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Mixed $\mathcal{H}_2/\mathcal{L}_1$ Control with Low Order Controllers: A Linear Matrix Inequality Approach. [1]

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Abstract

This paper addresses the problem of designing stabilizing controllers that minimize the $\mathcal{H}_2$ norm of a certain closed-loop transfer function while maintaining the $\mathcal{L}_1$ norm of a different transfer function below a prespecified level. This problem arises in the context of rejecting both stochastic as well as bounded persistent disturbances. Alternatively, in a robust control framework it can be thought as the problem of designing a controller that achieves good nominal $\mathcal{H}_2$ performance, while at the same time, guaranteeing stability against unmodeled dynamics with bounded induced $\mathcal{L}_\infty$ norm. The main result of this paper shows that, for the state feedback case, a suboptimal static feedback controller can be synthesized by a two stage process involving a finite-dimensional convex optimization problem and a line-search.

1. introduction

During the last decade a powerful robust control framework has been developed addressing the issues of stability and performance in the presence of norm-bound model uncertainty. Robust stability and performance are achieved by minimizing a suitably weighted norm (either $||.||_\infty$ [6, 7, 8, 15] or $||.||_1$ [2, 3, 4, 5, 12, 13]) of a closed-loop transfer function. This framework has gained wide acceptance among control engineers, since it embodies many desirable design objectives. However, it is limited by the fact that in this context, performance must be measured in the same norm used to assess stability. Clearly, a single norm is usually not enough to capture different, and often conflicting, design specifications, such as simultaneous rejection of disturbances having different characteristics (white noise, bounded energy, persistent); good tracking of classes of inputs; satisfaction of bounds on the peak values of some outputs; closed-loop bandwidth; etc.

This paper addresses the problem of designing stabilizing controllers that minimize the $\mathcal{H}_2$ norm of a certain closed-loop transfer function while maintaining the $\mathcal{L}_1$ norm of a different transfer function below a prespecified level. This problem arises in the context of rejecting both stochastic as well as bounded persistent disturbances. Alternatively, in a robust control framework it can be thought as the problem of designing a controller that achieves good nominal $\mathcal{H}_2$ performance, while at the same time, guaranteeing stability against unmodeled dynamics with bounded induced $\mathcal{L}_\infty$ norm.

Both the discrete-time mixed $\mathcal{H}_2/l^1$ problem (and its dual, mixed $l^1/\mathcal{H}_2$) can be solved by using the Youla parametrization to cast the problem into a (infinite-dimensional) constrained convex optimization form. It has been recently shown [14, 11] that for the case of a SISO plant having a single disturbance input and a single performance output, the closed loop system has finite impulse response. This property can be used to reduce the synthesis problem to a finite-dimensional quadratic optimization. However, as in the case of pure l^1 optimal control, the order of

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the controller is not bounded by the order of the plant, and could be arbitrarily high. While similar results are not yet available for the continuous–time counterpart of the problem, it is fair to assume (motivated by continuous–time $\mathcal{L}_1$ theory) that the problem is at least as hard as the discrete–time version, and that the optimal solution is likely to be non–rational, involving delay terms.

Motivated by the complexity of these controllers, in this paper we propose an alternative approach, based upon the use of upper bounds of the $\mathcal{H}_2$ and $\mathcal{L}_1$ norms given in terms of Linear Matrix Inequalities, to obtain a modified problem such that its solution is both feasible and upper–bounds the original problem. The main result of this paper shows that for the state feedback case, the optimal cost over the set of all stabilizing controllers can be achieved with static state–feedback. Moreover, this controller can be found through a two–stage procedure entailing a finite–dimensional convex optimization and a one–dimensional line search.

The paper is organized as follows: In section 2 we introduce the notation to be used and some preliminary results. In section 3 we show that, when suitably modified, the mixed $\mathcal{H}_2/\mathcal{L}_1$ problem can be reduced to a finite dimensional convex optimization and a one–dimensional line search. Finally, in section 4, we summarize our results and we indicate directions for future research.

Due to space limitations, all proofs have been omitted. They can be obtained by contacting the authors.

2. Preliminaries

2.1. Notation

R_+ denotes the set of nonnegative real numbers. $\mathcal{L}^\infty(R_+)$ denotes the space of measurable functions $f(t)$ equipped with the norm: $\|f\|_\infty = ess \sup_{R_+} |f(t)|$. $\mathcal{L}_1(R_+)$ denotes the space of Lebesgue integrable functions on R_+ equipped with the norm $\|f\|_1 \doteq \int_0^\infty |f(t)|dt < \infty$. Given a matrix Q_k with elements $q_{ij} \in \mathcal{L}_1$, representing a bounded linear operator defined by the usual convolution $y = Q * u$, its induced $\mathcal{L}_\infty$ to $\mathcal{L}_\infty$ norm is defined as:

$$\|Q\|_1 \doteq \sup_{\|u\|_\infty \leq 1} \|Q*u\|_\infty = \max_i \sum_{j=0}^{n} \|q_{ij}\|_1 < \infty$$

Given a function $f(t) \in \mathcal{L}^\infty(R_+)$, we will define, following [9], the norm $\|f(t)\|_{\infty,e} \doteq \sup_{t\geq 0}\{f'(t)f(t)\}^{1/2}$, i.e. the supremum over time of the pointwise *euclidian* norm of the vector $f(t)$. For an operator $H\colon \mathcal{L}^n_\infty \to \mathcal{L}^m_\infty$, we will denote the norm induced by $\|.\|_{\infty,e}$ as $\|H\|_{1,e}$, i.e.

$$\|H\|_{1,e} \doteq \sup_{\|v\|_{\infty,e}\leq 1} \|H * v\|_{\infty,e}$$

Note that for scalar signals these norms coincide with the usual $\|.\|_\infty$ and $\|.\|_1$ definitions, while in the general case we have:

$$\frac{1}{\sqrt{n}}\|H\|_1 \leq \|H\|_{1,e} \leq \sqrt{m}\|H\|_{1,e}$$

By $\mathcal{H}_2$ we denote the space of complex valued matrix functions $G(s)$ analytic on the right half plane and square integrable on the $j\omega$–axis, with $\|G(s)\|_2$ defined in the usual way as:

$$\begin{aligned}\|G\|_2^2 &\doteq \frac{1}{2\pi}\int_{-j\infty}^{+j\infty} \operatorname{trace}\left(G(j\omega)^* G(j\omega)\right) ds \\ &= \int_{-\infty}^{\infty} \operatorname{trace}\left(G(t)'G(t)dt\right)\end{aligned}$$

where $'$ denotes transpose

Throughout the paper we will use packed notation to represent state–space realizations, i.e. $G(s)$ will be written as

$$G(s) = C(sI - A)^{-1}B + D \doteq \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

Finally, given two transfer matrices $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ and Q with appropriate dimensions, the lower *linear fractional transformation* is defined as:

$$\mathcal{F}_l(T,Q) \doteq T_{11} + T_{12}Q(I - T_{22}Q)^{-1}T_{21}$$

2.2. Characterization of the Mixed $\mathcal{H}_2/\mathcal{L}_1$ Performance Measure

Next, we recall a result on the computation of an upper bound of the the $\mathcal{L}_{1,e}$ norm (and thus also on the $\mathcal{L}_1$ norm) of a stable, finite-dimensional, linear time-invariant (FDLTI) controllable system:

Lemma 1 ([9, 1]) *Consider the strictly proper, stable FDLTI system:*

$$G = \left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right) \tag{1}$$

Then:

$$\|G\|_{1,e}^2 \leq \|G\|_*^2 \doteq \inf_{\alpha>0,\, Q>0} \frac{1}{\alpha}\|CQC'\|_2 \tag{2}$$

subject to

$$AQ + QA' + \alpha Q + BB' \leq 0 \tag{3}$$

Remark 1 *It is shown in [9] that computing $\|G\|_*$ can be reduced to the problem of minimizing a convex function over an interval on the real line. Moreover, the function and its subgradient can be computed by solving Lyapunov equations. Thus, computing the $\|.\|_*$ is computationally simple.*

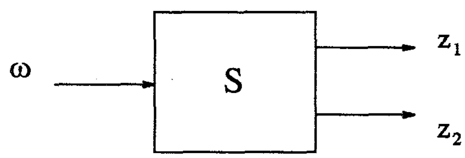

Figure 1: The Generalized Plant

By using the results of this Lemma we can characterize the mixed $\mathcal{L}_1/\mathcal{H}_2$ performance measure as follows. Consider the FDLTI stable system S shown in Figure 1, where ω represents an exogenous disturbance and where z_1 and z_2 represent performance outputs. Assume that S has the following state-space realization:

$$S = \left(\begin{array}{c|c} A & B \\ \hline C_1 & 0 \\ C_2 & 0 \end{array}\right) \tag{4}$$

Then it is well known that:

$$\|T_{z_2w}\|_2^2 = \text{trace}\,(C_2XC_2') \tag{5}$$

where $X > 0$ is the controllability Gramian of (A, B), i.e. it satisfies the following Lyapunov Equation:

$$AX + XA' + BB' = 0 \tag{6}$$

Moreover, the following Lemma can be easily proved:

Lemma 2 *Let Q denote any positive semi-definite solution to the Linear Matrix Inequality (3). Then the following two properties hold:*

1. $0 < X \leq Q$
2. $\|T_{z_2w}\|_2^2 \leq \text{trace}\,(C_2QC_2')$

Assume now that $\|T_{z_1,w}\|_* \leq 1$. From Lemma 1 this is equivalent to the existence of $\alpha > 0, Q > 0$ such that $\|C_1QC_1'\|_2 \leq \alpha$, where Q satisfies (3). Motivated by Lemma 2 we will consider the $\mathcal{H}_2$-type performance measure $J(T_{z_2w}) = \text{trace}\,(C_2QC_2')$.

2.3. The Mixed $\mathcal{H}_2/\mathcal{L}_*$ Control Problem

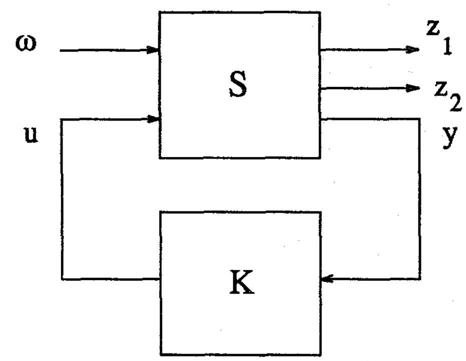

Figure 2: Setup for the Mixed $\mathcal{H}_2/\mathcal{L}_1$ Controller Synthesis

Consider now the system shown in Figure 2, where u and y represent the control action and the outputs available to the controller respectively. Then, the mixed $\mathcal{H}_2/\mathcal{L}_*$ problem can be stated as:

Problem 1 ($\mathcal{H}_2/\mathcal{L}_*$) *Given the system S with state-space realization:*

$$S = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & 0 & D_{22} \\ C_3 & 0 & 0 \end{array}\right) \tag{7}$$

find an internally stabilizing controller K such that $J(T_{z_2w})$ = trace(C_2QC_2') is minimized subject to $\|T_{z_1w}\|_* \leq 1$.

3. Main Result

In this section we analyze the structure of the optimal solutions to Problem 1. The main result of this section shows that in the state–feedback case, the optimal cost over the set of stabilizing controllers can be achieved using static state feedback.

3.1. The State Feedback Case

Theorem 1 *Assume that S has the following realization:*

$$S = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & 0 & D_{22} \\ I & 0 & 0 \end{array}\right) \quad (8)$$

where $C_2' D_{22} = 0$ and $D_{22}' D_{22} = I$. Then the following statement are equivalent:

(a) *There exists a finite dimensional LTI controller such that $J(T_{z_2w}) \leq \gamma^2$ and $\|T_{z_1w}\|_* \leq 1$.*

(b) *There exists a static control law $u = Kx$ such that $J(T_{z_2w}) \leq \gamma^2$ and $\|T_{z_1w}\|_* \leq 1$.*

(c) *There exists a scalar $\alpha > 0$ and matrices $Q > 0$, symmetric, and W such that the following LMIs have a solution:*

$$\begin{bmatrix} \alpha I & \begin{bmatrix} C_1 Q \\ W \end{bmatrix} \\ [QC_1' \quad W'] & Q \end{bmatrix} > 0$$

$$\begin{bmatrix} S - C_2QC_2' - C_2W'D_{22}' - D_{22}WC_2' & D_{22}W \\ W'D_{22}' & Q \end{bmatrix} \geq 0 \quad (9)$$

$$\text{Trace}(S) < \gamma^2$$

$$AQ + QA' + B_2W + W'B_2' + B_1B_1' + \alpha Q \leq 0$$

Moreover, the static controller $K = WQ^{-1}$ satisfies (b).

As in [9], this theorem can be used to reduce the controller synthesis problem to the problem of minimizing a real–valued function of a real variable. This can be accomplished as follows. Define the function $\Lambda: (0, \infty) \to R_+$ as:

$$\Lambda(\alpha) \doteq \{\min \gamma: (9) \text{ is feasible}\} \quad (10)$$

Then, the solution to problem 1 is given by $\mu = \min_\alpha \Lambda(\alpha)$ with the corresponding controller given by $K = WQ^{-1}$.

Remark 2 *For a given α, computing $\Lambda(\alpha)$ entails finding a solution to an LMI generalized eigenvalue problem. Thus, synthesizing the controller requires solving a finite–dimensional convex optimization problem, a task that can be efficiently accomplished (see [1, 10] for details), followed by a one–dimensional minimization. Moreover, consistent numerical experience suggests that the function $\Lambda(\alpha)$ is unimodal, although no formal proof of this fact exists at the present.*

4. A Simple Example

Consider the system given by the following state–space realization:

$$A = \begin{pmatrix} \frac{2}{3} & -\frac{8}{3} \\ \frac{4}{3} & \frac{2}{3} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 4 & 3 \\ 0 & 0 \end{pmatrix},$$

$$C_2 = (1 \quad -1), D_{12} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, D_{22} = 1$$

and assume that it is desired to minimize $\|T_{z_2w}\|_2$ subject to $\|T_{z_1w}\|_1 \sim\leq 10$.

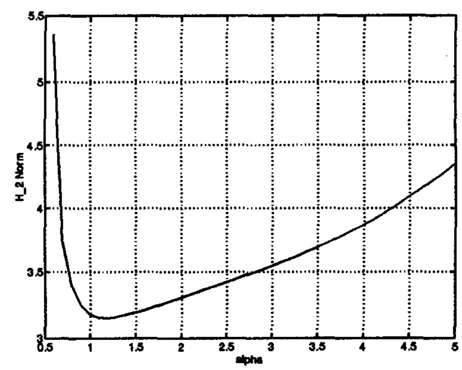

Figure 3: Upper bound of the $\mathcal{H}_2$ norm vs. α

Figure 3 shows the performance index $J(T_{z_2w})$ versus α for $\gamma = 10$. Note that this function is quasi–convex, with a global minimum $\sqrt{(J)} = 3.1484$ corresponding to $\alpha \approx 1.165$. The corresponding state–feedback controller is given by:

$$u = (-0.3445 \quad -3.7922)\, x$$

These results are summarized in Table 1.

Note that the gaps between the true $\mathcal{H}_2$ norm and its upper bound $\sqrt{J}$ and the $\mathcal{L}_1$ and $*$ norms are smaller than 7%.

$\|T_{z_2w}\|_2$	$\sqrt{J(T_{z_2w})}$	$\|T_{z_1w}\|_1$	$\|T_{z_1w}\|_*$
2.9605	3.1484	10.65	10

Table 1: Comparison of results

5. Conclusions

Mixed $\mathcal{H}_2/\mathcal{L}_1$ problems arise in the context of rejecting both stochastic as well as bounded persistent disturbances, or, in a robust control framework, in the context of nominal $\mathcal{H}_2$ performance, subject to robust stability against $\mathcal{L}_\infty$–bound perturbations.

It has been recently shown [14, 11] that in the SISO discrete time case these problems can be reduced to a finite–dimensional quadratic optimization. However, as in the case of pure l^1 optimal control, the order of the controller is not bounded by the order of the plant, and could be arbitrarily high. While similar results are not yet available for the continuous–time counterpart of the problem, it is fair to assume (motivated by continuous–time $\mathcal{L}_1$ theory) that the problem is at least as hard as the discrete–time version, and that the optimal solution is likely to be non–rational, involving delay terms.

Motivated by the complexity of these controllers, in this paper we propose an alternative approach, based upon the use of an upper bound of the mixed $\mathcal{H}_2/\mathcal{L}_1$ cost. The main result of the paper shows that for the state feedback case, the optimal solution over the set of all stabilizing controllers corresponds to static state–feedback. Moreover, this controller can be found through a two–stage procedure that entails solving a generalized eigenvalue problem and a one–dimensional minimization. While consistent numerical evidence suggests that the objective of this last minimization is unimodal, no formal proof is available at this moment. Research is presently being carried out towards establishing this fact and towards extending these results to the output feedback case.

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