_l(\cdot))$ denote the localizing matrices corresponding to the inequality and equality polynomial constraints in (16)-(17).

C. Computational Complexity Considerations and Low Order Relaxations

In this section, we discuss the computational complexity of the proposed method, and introduce a low cost algorithm for the desired controller design. We will show that, besides the *universal property*, the choice of polyhedral Lyapunov function also benefits our algorithm in the sense of providing a low cost SDP based algorithm. Note that the main computational cost stems from the size of the matrices $\boldsymbol{M}$ and $\boldsymbol{L}$, since the computational complexity of interior-point SDP solvers scales polynomially with the number of decision variables. Since the size of these matrices is $\binom{d+n}{n}$ (the number of moments of order up to d in n variables) in order to reduce the computational complexity, the idea is to make the value of d as small as possible while still attempting to enforce that the relaxation (18) be exact.

Since all the monomials in problems (16) and (17) are of degree 2 at most, it follows that the lowest achievable relaxation has order $d = 1$. Further, this relaxation is exact if the corresponding moment matrices have rank 1. These observations motivate considering the following rank constrained problem,

$$
\begin{array}{c}
\boldsymbol{M}_1(\boldsymbol{m}) \succeq 0 \\
\boldsymbol{L}_1(g_k(\cdot) \ge 0) \succeq 0 \\
\boldsymbol{L}_1(h_l(\cdot) = 0) = \boldsymbol{0} \\
rank[\boldsymbol{M}_1(\boldsymbol{m})] = 1
\end{array} \tag{19}
$$

Although this problem is non-convex, a convex optimization procedure that iteratively minimizing the reweighted nuclear norm can be applied on $\boldsymbol{M}_1$ to promote a low rank solution. For a discussion of convergence property, the interested reader is referred to [17]. The idea above leads to Algorithm 1 below

---

Algorithm 1 Information Structured Linear Constrained Regulation Design

---

Initialize: $iter = 0$, $\boldsymbol{W}^{(0)} = \boldsymbol{I}$
repeat
Solve

$$
\begin{array}{ll}
\min\limits_{\boldsymbol{m}} & \text{Trace}(\boldsymbol{W}^{(iter)}\boldsymbol{M}_1) \\
\text{subject to} & \\
\boldsymbol{M}_1(\boldsymbol{m}) \succeq 0 & \\
\boldsymbol{L}_1(g_k(\cdot) \ge 0) \succeq 0 & \\
\boldsymbol{L}_1(h_l(\cdot) = 0) = \boldsymbol{0} &
\end{array} \tag{20}
$$

Update

$$
\begin{array}{rcl}
\boldsymbol{W}^{(iter+1)} & = & \left(\boldsymbol{M}_1^{(iter)} + \sigma_2(\boldsymbol{M}_1^{(iter)})\right)^{-1} \\
iter & = & iter + 1
\end{array}
$$

until rank$\{\boldsymbol{M}_1\} = 1$.

D. Extension to Sparsity Promoting Design

In the previous section, we presented a computationally efficient algorithm for the case where the information structure is given. However in practice, there also exists situations where the goal is to maximize controller sparsity, rather than enforcing a specific structure. For instance, due to the cost of building communication channels, one may want to use as less as possible channels while designing the stabilizing gain. In this section, we extend Algorithm 1 to this sparsity promoting design problem.

Notice that for an information channel of $\boldsymbol{F}$ to be zero (non-available), we only need to enforce a localizing

[1] The detailed SDP formulation is omitted due to space constraints. It can be obtained, along with the code to solve it by contacting the authors.

matrix to be $\mathbf{0}$, which is numerically equivalent to enforcing a vector to be zero element-wise. Therefore, sparsity of information channels can be maximized by pursuing a group-lasso[29] problem on the elements of the localizing matrices corresponding to $\boldsymbol{F}_{[i,j]}$. Hence, we can simply involve a second term in the objective function of (20), which leads to Algorithm 2. Note that this algorithm includes a normalization step which aims at balancing the weights between the rank minimizing term and the sparsity promoting term, leading to improved rank convergence.

Algorithm 2 Design Stabilizing Static Controller with Maximal Sparse Pattern

Initialize: $iter = 0$, $\boldsymbol{W}^{(0)} = \boldsymbol{I}$, $w_{[i,j]}^{(0)} = 1$, $\forall i = 1, \cdots, m,\ j = 1, \cdots, n$.

repeat

Solve

$$\begin{aligned} \min_{\boldsymbol{m}} \quad & \mathrm{Trace}(\boldsymbol{W}^{(iter)}\boldsymbol{M}_1) \\ & + \lambda \sum_{i,j} w_{[i,j]}^{(iter)} \| \mathrm{vect}[\boldsymbol{L}_1(\boldsymbol{F}_{[i,j]})] \|_2 \\ \text{s.t.} \quad & \boldsymbol{M}_1(\boldsymbol{m}) \succeq 0 \\ & \boldsymbol{L}_1(g_k(\cdot) \geq 0) \succeq 0 \\ & \boldsymbol{L}_1(h_l(\cdot) = 0) = \mathbf{0} \end{aligned} \tag{21}$$

Update

$$\begin{aligned} \boldsymbol{W}^{(iter+1)} &= \left(\boldsymbol{M}_1^{(iter)} + \sigma_2(\boldsymbol{M}_1^{(iter)})\right)^{-1} \\ \boldsymbol{S}_{[i,j]} &= \| \mathrm{vect}[\boldsymbol{L}_1(\boldsymbol{F}_{[i,j]})] \|_2,\ \forall (i,j) \\ \boldsymbol{S} &= \sigma_1(\boldsymbol{M}_1^{(iter)}) \frac{\boldsymbol{S}}{\|\boldsymbol{S}\|_F} \\ w_{[i,j]}^{(iter+1)} &= \left(\boldsymbol{S}_{[i,j]} + \sigma_2(\boldsymbol{M}_1^{(iter)})\right)^{-1} \\ iter &= iter + 1 \end{aligned}$$

until $\mathrm{rank}[\boldsymbol{M}_1(\boldsymbol{m})] = 1$.

E. LP based Post-processing

By applying the Algorithm 1 or 2, an information structured linear constrained gain can be obtained with a $\bar{\delta}$-acceptable initial set. However, this feasible initial set may not be the tightest subset of $\mathcal{P}(\boldsymbol{F}, \boldsymbol{d})$. Once $\boldsymbol{G}_\epsilon$ and $\boldsymbol{F}$ are solved using either Algorithm 1 or 2, the maximum initial set can be estimated using the Extended Farkas Lemma to reduce the problem to the following LP:

$$\begin{aligned} \min_{\delta, \boldsymbol{Y}} \quad & \delta \\ \text{s.t.} \quad & \boldsymbol{Y} \begin{bmatrix} \boldsymbol{G}_\epsilon \\ -\boldsymbol{G}_\epsilon \end{bmatrix} = \begin{bmatrix} \boldsymbol{F} \\ -\boldsymbol{F} \end{bmatrix} \\ & \boldsymbol{Y} \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{w} \end{bmatrix} \leq \delta \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{d} \end{bmatrix} \\ & \boldsymbol{Y} \geq 0 \end{aligned} \tag{22}$$

Let δ^* denotes the minimum value achieved by the above problems, then $\boldsymbol{G}_\epsilon \leftarrow \delta^* \boldsymbol{G}_\epsilon$ and the tightest initial set will be $\mathcal{P}(\boldsymbol{G}_\epsilon, \boldsymbol{w})$.

F. Extension to Continuous-time Case

In order to generalize our algorithms into continuous-time case,

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{u}(t) &= \boldsymbol{F}\boldsymbol{x}(t) \end{aligned} \tag{23}$$

we recall that its Euler Approximation System (EAS) is given by

$$\boldsymbol{x}(t+1) = [\boldsymbol{I} + \tau\boldsymbol{A}]\boldsymbol{x}(t) + \tau\boldsymbol{B}\boldsymbol{u}(t) \tag{24}$$

It has been shown that stability of the EAS implies the stability of the continuous-time system, and that any positively invariant set of EAS is also positively invariant for its continuous-time counterpart [4]. The stability of (23) conversely implies the existence of some $\tau > 0$ such that its EAS is stable. Hence, given a continuous-time plant, the algorithms we proposed can be adopted according to the corresponding Euler approximation with a small enough τ.

IV. Numerical Examples

In this section, we illustrate the proposed method with a discrete-time state feedback example. Since, to the best of our knowledge, no existing constrained control method can accommodate sparsity constraints, we compare with two existing information structured gain design methods: an SDP based method [26] and an LMI based method[18].

Consider the following discrete-time system

$$\begin{aligned} \boldsymbol{A} &= \begin{bmatrix} 0.6593 & -0.3911 & 0.2948 \\ 0.1906 & 0.7658 & -0.3475 \\ 2.2012 & 1.5873 & -0.8751 \end{bmatrix} \\ \boldsymbol{B} &= \begin{bmatrix} 0.6617 & 0.8344 & 0.5075 \\ 0.1705 & -0.4283 & -0.2391 \\ 0.0994 & 0.5144 & 0.1356 \end{bmatrix} \\ \boldsymbol{C} &= \boldsymbol{I} \end{aligned} \tag{25}$$

the goal is to find a constrained (with $\boldsymbol{d} = \mathbf{1}$, $\boldsymbol{w} = \mathbf{1}$) state feedback gain subject to the information structure

$$\boldsymbol{S} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{26}$$

Applying Algorithm 1 with parameters $\epsilon = 0.1$ and $\bar{\delta} = 0.5$ results in a stabilizing gain

$$\boldsymbol{F} = \begin{bmatrix} -0.0453 & -0.0103 & 0 \\ -0.0297 & 0.0202 & -0.0113 \\ 0 & 0.0473 & -0.0271 \end{bmatrix} \tag{27}$$

with $\|\boldsymbol{G}_\epsilon\|_\infty = 0.5425$. Further applying the LP based post-processing, the maximum set induced by this polyhedral Lyapunov function is $\|\bar{\boldsymbol{G}}_\epsilon\|_\infty = 0.2590$.

Using Algorithm 2 with parameter $\epsilon = 0.1$ and $\bar{\delta} = 1$ we trade in some volume of the initial set to achieve a much sparser stabilizing gain

$$\boldsymbol{F} = \begin{bmatrix} 0 & 0 & -0.0279 \\ -0.0788 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{28}$$

with $\|\boldsymbol{G}_\epsilon\|_\infty = 0.7928$. Further applying the LP based post-processing, the maximum set induced by this polyhedral Lyapunov function is $\|\bar{\boldsymbol{G}}_\epsilon\|_\infty = 0.6246$.

For the proposed stabilizing gain (27) and (28), we randomly sample 10000 initial state $\boldsymbol{x}_0$, such that $\|\bar{\boldsymbol{G}}_\epsilon \boldsymbol{x}_0\|_\infty \leq 1$. Notice that, for (27) and (28), the matrix $\bar{\boldsymbol{G}}_\epsilon$ is different. In Figure 1, we show the maximum absolute value of the 10000 control inputs sequences at each time instance for both proposed gain, respectively. The goal is to show that the control inputs $\boldsymbol{u}$ never exceeds the upper bound $\boldsymbol{d} = \mathbf{1}$, once the state $\boldsymbol{x}_0$ is initialized inside the corresponding set $\mathcal{P}(\bar{\boldsymbol{G}}_\epsilon, \boldsymbol{w})$.

In order to achieve a fair comparison, for the static gain designed by different methods, we show the $\|\boldsymbol{F}\|_\infty$ in Table I. This value, regardless of the shape of $\boldsymbol{G}_\epsilon$, reflects the volume of the trivial initial set $\{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_\infty \leq \frac{1}{\|\boldsymbol{F}\|_\infty}\}$. It can be observed that, the trivial initial set associated with the proposed static gains possess much larger volume as compared to the existing methods.

Method	$\|\boldsymbol{F}\|_\infty$
(27)	0.0744
(28)	0.0788
[26]	0.9754
[18]	15.3804

TABLE I

THE $\mathcal{H}_\infty$ NORM OF THE STATIC GAIN DESIGNED BY DIFFERENT METHODS

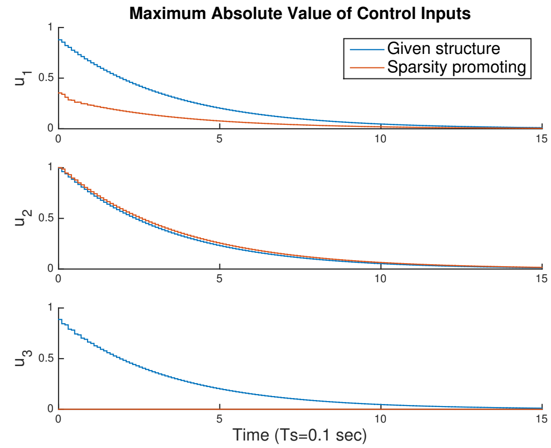

Fig. 1. Maximum absolute value over time of control inputs from 10000 trials

V. Conclusions

In this paper, we consider the problem of finding stabilizing state feedback gain under constraints on both the information structure and the control effort. A semi-algebraic formulation is introduced to parametrizing all admissible controllers. Then, a convex optimization based synthesis approach is proposed that exploits recent results in polynomial optimization. Finally, we discuss the computational complexity of the proposed method, and present efficient algorithms with reduced complexity for two practical cases: (i) enforcing a given information structure; and (ii) maximizing controller sparsity. A numerical example was presented to illustrate the advantages of the proposed methods.

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