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Is set modeling of white noise a good tool for robust $\mathcal{H}_2$ analysis?

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Abstract

Recently a new deterministic characterization of the $\mathcal{H}_2$ norm has been proposed, using a new norm ($\|.\|_{W_\eta}$), based on (approximate) set membership modeling of white noise. The main result in [10, 11, 12, 13] shows that under mild conditions, for a *fixed* system the gap between the $\mathcal{H}_2$ and W_η norms can be made arbitrarily small. Motivated by these results it has been argued that the $\|.\|_{W_\eta}$ norm provides a useful tool for analyzing robust $\mathcal{H}_2$ controllers, specially since in this context LMI based necessary and sufficient conditions for robust performance are available. Unfortunately, as we show here with an example involving a very simple plant, the worst case $\|.\|_{W_\eta^m}$ norm can conservative by at least a factor of $\sqrt{m}$ (where m denotes the dimension of the exogenous signal) for the original robust $\mathcal{H}_2$ problem. Thus, at this point the problem of finding non-conservative bounds on the worst $\mathcal{H}_2$ norm under LTI or slowly-varying LTV perturbations still remains open.

1 Introduction

A large number of control problems of practical importance involve designing a controller capable of stabilizing a given linear time invariant system while minimizing the worst case response to some exogenous disturbances. Depending on the choice of models for the input signals and on the criteria used to assess performance, this prototype problem leads to different mathematical formulations. The case where the exogenous disturbances w belong to the set of signals with spectral density bounded by one and the objective is to minimize the worst-case "size" of the output z measured using the power seminorm* leads to the well known $\mathcal{H}_2$ control problem.

$\mathcal{H}_2$ control is appealing since there is a well established connection between the performance index being optimized and performance requirements encountered in practical situations. Moreover, the resulting controllers are easily found by solving two Riccati equations, and in the state-feedback case exhibit good robustness properties ([1]). However, as the classical paper [4] established, these margins vanish in the output feedback case, where infinitesimal model perturbations can destabilize the closed-loop system.

Following this paper, several attempts were made to incorporate robustness into the $\mathcal{H}_2$ framework, at least for the case of minimum phase (or mildly non-minimum phase) plants [16, 19]. More recently these efforts led to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem [2, 5, 20, 7, 18, 15, 3], where the resulting controller guarantees optimal performance for the *nominal* controller and stability against LTI dynamic uncertainty. While these results represent significant progress towards obtaining robust $\mathcal{H}_2$ controllers, they suffer from the fact that performance is only guaranteed for the nominal plant. Moreover, the resulting controllers have potentially high order (in fact, the optimal $\mathcal{H}_2/\mathcal{H}_\infty$ controller is infinite dimensional [9]).

Robust $\mathcal{H}_2$ performance was analyzed in [17] where bounds on the worst-case performance

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*An alternative stochastic interpretation can be given by considering the input signal w to be white Gaussian noise with unit covariance and having as design objective the minimization of the RMS value of the output, $\lim_{t\to\infty} \mathcal{E}[z^T(t)z(t)]$, where $\mathcal{E}$ denotes expectation.

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were obtained. These bounds are related to the auxiliary problem introduced in [7] and lead to tractable synthesis problems. However, they are obtained assuming non–causal, non–linear time varying model uncertainty. Thus, they are potentially conservative for the case of causal, LTI perturbations.

Recently [6] an upper bound on the worst case $\mathcal{H}_2$ norm under passive uncertainty has been proposed. This upper bound is obtained using an impulse response based interpretation of the $\mathcal{H}_2$ norm and dynamic (non–causal) stability multipliers. This approach is appealing since it takes into account, to some extent, causality. However, in order to obtain tractable problems, these multipliers must be restricted to the span of some basis, selected a–priori. Moreover, the complexity of this basis is limited by the fact that the computational complexity of the resulting LMI problem grows roughly as the 10-th power of the state dimension [14].

Alternatively, a new research line has emerged [10] based upon (approximate) set membership modeling of white signals. As shown in [10, 11, 12, 13], for a ***fixed, given plant*** this alternative formulation can capture the $\mathcal{H}_2$ norm with arbitrary precision. Motivated by these results it has been argued [11, 12, 13] that this approach provides a useful tool for analyzing robust $\mathcal{H}_2$ controllers, specially since in this context LMI based necessary and sufficient conditions for robust performance are available. Moreover, these conditions are no more complex than comparable $\mathcal{H}_\infty$ conditions for the same problem. Unfortunately, as we show here with an example, these conditions can be conservative by at least a factor of $\sqrt{m}$, where m denotes the dimension of the exogenous input, even for very simple plants. Thus, at the present time the problem of robust $\mathcal{H}_2$ analysis for general MIMO system still remains open.

2 Preliminaries

2.1 Notation and Definitions

$\mathcal{L}_\infty$ denotes the Lebesgue space of complex valued matrix functions which are essentially bounded on the unit circle, equipped with the norm $\|G(z)\|_\infty \doteq ess\sup_{|z|=1} \overline{\sigma}(G(z))$, where $\overline{\sigma}$ denotes the largest singular value. By $\mathcal{H}_\infty$ we denote the subspace of functions in $\mathcal{L}_\infty$ with a bounded analytic continuation outside the unit disk. The norm on $\mathcal{H}_\infty$ is defined by $\|G(z)\|_\infty \doteq ess\sup_{|z|>1} \overline{\sigma}(G(z))$. By $\mathcal{H}_2$ we denote the space of complex valued matrix functions $G(z)$ with analytic continuation outside the unit disk and square integrable there, equipped with the usual $\mathcal{H}_2$ norm:

$$\|G\|_2^2 \doteq \sup_{\gamma>1} \frac{1}{2\pi} \oint_{|z|=\gamma} |G(z)|_F^2 \frac{dz}{z},$$

where $\|.\|_F$ denotes the Frobenious norm.

Given two matrices M and Δ of compatible dimensions we denote by $\Delta \star M$ the upper LFT $\mathcal{F}_u(M, \Delta)$, i.e:

$$\Delta \star M = M_{22} + M_{21}\Delta\left(I - M_{11}\Delta\right)^{-1} M_{12}$$

Let $\mathcal{L}(\ell^2)$ denote the set of linear bounded operators in ℓ^2. In the sequel we will consider the following set of structured bounded operators in $\mathcal{L}(\ell^2)$:

$$\mathcal{B}\mathbf{\Delta} = \{\Delta \in \mathcal{L}(\ell^2) \colon \Delta = \text{diag}[\delta_1 I_{r_1}, \ldots, \delta_L I r_L, \Delta_{L+1}, \ldots, \Delta_{L+F}] \colon \|\Delta\|_{\ell^2\to\ell^2} \le 1\}$$

The subsets of $\mathcal{B}\mathbf{\Delta}$ formed by Linear Time Invariant and (arbitrarily) Slowly Linear Time Varying operators will be denoted by $\mathcal{B}\mathbf{\Delta}^{\text{LTI}}$ and $\mathcal{B}\mathbf{\Delta}^{\text{SLTV}}$ respectively. Finally, we will also make use of the following set of scaling matrices which commute with the elements in $\mathcal{B}\mathbf{\Delta}$:

$$\mathbf{X} = \text{diag}\left[X_1, \ldots, X_L, x_{L+1} I_{m_1}, \ldots, x_{L+F} I_{m_F}\right]$$

Definition 1 ((Robust $\mathcal{H}_2$ performance)) *The uncertain system (M, Δ) with input u in l_2^m has robust $\mathcal{H}_2$ performance against LTI perturbations if it is robustly stable and*

$$\sup_{\Delta\in\mathcal{B}\mathbf{\Delta}^{\text{LTI}}} \|\Delta \star M\|_2 \le 1 \tag{1}$$

Definition 2 ([8]) *A real function $f : [a, b] \to \mathcal{R}$ is said to be of bounded variation if there exists a constant K such that for any partition of $[a, b]$*

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \le K$$

The total variation of f, denoted as $TV(f)$ is defined as

$$TV(f) = \sup \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

where the supremum is taken over all partitions of $[a, b]$.

The approach proposed in [10] is based upon approximating white noise by a subset of ℓ^2 composed by "approximately" white signals, defined as follows:

Definition 3 ([10]) *Given $\eta > 0$, the set of "white up to accuracy η" signals is given by:*

$$W_\eta^m = \left\{ f \in \ell^2 : \left\| \int_0^s f(\omega) f(\omega)^* \frac{d\omega}{2\pi} - \frac{s}{2\pi} \frac{\|f\|^2}{m} I_m \right\|_\infty < \eta \right\} \quad (2)$$

where the $\|.\|_\infty$ norm denotes the maximum across the coordinates of the supremum norm.

Thus this set includes all signals in ℓ^2 such that their cumulative spectrum only deviates a small amount from the spectrum of a "true" white signal. Given an ℓ^2 stable system (not necessarily LTI) one can look then at the worst case value of the output (in the ℓ^2 sense) in response to signals in W_η^m and use this to define an induced norm as follows:

Definition 4 ([10]) *Given an ℓ^2 stable operator H, its W_η^m norm is defined as:*

$$\|H\|_{W_\eta^m} := \sup \left\{ \|Hf\| : f \in W_\eta^m, \frac{1}{m}\|f\|_2^2 \le 1 \right\} \quad (3)$$

Since the set W_η^m is formed by signal that are close to being white, one can expect that $\|H\|_{W_\eta^m}$ is close to $\|H\|_2$ in some sense. The following theorem shows that this is indeed the case as long as H is a fixed, given system.

Theorem 1 ([10]) *For an ℓ^2 stable LTI system H the following inequality holds:*

$$\|H\|_2^2 \le \|H\|_{W_\eta^m}^2 \le \|H\|_2^2 + \eta TV(|H|^2)$$

Corollary 1 *For a given, fixed system H, $\|H\|_{W_\eta^m} \stackrel{\eta \to 0+}{\longrightarrow} \|H\|_2$*

2.2 Robust $\|.\|_{W_\eta^m}$ Performance

Consider now the problem of assessing the worst case performance (in the W_η^m sense) of the interconnection of a nominal LTI system M and bounded structured uncertainty.

Definition 5 *(Robust $\|.\|_{W_\eta^m}$ performance) The uncertain system (M, Δ) with input u in l_2^m has robust $\|.\|_{W_\eta^m}$ performance if it is robustly stable, and there exists $\eta > 0$ such that*

$$\sup_{\Delta \in \mathcal{B}\mathbf{\Delta}^{SLTV}} \|\Delta \star M\|_{W_\eta^m} \le 1 \quad (4)$$

The following result shows that this definition leads to a tractable necessary and sufficient condition that can be checked numerically to any desired degree of accuracy.

Condition 1 ([10]) *The interconnection $\Delta \star M$ achieves robust W_η^m performance against $\Delta \in \mathcal{B}\mathbf{\Delta}^{SLTV}$ (not necessarily causal) if and only if there exists $X(\omega) \in \mathbf{X}$, and a matrix function $Y(\omega) = Y^*(\omega) \in \mathbf{C}^{m \times m}$, such that*

$$M(e^{j\omega})^* \begin{bmatrix} X(\omega) & 0 \\ 0 & I \end{bmatrix} M(e^{j\omega}) - \begin{bmatrix} X(\omega) & 0 \\ 0 & Y(\omega) \end{bmatrix} < 0 \quad (5)$$

holds for all $\omega \in [0, 2\pi]$, and

$$\int_0^{2\pi} trace(Y(\omega)) \frac{d\omega}{2\pi} < 1 \quad (6)$$

Clearly, from Corollary 1, having robust $\|.\|_{W_\eta^m}$ performance in the sense of Definition 5 is a **sufficient** condition for achieving robust $\mathcal{H}_2$ performance in the usual sense. Moreover, motivated by Theorem 1 one may think that, as argued in [10], this is also necessary. However, as we show in the sequel, in the case of MIMO systems this condition is **only** sufficient and potentially conservative by at least a factor of $\sqrt{m}$, where m is the dimension of the exogenous ℓ^2 disturbance. Thus, it follows that robust $\mathcal{H}_2$ performance (in the usual sense) and robust W_η^m performance are different problems, and solving the latter does not necessarily solves the former.

3 A Simple Counterexample

Consider the following plant:

$$\begin{aligned} \begin{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ z \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \end{bmatrix} \\ &= M \begin{bmatrix} w \\ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \end{bmatrix} \\ \Delta &= [\Delta_1 \quad \Delta_2] \in \mathcal{B}\mathbf{\Delta}^{LTI} \\ z &= (\Delta \star M) \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \end{aligned} \quad (7)$$

Clearly for this plant we have that

$$\sup_{\Delta \in \mathcal{B}\mathbf{\Delta}^{\mathrm{LTI}}} \|\Delta \star M\|_2 = 1 \qquad (8)$$

Moreover, expanding the set of Δ's to include non–causal perturbation does not increase this worst–case norm. On the other hand, consider now the following perturbation $\Delta^n \doteq [\Delta_1^n \quad \Delta_2^n]$:

$$\Delta_1^n(j\omega) \doteq \begin{cases} 1 & \text{if } \omega \in \left[\frac{k\Pi}{2n}, \frac{(k+1)\Pi}{2n}\right),\ k \text{ even}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\Delta_2^n(j\omega) \doteq \begin{cases} 1 & \text{if } \omega \in \left[\frac{k\Pi}{2n}, \frac{(k+1)\Pi}{2n}\right),\ k \text{ odd}, \\ 0 & \text{otherwise.} \end{cases} \qquad (9)$$

By construction Δ^n is such that $\|\Delta^n\|_\infty = 1$ for all n.

Finally, consider the following input:

$$f^n = \begin{bmatrix} f_1^n \\ f_2^n \end{bmatrix}$$

$$f_1^n(j\omega) \doteq \begin{cases} \sqrt{2} & \text{if } \omega \in \left[\frac{k\Pi}{2n}, \frac{(k+1)\Pi}{2n}\right),\ k \text{ even}, \\ 0 & \text{otherwise.} \end{cases}$$

$$f_2^n(j\omega) \doteq \begin{cases} \sqrt{2} & \text{if } \omega \in \left[\frac{k\Pi}{2n}, \frac{(k+1)\Pi}{2n}\right),\ k \text{ odd}, \\ 0 & \text{otherwise.} \end{cases} \qquad (10)$$

By construction f^n is an ℓ^2 signal with $\|f\|_2^2 = 2$. Moreover, as we show next, given any $\eta > 0$, f^n is in the set W_η^2 of signals "white up to η" for an appropriate choice of n. To this effect consider the following quantity:

$$F_\eta(s) = \frac{1}{2\pi} \int_0^s \left[f^n(j\omega)(f^n)^*(j\omega) - \frac{1}{2} I_2 \|f^n\|_2^2 \right] d\omega \qquad (11)$$

Using the fact that

$$f^n(j\omega)(f^n)^*(j\omega) - \tfrac{1}{2} I_2 \|f^n\|_2^2 =$$

$$\begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } \omega \in \left[\frac{k\Pi}{2n}, \frac{(k+1)\Pi}{2n}\right),\ k \text{ even} \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \omega \in \left[\frac{k\Pi}{2n}, \frac{(k+1)\Pi}{2n}\right),\ k \text{ odd} \end{cases} \qquad (12)$$

it can be easily shown that

$$\sup_{s \in [0, 2\pi)} \|F(s)\|_\infty = \frac{1}{4n} \qquad (13)$$

It follows that $f^n \in W_\eta^2$ for all $n > \frac{1}{4\eta}$. Finally, the output z^n corresponding to the uncertainty Δ^n and signal f^n satisfies:

$$\|z^n\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} \left[(\Delta_1^n f_1^n)^2 + (\Delta_2^n f_2^n)^2 \right] d\omega = 2 \qquad (14)$$

Hence in this case we have that, **for any** $\eta > 0$:

$$\begin{aligned} 1 &= \sup_{\Delta \in \mathcal{B}\mathbf{\Delta}^{\mathrm{LTI}}} \|\Delta \star M\|_2 < \sqrt{2} \\ &\leq \sup_{\Delta \in \mathcal{B}\mathbf{\Delta}^{\mathrm{LTI}}} \left\{ \sup_{f \in W_\eta^2} \|(\Delta \star M) f\|_2 \right\} \\ &= \sup_{\Delta \in \mathcal{B}\mathbf{\Delta}^{\mathrm{LTI}}} \|\Delta \star M\|_{W_\eta^2} \end{aligned} \qquad (15)$$

This last equation shows that in the case of MISO systems having m inputs the $\|.\|_{W_\eta^m}$ norm may be conservative by at least a factor of $\sqrt{m}$ and thus Definition 5 does not coincide in general with the standard definition of Robust $\mathcal{H}_2$ performance.

Note that the perturbation Δ^n is not square. An example with square perturbations and having exactly the same gap can be obtained by simply setting:

$$M_s = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\Delta_s = \begin{bmatrix} \Delta_1 & \Delta_2 \\ \Delta_3 & \Delta_4 \end{bmatrix} \in \mathcal{B}\mathbf{\Delta}^{\mathrm{LTI}} \qquad (16)$$

Clearly $\Delta_s \star M_s = [\Delta_1 \quad \Delta_2]$ and the example reduces to the previous one.

Further insight into the conservatism of the $\|.\|_{W_\eta^m}$ can be gained by using condition 1. Since in order to apply this condition M should be square, we add a row of zeros to M_s, yielding:

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (17)$$

Obviously, this does not change any of the features discussed previously. Since Δ is a full block,

$$X = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix},$$

and Equation (5) becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{bmatrix} - \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y_1 & y_2 \\ 0 & 0 & y_3 & y_4 \end{bmatrix} < 0. \qquad (18)$$

Equation (18) implies the following inequalities

$$\begin{aligned} 1 - x &< 0 \\ x - y_1 &< 0 \\ x - y_4 &< 0, \end{aligned}$$

from which it follows that $y_1 > 1$, $y_4 > 1$ and thus trace$(Y) > 2$. Since Y does not depend on ω,

$$\int_0^{2\pi} trace(Y(\omega))\frac{d\omega}{2\pi} > 2$$

and the inequality can be achieved up to arbitrarily small ϵ. In this case the worst case W_η^m norm with respect to slowly time varying Δ's coincides with the worst case W_η^m norm with respect to LTI Δ's and is $\sqrt{2}$ times bigger than the worst case 2-norm with respect to LTI Δ's. The conservativeness of condition 1 does not come from using slowly time varying perturbations, but from the definition of the worst case W_η^m norm itself.

A few comments are in order before closing this section. Note that the worst–case model uncertainty Δ^n as defined in (9) is non-causal. However, it is easily seen that the only property that we exploited in the example is that $|\Delta_i^n(j\omega)| = 1$. Therefore causal perturbations Δ_ϵ^n can be constructed by "rounding" the corners of $|\Delta^n|$ (approximating it for instance using a Butterworth filter) so that the resulting Δ_ϵ^n has:

$$\int_0^{2\pi} \left||\Delta_i^n(j\omega)| - |\Delta_{i,\epsilon}^n(j\omega)|\right| d\omega \leq \epsilon, \; i = 1,2 \tag{19}$$

leading to causal perturbations. A similar technique yields a causal ℓ^2 worst case signal. Thus, contrary to what it was conjectured in [14] the gap (15) does not change when the model uncertainty is restricted to causal LTI operators. Note also that the worst case 2-norm of $[\Delta_1\Delta_2]$ for causal and noncausal LTI operators is:

$$\begin{aligned} & \int_0^{2\pi} \sup_{\Delta\in\mathcal{B}\mathbf{\Delta}^{LTI}} trace\left([\Delta_1(e^(j\omega)\Delta_2(e^{j\omega})]^* \right. \\ & \left. [\Delta_1(e^{j\omega})\Delta_2(e^{j\omega})]\right)\frac{d\omega}{2\pi} \\ = & \int_0^{2\pi} \sup_{\Delta\in\mathcal{B}\mathbf{\Delta}^{LTI}} trace\left([\Delta_1(e^(j\omega)\Delta_2(e^{j\omega})] \right. \\ & \left. [\Delta_1(e^{j\omega})\Delta_2(e^{j\omega})]^*\right)\frac{d\omega}{2\pi} \\ = & \int_0^{2\pi} \|[\Delta_1\Delta_2]\|_\infty \frac{d\omega}{2\pi} \\ = & \; 1 \end{aligned}$$

4 Where the problem lies

From Theorem 1, one can conclude:

$$\begin{aligned} & \sup_{\Delta\in\mathcal{B}\mathbf{\Delta}^{LTI}} \|\Delta\star M\|_2^2 \leq \\ & \sup_{\Delta\in\mathcal{B}\mathbf{\Delta}^{LTI}} \|\Delta\star M\|_{W_\eta^m}^2 \leq \\ & \sup_{\Delta\in\mathcal{B}\mathbf{\Delta}^{LTI}} (\|\Delta\star M\|_2^2 \\ & +\eta TV(|\Delta\star M|^2)) \end{aligned} \tag{20}$$

and therefore

$$\begin{aligned} & \sup_{\Delta\in\mathcal{B}\mathbf{\Delta}^{LTI}} \|\Delta\star M\|_2^2 \leq \\ & \lim_{\eta\to 0} \sup_{\Delta\in\mathcal{B}\mathbf{\Delta}^{LTI}} \|\Delta\star M\|_{W_\eta^m}^2 \\ & \leq \lim_{\eta\to 0} \sup_{\Delta\in\mathcal{B}\mathbf{\Delta}^{LTI}} (\|\Delta\star M\|_2^2 \\ & +\eta TV(|\Delta\star M|^2)) \end{aligned} \tag{21}$$

For Definition 5 to be equivalent to the standard definition of Robust $\mathcal{H}_2$ performance, the following would have to hold:

$$\begin{aligned} & \sup_{\Delta\in\mathcal{B}\mathbf{\Delta}^{LTI}} \|\Delta\star M\|_2^2 \leq \\ & \lim_{\eta\to 0} \sup_{\Delta\in\mathcal{B}\mathbf{\Delta}^{LTI}} \|\Delta\star M\|_{W_\eta^m}^2 \\ & \leq \sup_{\Delta\in\mathcal{B}\mathbf{\Delta}^{LTI}} (\|\Delta\star M\|_2^2) \end{aligned} \tag{22}$$

however, since in general

$$\lim_{\eta\to 0} \sup_{\Delta\in\mathcal{B}\mathbf{\Delta}^{LTI}} \eta TV(|\Delta\star M|^2) \neq 0 \tag{23}$$

Equation (22) does not necessarily follow from Equation (21). Indeed, as the example presented earlier in this paper shows, Equation (22) does not hold in general. By switching the lim and sup operators in Definition 5, the worst case W_η^m penalizes not only the 2-norm of the operator $\Delta\star W$, but also its total variation. This fact causes the worst case W_η^m to be achieved by Δ's that are not a good abstraction of physical uncertainty, and thus artificially inflates the worst case norm with respect to the worst case $\mathcal{H}_2$ norm.

5 Conclusions

While the $\|.\|_{W_\eta^m}$ norm provides a useful tool for analyzing the $\mathcal{H}_2$ norm of a fixed given system, the simple counterexample presented here indicates that these results cannot be used in general to assess the worst case $\mathcal{H}_2$ norm of uncertain systems, since there exists at least a $\sqrt{m}$ gap between the worst case $\mathcal{H}_2$ norm and the worst case $\|.\|_{W_\eta^m}$. Thus, necessary and sufficient conditions for robust performance in the W_η^m sense are only sufficient for robust $\mathcal{H}_2$ performance. Hence, at the present time the problem of obtaining tight bounds on the worst case $\mathcal{H}_2$ performance in the presence of LTI (or slowly LTV) uncertainty is still open.

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