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Brief paper

A risk adjusted approach to robust simultaneous fault detection and isolation ☆

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Abstract

This paper addresses the problem of detecting and isolating faults in noisy MIMO uncertain-systems, subject to structured dynamic uncertainty. Its main result shows that this problem can be efficiently solved using a combination of sampling and LMI optimization tools. These results are illustrated with two examples and benchmarked against existing methods.

Keywords: Fault detection; Fault isolation; Model (in)validation

1. Introduction

Fault detection and isolation (FDI) has been the subject of intense recent research, leading to a variety of methods (see for instance Frank & Ding, 1997; Gertler, 1998 and references therein). Amongst these methods, *model-based* approaches are specially appealing, since do not require additional hardware. However, a drawback of these approaches is their (potential) *fragility*: a mismatch between the actual plant and the model used in the FDI algorithm can result in false alarms. Robust FDI methods have been well studied (see for instance Collins & Song, 2000; Emaimi-Naeimi, Akhter, & Rock, 1998; Frank & Ding, 1997; Henry & Zolghadri, 2005; Jiang, Wang, & Soh, 2002; Saberi, Stoorvogel, Sannuti, & Niemann, 2000; Stoustrup & Niemann, 2003; Zhong, Ding, Lam, & Wang, 2003 and references therein). A potential disadvantage of these methods is the difficulty in isolating the exact location of the fault and in detecting simultaneous faults. Motivated by Shim and Sznaier (2003), in this paper we propose to address these issues by recasting the problem into a robust model (in)validation form. A similar approach was pursued in Henry, Zolghadri, Monsion, and Ygorra (2002), where, for frequency domain data, the problem of fault detection was recast as a μ analysis one and reduced to an LMI optimization using the well known μ upper bound. However, when in addition to detecting a fault it is desired to isolate its location, the resulting problem becomes NP-hard (Shim & Sznaier, 2003; Toker & Chen, 1996). In order to obtain tractable solutions, we propose (i) an efficient deterministic convex relaxation for the case multiplicative or additive uncertainty, and (ii) a risk-adjusted one, motivated by Ding, Zhang, and Frank (2003), for general uncertainty structures and fault dynamics. Both relaxations have the ability to estimate the location and strength of the fault(s). In addition, the computational complexity of the stochastic one grows only polynomially with the dimension of the plant.

2. Preliminaries

$\mathscr{H}_\infty$ denotes the subspace of transfer matrices analytic in $|\lambda| \leqslant 1$ equipped with the norm: $\|G\|_\infty \doteq ess\,\sup_{|\lambda|<1} \overline{\sigma}(G(\lambda))$, where $\overline{\sigma}(.)$ denotes maximum singular value. $\mathscr{BH}_\infty$ and $\mathscr{BH}_\infty^n$ denote the unit ball in $\mathscr{H}_\infty$ and the set of $(n-1)$th order FIR transfer matrices that can be completed to belong to $\mathscr{BH}_\infty$, i.e. $\mathscr{BH}_\infty^n \doteq \{H(\lambda) = \mathbf{H}_0 + \cdots + \mathbf{H}_{n-1}\lambda^{n-1} : H(\lambda) + \lambda^n G(\lambda) \in \mathscr{BH}_\infty,$ for some $G(\lambda)\}$, respectively. ℓ_2 denotes the space of real sequences equipped with the norm

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$\|x\|_2^2 \doteq \sum_{i=0}^{\infty} x_i^2 < \infty$. As usual, $\mathbf{M} \geqslant 0$ indicates a positive semi-definite matrix. Finally, to any finite sequence $\{\mathbf{x}_k\}_{k=0}^{n-1}$, we will associate the Toeplitz matrix:

$$\mathbf{T}_x^n = \begin{bmatrix} \mathbf{x}_0 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{x}_1 & \mathbf{x}_0 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_{n-1} & \mathbf{x}_{n-2} & \dots & \mathbf{x}_0 \end{bmatrix}.$$

The following algorithm generates N_s uniformly distributed samples $\mathbf{h}^i = \{\mathbf{H}_0^i, \mathbf{H}_1^i, \dots, \mathbf{H}_n^i\}$ from the set $\mathscr{BH}_\infty^n$.

Algorithm 1 (*Sznaier, Lagoa, and Mazzaro, 2005*). *Let $k = 0$. Generate N_1 uniform samples from the set $\{\mathbf{H}_0 : \overline{\sigma}(\mathbf{H}_0) \leqslant 1\}$.*

1. *Let $k := k + 1$. For every sample $(\mathbf{H}_0^i, \mathbf{H}_1^i, \dots, \mathbf{H}_{k-1}^i)$, consider the partition*

$$\begin{bmatrix} \mathbf{H}_k^i & \cdots & \mathbf{H}_1^i & \mathbf{H}_0^i \\ \mathbf{H}_{k-1}^i & \cdots & \mathbf{H}_0^i & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{H}_0^i & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_k^i & \mathbf{B} \\ \mathbf{C} & \mathbf{A} \end{bmatrix} \tag{1}$$

   and let $\mathbf{Y}$ and $\mathbf{Z}$ be a solution of the linear equations

$$\mathbf{B} = \mathbf{Y}(\mathbf{I} - \mathbf{A}^{\mathrm{T}}\mathbf{A})^{1/2}; \quad \mathbf{C} = (\mathbf{I} - \mathbf{A}\mathbf{A}^{\mathrm{T}})^{1/2}\mathbf{Z}.$$

2. *Let $\mathbf{J}(\mathbf{H}_0, \dots, \mathbf{H}_{k-1}) \doteq |(\mathbf{I} - \mathbf{Y}\mathbf{Y}^{\mathrm{T}})^{1/2}|^m |(\mathbf{I} - \mathbf{Z}^{\mathrm{T}}\mathbf{Z})^{1/2}|^s$. Generate $\lfloor N_1 \mathbf{J}(\mathbf{H}_0^i, \dots, \mathbf{H}_{k-1}^i) \rfloor$ samples uniformly over the set $\{\mathbf{W} : \overline{\sigma}(\mathbf{W}) \leqslant 1\}$ and for each of those samples $\mathbf{W}^i$, compute*

$$\mathbf{H}_k^i = -\mathbf{Y}\mathbf{A}^{\mathrm{T}}\mathbf{Z} + (\mathbf{I} - \mathbf{Y}\mathbf{Y}^{\mathrm{T}})^{1/2}\mathbf{W}^i(\mathbf{I} - \mathbf{Z}^{\mathrm{T}}\mathbf{Z})^{1/2}.$$

3. *If $k \leqslant n$ go to step 1. Otherwise, stop.*

Lemma 1 (*Carathéodory–Fejér Foias and Frazho, 1900*). *Given two sequences $\mathbf{u} = \{u_o, \dots, u_{n-1}\}$ and $\mathbf{y} = \{y_0, \dots, y_{n-1}\}$, there exists $\Delta \in \mathscr{H}_\infty$, $\|\Delta\|_\infty \leqslant \gamma$ such that $\Delta\mathbf{u} = \mathbf{y}$ if and only if $(\mathbf{T}_y^n)^{\mathrm{T}}\mathbf{T}_y^n - \gamma^2(\mathbf{T}_u^n)^{\mathrm{T}}\mathbf{T}_u^n \leqslant 0$.*

3. Robust FDI

3.1. Problem formulation

In this paper we consider the problem of FDI for systems represented by the following parameterized fault model:

$$\mathbf{y} = \left[G_0(\lambda, \Delta_o) + \sum_{i=1}^{r} f_i G_i(\lambda, \Delta_i) \right] \mathbf{u} + \mathbf{d},$$
$$\Delta_i \in \mathbf{\Delta}_i \subseteq \mathscr{BH}_\infty, \quad \|\mathbf{d}\|_2 \leqslant \eta. \tag{2}$$

Here the transfer matrices $G_0(\lambda, \Delta_o)$ and $G_i(\lambda, \Delta_i)$, $i = 1, \dots, r$ represent the plant under normal (e.g. non-failure) conditions and dynamic fault models, respectively, Δ_i represent (structured) dynamic model uncertainty and d represents ℓ^2 bounded measurement noise. The scalars $f_i \in [0, 1]$ are fault indicators, with $f_i = 0$ corresponding to the case of no failure. In this context, the FDI problem can be stated as

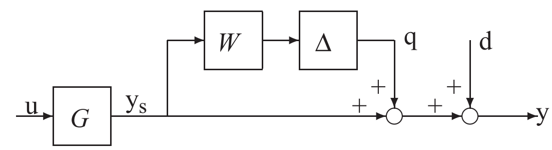

Fig. 1. Setup for robust FDI with multiplicative uncertainty.

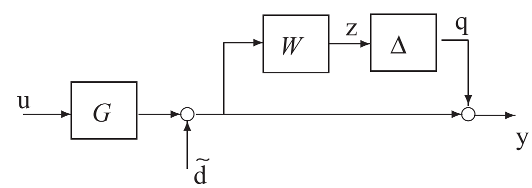

Fig. 2. Jointly convex FDI setup.

Problem 1. *Given the nominal model $G_o(\lambda, \Delta_o)$, failure dynamics $G_i(\lambda, \Delta_i)$, a bound η on the measurement noise, uncertainty sets $\mathbf{\Delta}_i$, and n input/output experimental measurements determine*: (i) *whether a fault has occurred, and* (ii) *in that case isolate it and determine its strength.*

Note that there may exist more than one triple $\{\Delta_i, \mathbf{d}, \mathbf{f}\}$ that explains the data. To avoid ambiguities, we will select the solution that minimizes $\|\mathbf{f}\|_2$. This choice minimizes the number of false alarms, since it tries to explain the experimental data as being produced by the nominal dynamics, possibly affected by uncertainty and noise. With this choice, Problem 1 can be recast in the following optimization form:

Problem 2. *Given the* a priori *information $G_i(\lambda, \Delta_i)$, η and measurements* $\mathbf{u}$ *and* $\mathbf{y}$ *find* $\min_{\Delta_i, \mathbf{d}} \|\mathbf{f}\|_2$, *subject to* (2). *If $\mathbf{f} = 0$ then no fault is present. Otherwise, the fault location/strength is identified by the elements of* $\mathbf{f}$.

Unfortunately, as stated Problem 2 leads to a generically NP-hard bilinear matrix inequality (BMI) optimization in $\mathbf{d}, \mathbf{f}$ (Shim & Sznaier, 2003). To avoid this difficulty, in the sequel we propose two convex relaxations.

3.2. A deterministic convex relaxation for multiplicative uncertainty

Consider the case shown in Fig. 1, where the nominal and failure dynamics are subject to *multiplicative*, *unstructured* uncertainty. While the problem is still not jointly convex in all the variables involved, a convex relaxation can be obtained by considering the alternative setup shown in Fig. 2, where $\mathbf{d}$ is also affected by the unknown dynamics Δ:

$$\mathbf{y} = (I + \Delta W) \left[\left(G_0 + \sum_{i=1}^{r} f_i G_i \right) \mathbf{u} + \tilde{\mathbf{d}} \right]. \tag{3}$$

Note that the only difference in the two setups is in the measurement noise level. Specifically, assume that there exists a triple $(\mathbf{f}, \tilde{\mathbf{d}}, \Delta)$ satisfying Eq. (3) with $\|\tilde{\mathbf{d}}\|_2 \leqslant \tilde{\eta} \doteq \eta/(1 +$

$\|W\|_\infty\|\Delta\|_\infty$), and let $\mathbf{d} \doteq (1+\Delta W)\tilde{\mathbf{d}}$. Then the triple $(\mathbf{f}, \mathbf{d}, \Delta)$ satisfies

$$\mathbf{y} = (I + \Delta W)\left(G_0 + \sum_{i=1}^{r} f_i G_i\right)\mathbf{u} + \mathbf{d} \tag{4}$$

and $\|\mathbf{d}\|_2 \leqslant \eta$. Thus, one can attempt to find a solution to the original problem by searching for a solution to the FDI problem shown in Fig. 2, with noise level $\tilde{\eta}$. As shown next, this leads to a convex problem. In addition, if $\|W\Delta\| \ll 1$ then this approximation is not too conservative.

Theorem 1. *There exist a feasible tripe* $(\mathbf{f}, \tilde{\mathbf{d}}, \Delta)$ *that satisfies Eq.* (3) *if and only if there exists at least an admissible vector* $\mathbf{f}$, $0 \leqslant f_i \leqslant 1$ *and a finite sequence* $\mathbf{q} = \{\mathbf{q}_0, \mathbf{q}_1, \ldots, \mathbf{q}_n\}$ *such that the following LMIs hold*:

$$\begin{bmatrix} \mathbf{X}(\mathbf{q}) & (\mathbf{T}^n_{\mathbf{q}})^{\mathrm{T}} \\ \mathbf{T}^n_{\mathbf{q}} & \left[\dfrac{\mathbf{I}}{\gamma^2} - (\mathbf{T}^n_W)^{\mathrm{T}}\mathbf{T}^n_W\right]^{-1} \end{bmatrix} \geqslant 0,$$
$$\begin{bmatrix} \tilde{\eta}^2\mathbf{I} & \mathbf{Y}^{\mathrm{T}}(\mathbf{q},\mathbf{f}) \\ \mathbf{Y}(\mathbf{q},\mathbf{f}) & \mathbf{I} \end{bmatrix} \geqslant 0 \tag{5}$$

with

$$\mathbf{X}(\mathbf{q}) \doteq (\mathbf{T}^n_W\mathbf{T}^n_y)^{\mathrm{T}}\mathbf{T}^n_W\mathbf{T}^n_y - (\mathbf{T}^n_W\mathbf{T}^n_y)^{\mathrm{T}}\mathbf{T}^n_W\mathbf{T}^n_q - (\mathbf{T}^n_W\mathbf{T}^n_q)^{\mathrm{T}}\mathbf{T}^n_W\mathbf{T}^n_y,$$
$$\mathbf{Y}(\mathbf{q},\mathbf{f}) \doteq \left[\mathbf{T}^n_y - \mathbf{T}^n_{\mathbf{q}} - \left(\mathbf{T}^n_{G_o} + \sum_i f_i\mathbf{T}^n_{G_i}\right)\mathbf{T}^n_u\right],$$

where, by a slight notational abuse, $\mathbf{T}^n_W$, *denotes the Toeplitz matrix associated with the impulse response of W.*

Proof. From Eq. (3) we have that (see Fig. 2):

$$\mathbf{T}^n_z = \mathbf{T}^n_W(\mathbf{T}^n_y - \mathbf{T}^n_q), \quad \mathbf{T}^n_{\tilde{d}} = \mathbf{T}^n_y - \mathbf{T}^n_q - \mathbf{T}^n_G\mathbf{T}^n_u, \tag{6}$$

From Lemma 1, there exists $\Delta \in \mathbf{\Delta}$ such that $\mathbf{q} = \Delta\mathbf{z}$ if and only if $(\mathbf{T}^n_z)^{\mathrm{T}}\mathbf{T}^n_z \geqslant \gamma^{-2}(\mathbf{T}^n_q)^{\mathrm{T}}\mathbf{T}^n_q$. Combining this inequality with (6) and using Schur complements, gives the first LMI in (5). The second LMI just restates $\|\tilde{\mathbf{d}}\|_2 \leqslant \tilde{\eta}^2$. □

Remark 1. From the results above it follows that finding minimum $\|\mathbf{f}\|$ such that (3) holds reduces to a convex LMI minimization problem.

3.3. A general risk adjusted relaxation

In this section we propose a risk-adjusted relaxation of Problem 2 that has polynomial, rather than exponential, computational complexity growth with the problem data (Tempo, Bai, & Dabbene, 1996). In addition, this approach can handle arbitrary uncertainty structures. The main idea of the method is to uniformly sample the set of admissible uncertainties $\mathbf{\Delta}_i$, in an attempt to find at least one element $\tilde{\Delta}_o \in \mathbf{\Delta}_o$ and r pairs $\{\tilde{\Delta}_i, \tilde{f}_i\} \in \mathbf{\Delta}_i \times [0,1]$, $i = 1, \ldots, r$ so that the model $G_o(\lambda, \tilde{\Delta}_o) + \sum f_i G_i(\lambda, \tilde{\Delta}_i)$ together with an admissible noise $\tilde{\mathbf{d}}$, $\|\tilde{\mathbf{d}}\|_2 \leqslant \eta$ can explain the experimental data $\mathbf{y}$. As we show next, this removes the interpolation constraint that renders the problem non-convex in $(\mathbf{f}, \mathbf{d}, \Delta_i)$.

Lemma 2. *For fixed* Δ_i, $i = 0, \ldots, r$, *Problem* 2 *is equivalent to the following LMI optimization problem*:

$$\min \quad \alpha \tag{7}$$
$$\text{s.t.} \quad \begin{bmatrix} \alpha & \mathbf{f}^{\mathrm{T}} \\ \mathbf{f} & \mathbf{I} \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} \eta^2 & \mathbf{X}^{\mathrm{T}} \\ \mathbf{X} & \mathbf{I} \end{bmatrix} \geqslant 0,$$
$$\mathbf{X} = \mathscr{Y} - \left[T_{G_{0,\Delta_o}} + \sum f_i T_{G_{i,\Delta_i}}\right]\mathscr{U}, \tag{8}$$

where $T_{G_{i,\Delta_i}}$ *denotes the Toeplitz matrix associated with the impulse response of* $G_i(\lambda, \Delta_i)$, $\mathscr{U} = [\mathbf{u}^{\mathrm{T}}_o, \ldots, \mathbf{u}^{\mathrm{T}}_{n-1}]^{\mathrm{T}}$ *and* $\mathscr{Y} = [\mathbf{y}^{\mathrm{T}}_o, \ldots, \mathbf{y}^{\mathrm{T}}_{n-1}]^{\mathrm{T}}$.

Proof. Follows from (2) by applying a Schur complement argument to the inequalities $\alpha \geqslant \mathbf{f}^{\mathrm{T}}\mathbf{f}$, and $\eta^2 \geqslant \sum_{i=0}^{n-1}\mathbf{d}^{\mathrm{T}}_i\mathbf{d}_i = \mathbf{X}^{\mathrm{T}}\mathbf{X}$. □

The main difficulty with the approach outlined above is that the sets $\mathbf{\Delta}_i$ are *infinite dimensional*. However, since Δ_i are causal operators *only their first n Markov parameters* affect the output $\mathbf{y}$. Thus, rather than having to sample $\mathscr{BH}_\infty$, we only need to (i) sample the set $\mathscr{BH}^n_\infty$, e.g. using Algorithm 1, and (ii) combine the samples.[1] This observation leads to the following robust FDI algorithm:

Algorithm 2. *Given* $G_i(\lambda, \Delta_i)$ *and n output measurements* $\{\mathbf{y}\}_{i=0}^{n-1}$, *choose* N_1 *and generate* N_s *samples* $\{\Delta^j_i(\lambda)\}_{j=1}^{N_s}$ *from the set* $\mathscr{BH}^r_i$ *using Algorithm* 1.

0. *Set* $f_{\min} = \infty$.
1. *For each* Δ^j_i, *solve the following convex problem in* $\mathbf{f}$:

$$\min \quad \|\mathbf{f}\|$$
$$\text{s.t.} \quad \begin{bmatrix} \eta^2\mathbf{I} & \mathbf{X}^{\mathrm{T}} \\ \mathbf{X} & \mathbf{I} \end{bmatrix} \geqslant 0 \ \textit{where}$$
$$\mathbf{X} = \mathscr{Y} - \left[T_{G_{0,\Delta^j_o}} + \sum f_i T_{G_{i,\Delta^j_i}}\right]\mathscr{U}. \tag{9}$$

2. *If* $\|\mathbf{f}\| < \|\mathbf{f}_{\min}\|$ *set* $\mathbf{f}_{\min} = \mathbf{f}$.
3. *Set* $j = j + 1$. *If* $j \leqslant N_s$ *go back to step* 1.

Remark 2. Let (ε, ν) be two constants in $(0, 1)$, and, for a fixed Δ_i, denote by $f(\Delta_i)_{\min}$ the minimum norm solution to the LMIs (9). Theorem 3.1 in Tempo et al. (1996) shows that if N_1 in Algorithm 1 is chosen to satisfy

$$N_1 \geqslant \frac{\ln(1/\nu)}{\ln(1/(1-\varepsilon))}, \tag{10}$$

then $Prob\{Prob[\|\mathbf{f}(\Delta_i)_{\min}\|_2 < \|\mathbf{f}^{N_1}_{\min}\|_2] \leqslant \nu\} \geqslant (1-\delta)$, where $\mathbf{f}^{N_1}_{\min}$ denotes the solution found by Algorithm 2. Roughly speaking, with confidence $1 - \nu$, the algorithm will find, with probability $1 - \varepsilon$, the solution to Problem 2.

Thus, by introducing an (arbitrarily small) risk of a false alarm, we can substantially alleviate the computational com-

[1] In the case of structured uncertainty, the same construction can be used block-wise.

plexity entailed in robustly detecting faults in plants subject to structured uncertainty and measurement noise. In addition, it can be argued that a purely deterministic approach to FDI could be potentially overly optimistic, since the system will be deemed to be operating under no-fault conditions even if there exist a *single* combination of uncertainty and noise such that the corresponding $\|\mathbf{f}\|_2=0$. On the other hand, in such cases the approach proposed here will indicate, (with probability close to 1) the existence of a fault.

4. Illustrative examples

Example 1. Consider the following system subject to multiplicative uncertainty:

$$y = (I + \Delta)\left(G_0 + \sum_{i=1}^{3} f_i G_i\right) u + d, \tag{11}$$

where

$$G_0 = \frac{s^4 - 4.75s^3 - 2.48s^2 - 1.19s - 0.56}{s^4 + 1.92s^3 + 1.61s^2 + 0.83s + 0.16},$$

$$G_1 = \frac{5.07s^4 + 3.91s^2 + 0.94}{s^4 + 2.55s^3 + 3.76s^2 + 4.16s + 3.18},$$

$$G_2 = \frac{31.75s^3 + 1.8s}{s^4 + 2.55s^3 + 3.76s^2 + 4.16s + 3.18},$$

$$G_3 = \frac{75.75s^2 + 65}{s^4 + 2.55s^3 + 3.76s^2 + 4.16s + 3.18}$$

and assume that the available *a priori* information is $\|\Delta\|_\infty \leqslant 0.3$ and $\|d\|_2 \leqslant 0.94$.[2] Here, to facilitate comparisons with existing approaches, we have formulated the problem in the continuous time domain and we will assume parametric uncertainty (e.g. constant Δ). Thus, in order to apply our techniques, a discrete-time model of the system above was obtained by using samplers and zero order holds with a sampling time of 0.1 s.

The second and third columns in Table 1 show the results of applying the proposed relaxations to several faults. The experimental data consisted of $n = 20$ samples of the step response, corrupted by noise d, with $\|d\|_2=0.84$, of the model (11) corresponding to $\Delta=0.28$. For the risk-adjusted relaxation the value $N_1 = 250$ was used in Algorithm 1, which guarantees, with confidence 0.99, a probability of 0.98 of finding the minimum $\|\mathbf{f}\|_2$ that explains the experimental data, resulting on a computation time of 42 s using Matlab's LMI toolbox on a 1.7 GHz Xeon processor. For comparison, the deterministic relaxation required 4 s. Both methods successfully identified and isolated faults in all cases, with the risk-adjusted relaxation slightly outperforming the deterministic one, due to the moderately large uncertainty level.

Next, we compare the proposed approaches against the probabilistic residual based method proposed in Ding et al. (2003). To this effect, consider the state space realizations $G_i = (A_i, B_i, C_i, D_i)$, $i = 0, \ldots, 3$, and note that for the case of constant Δ, the model (11) can be written as

$$\begin{aligned} \dot{x} &= (A + \Delta A + \Delta A_F)x + (B + \Delta B + \Delta B_F)u, \\ y &= (C + \Delta C + \Delta C_F)x + (D + \Delta D + \Delta D_F)u + d, \end{aligned} \tag{12}$$

where

$$A = \begin{bmatrix} A_o & 0 \\ 0 & 0 \end{bmatrix}, \quad B = [\,B_o^{\mathrm{T}} \quad 0\,]^{\mathrm{T}}, \quad C = [\,C_o \quad 0\,], \quad D = D_o,$$

$$\Delta A = 0, \quad \Delta B = 0, \quad \Delta C = \delta \cdot C, \quad \Delta D = \delta \cdot D, \quad |\delta| \leqslant 0.1,$$

$$\Delta A_F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_1 & & \\ \vdots & & \ddots & \\ 0 & & & A_3 \end{bmatrix}, \quad \Delta B_F = [\,0 \quad B_1^{\mathrm{T}} \quad \ldots \quad B_3^{\mathrm{T}}\,]^{\mathrm{T}},$$

$$\Delta C_F = (1 + \delta)[\,0 \quad f_1 C_1 \quad \ldots \quad f_3 C_3\,],$$

$$\Delta D_F = (1 + \delta) \sum f_i D_i. \tag{13}$$

In its simplest form, the approach proposed by Ding et al. (2003) uses an observer to generate a residual r:

$$\dot{\hat{x}} = (A - LC)\hat{x} + (B - LD)u,$$

$$r = y - C\hat{x} - Du. \tag{14}$$

A fault is deemed to have occurred whenever $\|r\|_2 > J_{\mathrm{th}}$, where the threshold J_{th} is selected so that the probability of false alarms, that is $P(\|r|_2 > J_{\mathrm{th}}|$ no fault is present) $\leqslant \alpha$, where α is given. Using a gain $L = 0$ (corresponding to both the optimal $\mathscr{H}_2$ and $\mathscr{H}_\infty$ filters) and setting the false alarm rate (FAR) to 2%, yields $J_{\mathrm{th}} = 17.45$. As shown in Table 1, while for larger values of $\|\mathbf{f}\|$ all methods have comparable performance, both invalidation based approaches outperform the residual based one in the case of smaller $\|\mathbf{f}\|$. Note also that the former approaches were able to accurately estimate the fault components, albeit at the cost of increased on-line computational time.

Example 2. Consider the problem of FDI in the yaw damper system of a jet transport. The transfer function from the rudder and aileron deflections to the yaw rate and bank angle can be represented by a model of the form (11) with (see Shim & Sznaier, 2003) $\|\Delta\|_\infty \leqslant 0.3$, $\|d\|_2 \leqslant 2.6$, and $G_i = N_i/D_i$, $i = 0, \ldots, 3$,

$$N_i(z) = \begin{pmatrix} N_{11}^i & N_{12}^i \\ N_{21}^i & N_{22}^i \end{pmatrix},$$

where G_o denotes the nominal plant, and where

$$D_o(z) = z^4 - 3.81z^3 + 5.45z^2 - 3.46z + 0.83,$$

$$D_i(z) = z^8 - 7.7z^7 + 26.2z^6 - 50.8z^5 + 61.4z^4 - 47.6z^3 + 23.1z^2 - 6.4z + 0.78, \quad i = 1, 2, 3,$$

$$N_{11}^o = -0.44z^3 + 1.3z^2 - 1.3z + 0.42,$$

$$N_{12}^o = 0.11z^3 - 0.34z^2 + 0.33z - 0.11,$$

[2] This noise level corresponds to 10% of the energy of the step response of the plant.

Table 1
Example 1. Fault estimates corresponding to the case $\|\Delta\|_\infty \leqslant 0.3$ and 10% noise level

Real fault			Risk adjusted ($N_1 = 250$, $prob = 98\%$)			Deterministic			Observer-based ($FAR = 2\%$)
0.0	0.0	0.0	10^{-5}*[0.10	0.11	0.11]	10^{-5}*[0.11	0.12	0.12]	No Fault
1.0	0.0	0.0	0.95	0.00	0.00	0.92	0.00	0.00	Fault
0.0	1.0	0.0	0.04	0.95	0.00	0.05	0.94	0.00	Fault
0.0	0.0	1.0	0.00	0.00	0.97	0.00	0.00	0.96	Fault
0.45	0.21	0.56	0.36	0.21	0.54	0.33	0.21	0.53	Fault
0.24	0.10	0.05	0.16	0.10	0.04	0.13	0.10	0.03	No Fault
0.20	0.10	0.05	0.12	0.10	0.04	0.10	0.09	0.03	No Fault
0.10	0.05	0.1	0.03	0.04	0.08	0.02	0.04	0.07	No Fault

Table 2
Risk adjusted versus deterministic relaxation for Example 2

Real fault			Risk adjusted estimate ($N_1 = 250$)			Deterministic relaxation		
0.0	0.0	0.0	10^{-6}*[0.7814	0.8356	0.8715]	10^{-3}*[0.4572	0.4592	0.4514]
1.0	0.0	0.0	0.7313	0.0194	0.0222	0.6336	0.0797	0.0194
0.0	1.0	0.0	0.0400	0.5868	0.0261	0.0755	0.5118	0.0406
0.0	0.0	1.0	0.0215	0.0000	0.6567	0.0447	0.0000	0.5647
0.85	0.78	0.001	0.6694	0.4318	0.0500	0.6113	0.4183	0.0402
0.45	0.21	0.56	0.3248	0.1749	0.3424	0.2966	0.0876	0.3058

$$N_{21}^o = 10^{-3} * (5z^3 - 7z^2 - 4.3z + 5.3),$$

$$N_{22}^o = 10^{-2} * (5.3z^3 - 4.6z^2 - 5.1z + 4.5),$$

$$N_{11}^1 = -0.28z^7 - 1.42z^6 + 2.64z^5 - 1.73z^4 - 0.93z^3 + 2.16z^2 - 1.26z + 0.26,$$

$$N_{12}^1 = -0.07z^7 + 0.37z^6 - 0.68z^5 + 0.45z^4 + 0.24z^3 - 0.56z^2 + 0.32z - 0.07,$$

$$N_{21}^1 = 10^{-5} * (7.1z^7 + 46.6z^6 - 181.8z^5 + 138.2z^4 + 106.6z^3 - 169.6z^2 + 46.9z + 6.1),$$

$$N_{22}^1 = -10^{-4} * (0.2z^7 + 1.4z^6 - 5.8z^5 + 4.4z^4 + 3.4z^3 - 5.4z^2 + 1.5z + 0.2),$$

$$N_{11}^2 = 10^{-2} * (z^7 - 1.2z^6 - 7.8z^5 + 22.9z^4 - 23.5z^3 + 8.9z^2 + 0.6z - 0.9),$$

$$N_{12}^2 = -10^{-3} * (1.9z^7 - 2.4z^6 - 14.9z^5 + 43.7z^4 - 44.8z^3 + 16.9z^2 + 1.2z - 1.7),$$

$$N_{21}^2 = -10^{-3} * (2.3z^7 - 2.9z^6 - 18z^5 + 53z^4 - 54z^3 + 21z^2 + 1.5z + 2),$$

$$N_{22}^2 = 10^{-4} * (6z^7 - 7z^6 - 47z^5 + 137z^4 - 141z^3 + 53z^2 + 4z - 5),$$

$$N_{11}^3 = 10^{-6} * (2.4z^7 + 53z^6 - 27z^5 - 206z^4 + 215z^3 + 13z^2 - 49z - 2),$$

$$N_{12}^3 = -10^{-6} * (z^7 + 24z^6 - 12z^5 - 93z^4 + 97z^3 + 6z^2 - 22z - 0.9),$$

$$N_{21}^3 = 10^{-5} * (1.5z^7 + 32z^6 - 16z^5 - 126z^4 + 132z^3 + 8.2z^2 - 30z - 1.2),$$

$$N_{22}^3 = -10^{-5} * (0.36z^7 + 7.8z^6 - 4.0z^5 - 30z^4 + 32z^3 + 2.0z^2 - 7.2z - 0.3).$$

The experimental data consist of 20 samples of the response of $(I + \tilde{\Delta})G_f$ [3] to a 0.69 Hz square wave with amplitude ±1, where $\tilde{\Delta}$ is given by[4]

$$\tilde{\Delta} = \frac{0.018}{D_\Delta}\begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix},$$
$$D_\Delta = z^4 + 1.87z^3 + 1.27z^2 + 0.37z + 0.04,$$

$$\Delta_{11} = 1.9z^4 + 2.5z^3 - 0.24z^2 - 1.04z - 0.25,$$
$$\Delta_{12} = 0.5z^4 + 0.8z^3 + 0.25z^2 - 0.09z - 0.03,$$
$$\Delta_{21} = 2.9z^4 + 3.0z^3 - 1.66z^2 - 2.3z - 0.51,$$
$$\Delta_{22} = 3z^4 + 3.5z^3 - 1.12z^2 - 2.1z - 0.47. \quad (15)$$

[3] Here G_f denotes the transfer function of the failure mode under consideration.

[4] This corresponds to a randomly generated uncertainty with $\|\tilde{\Delta}\|_\infty \leqslant 0.297$.

As shown in Table 2, both relaxations were able to provide good estimates of the fault indicators. Similar results were obtained with lower uncertainty and noise levels. In this case, as expected, the performance of both methods becomes closer with decreasing uncertainty and noise levels.

5. Conclusion

This paper considered the problem of robust fault detection and isolation (FDI) for systems described by a parameterized fault model, subject to arbitrary dynamic uncertainty. In general this setup leads to non-convex, NP hard problems. To remove this limitation, we propose two convex relaxations: one deterministic and one stochastic. The deterministic relaxation reduces the problem to a conventional LMI optimization, but is limited to the case of unstructured multiplicative (or additive) uncertainty. On the other hand, the risk-adjusted relaxation can, in return for an (arbitrarily small) probability of a false alarm, handle completely general uncertainty structures. Further, this approach also entails a substantial reduction of the computational complexity of the problem. Since the number of samples needed for reliable fault estimation is relatively small, it is feasible to generate and store these samples off-line, leading to further reduction of the computational complexity of the problem that needs to be solved on-line. Both relaxations have comparable performance for relatively low uncertainty levels, with the stochastic relaxation outperforming the convex one as the uncertainty level increases.

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