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Robust Control of Constrained Systems via Convex Optimization

Mario Sznaier[1]
Department of Electrical Engineering
The Pennsylvania State University
University Park, PA 16802

Franco Blanchini
Dipartimento di Matematica e Informatica
Università degli Studi di Udine
Via Zannon 6, 33100, Udine, Italy

Abstract

A successful controller design paradigm must take into account both model uncertainty and design specifications. In this paper we propose a design procedure, based upon the use of convex optimisation, that takes explicitly into account both time and frequency domain specifications. The main result of the paper shows that these controllers can be obtained by solving a sequence of problems, each one consisting of a finite–dimensional convex optimisation and a standard, unconstrained $\mathcal{H}_\infty$ problem. Additionally, the paper serves as a brief tutorial on the issues involved in addressing design problems with multiple design specifications via convex optimization.

I. Introduction

The control of systems under input/output time–domain constraints is a long–standing problem in control theory (see for instance [1–6]). However, most of the design techniques currently available assume that the dynamics of the system are completely known. Clearly, such an assumption is too restrictive, resulting in controllers with limited application.

During the last decade a powerful robust control framework has been developed addressing the issues of stability and performance in the presence of norm–bound model uncertainty. Robust stability and performance are achieved by minimizing a suitable norm (either $\|.\|_1$ or $\|.\|_\infty$) of a closed–loop transfer function. However, despite its significance, this framework is limited by the fact that in this context, performance must be measured in the same norm used to assess stability. Clearly, a single norm is usually not enough to capture different (and often conflicting) design specifications, such as mixed time/frequency domain specifications. Thus, designers are forced to use weighting functions and similarity scaling, coupled with extensive trial and error, to translate the specifications into a form amenable to the theoretical framework.

Recently, some progress has been made towards solving problems involving mixed time/frequency domain constraints for SISO discrete [7–9] and continuous [10–11] time systems. In this paper we extend these results to the MIMO case. The proposed design method is based upon solving an auxiliary discrete–time problem, obtained using the simple transformation $z = 1 + \tau s$, and then transforming back the resulting controller to the s domain. Noteworthy, solving this auxiliary problem only entails solving a *finite dimensional* convex constrained optimization problem and an *unconstrained* $\mathcal{H}_\infty$ problem. Thus, solving these problems is no more demanding computationally than solving a finite dimensional convex optimization problem and two Riccati equations.

The paper is organized as follows: In section II we introduce the notation to be used, we give a precise statement of the problem and we present some preliminary results. Section IV contains the proposed synthesis method. Here we first extend the results of [7] to discrete–time MIMO systems (by using an all-pass embedding argument) and then combine these results with the properties of the EAS to obtain a design procedure for MIMO continuous time systems. In section V we use our theory to design controllers for a two mass system widely used as a benchmark for robust control. Finally, in section VI, we summarize our results and we indicate directions for future research. Due to space limitations all proofs have been ommitted. They can be obtained by contacting the authors.

II. Problem Formulation

2.1 Notation

By $\mathcal{L}_\infty(j\mathcal{R})$ we denote the Lebesgue space of complex valued transfer matrices which are essentially bounded on the imaginary axis, with norm $\|T(s)\|_{\mathcal{H}_\infty} \triangleq \sup_\omega \overline{\sigma}(T(j\omega))$, where $\overline{\sigma}$ denotes the maximum singular value. $\mathcal{H}_\infty(j\mathcal{R})$ ($\mathcal{H}_\infty(j\mathcal{R})^-$) denotes the set of stable (antistable) complex matrices $G(s) \in \mathcal{L}_\infty(j\mathcal{R})$. Similarly, $\mathcal{L}_\infty(T)$ denotes the Lebesgue space of complex valued transfer matrices which are essentially bounded on the unit circle with norm $\|T(z)\|_{\mathcal{H}_\infty} \triangleq \sup_\omega \overline{\sigma}(T(e^{j\omega}))$. $\mathcal{L}_p^\infty(R_+)$ (l_p^∞) denotes the space of measurable vector functions $f(t)$ (bounded sequences) equipped with the norm: $\|f\|_{\mathcal{L}^\infty} = \max_i ess.\sup_{R_+} |f(t)|$ ($\|f\|_{l^\infty} = \max_i \sup_k |f_k|$). Given a sequence h (a function $h(t)$) we will denote its z–transform (Laplace transform) by $H(z)$ ($H(s)$) and, by a slight abuse of notation, we will denote as $\|H(z)\|_{l^\infty} \triangleq \sup_k |h_k|$ ($\|F(s)\|_{\mathcal{L}^\infty} \triangleq ess.\sup_{R_+} |f(t)|$). $\mathcal{A}$ denotes the space whose elements have the form:

$$h = h^L(t) + h_0\delta(t)$$

where $h^L(t) \in \mathcal{L}^1(R_+)$ and $\delta(t)$ is the Dirac function, equipped with the norm $\|h\|_{\mathcal{A}} \triangleq \|h^L\|_{L_1} + |h_0|$. We denote by $\mathcal{A}_m$ the space of vector functions having m components in $\mathcal{A}$.

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2.2 Statement of the Problem

Consider the system S shown in figure 1, where the signals $w_f \in \mathcal{L}^2_{n_f}$ (a bounded energy signal), $w_t \in \mathcal{D}_t$ and $u \in \mathcal{L}^{\infty}{}_p$ represent exogenous inputs and the control action respectively, $\mathcal{D}_t$ is a given set of test inputs; and where $\zeta_f \in \mathcal{L}^{\infty}{}_{m_f}$, $\zeta_t \in \mathcal{L}^{\infty}{}_{m_t}$ and $y \in \mathcal{L}^{\infty}{}_{m_y}$ represent the regulated outputs and the measurements respectively. Then, the mixed $\mathcal{L}^\infty/\mathcal{H}_\infty$ control problem can be stated as: *Given the nominal system* (S), *find an internally stabilizing controller* $K(s)$ *such that worst case peak amplitude of the performance output* $\|\zeta_t\|_{\mathcal{L}^\infty}$ *due to signals in the set* $\mathcal{D}_t$ *is minimized, subject to the constraint* $\|T_{\zeta_f w_f}\|_{\mathcal{H}_\infty} \leq \gamma$.

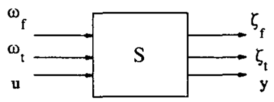

Figure 1. The Plant

In the sequel we will consider the case in which the set of test signals $\mathcal{D}_t$ is finite, i.e. $\mathcal{D}_t = \{w_{t_1}, w_{t_2} \dots w_{t_n}\}$. This case corresponds to the common practice of specifying some of the performance requirements in terms of the response of the closed-loop system to a given set of test inputs.

Remark 1: In the sequel we will assume, for simplicity, that the test signals are stable, i.e. $w_{t_i} \in \mathcal{A}_{n_t}$. However, this does not entail any loss of generality, since *unstable* signals can be accommodated by absorbing their unstable poles in the plant (see [7] for details).

2.3 Problem Transformation

Assume that the system S has the following state–space realization:

$$\left(\begin{array}{c|ccc} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ C_3 & D_{31} & D_{32} & D_{33} \end{array}\right) \qquad (S)$$

where D_{13} has full column rank, D_{31} has full row rank, and where the pairs (A, B_3) and (C_3, A) are stabilizable and detectable respectively. It is well known (see for instance [15]) that the set of all internally stabilizing controllers can be parametrized in terms of a free parameter $Q \in \mathcal{H}_\infty$ as:

$$K = \mathcal{F}_l(J, Q) \qquad (1)$$

By using this parametrization, the closed-loop transfer matrices $T_{\zeta_f w_f}$ and $T_{\zeta_t w_t}$ can be written as:

$$\begin{aligned} T_{\zeta_f w_f}(z) &= T_{11}(s) + T_{12}(s)Q(s)T_{21}(s) \\ T_{\zeta_t w_t}(s) &= V_{11}(s) + V_{12}(s)Q(s)V_{21}(s) \end{aligned} \qquad (2)$$

where T_{ij}, V_{ij} are stable transfer matrices. Hence the problem can be now precisely stated as:

Problem 1 *(Mixed* $\mathcal{L}^\infty/\mathcal{H}_\infty$ *control problem:)*
Find the optimal value of the performance measure:

$$\nu^o = \inf_{Q \in \mathcal{RH}_\infty} \left\{ \sup_{w_t \in \mathcal{D}_t} \{\|T_{\zeta_t w_t}(s) W_t(s)\|_{\mathcal{L}^\infty}\} \right\} \qquad (\mathcal{L}^\infty/\mathcal{H}_\infty)$$

$$subject\ to: \|T_{11}(s) + T_{12}(s)Q(s)T_{21}(s)\|_{\mathcal{H}_\infty} \leq \gamma \qquad (3)$$

where $W_t(s)$ denotes the Laplace transform of w_t.

Remark 2: By defining:

$$T^a_{\zeta_t w_t} \triangleq \begin{bmatrix} T_{\zeta_t w_t} & 0 & \dots & 0 \\ 0 & T_{\zeta_t w_t} & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & T_{\zeta_t w_t} \end{bmatrix} \qquad (4)$$
$$W^a \triangleq \left[W'_{t_1}(s) \quad W'_{t_2}(s) \quad \dots \quad W'_{t_n}(s) \right]'$$

and absorbing $W^a(s)$ into $T^a_{\zeta_t w_t}$, Problem 1 can be restated as:

$$\nu^o = \inf_{Q \in \mathcal{RH}_\infty} \{\|T_{\zeta_t w_t}(s)\|_{\mathcal{L}^\infty}\} \qquad (5)$$

$$subject\ to: \|T_{11}(s) + T_{12}(s)Q(s)T_{21}(s)\|_{\mathcal{H}_\infty} \leq \gamma \qquad (6)$$

Remark 3: It is well known (see for instance [15]), that it is possible to perform the parametrization in such a way that T_{12} is inner and T_{21} is co-inner. If T_{12} (T_{21}) is not square, it is possible to choose $T_{12\perp}$ $(T_{21\perp})$ such that $T_{12a} \triangleq [T_{12} \quad T_{12\perp}]$ ($T_{21a}^{\sim} \triangleq [T_{21}^{\sim} \quad T_{21\perp}^{\sim}]$) is a unitary matrix. Since the $\|.\|_{\mathcal{H}_\infty}$ is invariant under multiplication by unitary matrices, it follows that $\|T_{\zeta_f w_f}\|_{\mathcal{H}_\infty}$ can be reduced to the form:

$$\|T_{\zeta_f w_f}\|_{\mathcal{H}_\infty} = \left\| T_{11} + T_{12a} \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} T_{21a} \right\|_{\mathcal{H}_\infty} = \left\| R + \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \right\|_{\mathcal{H}_\infty} \qquad (7)$$

where $R \triangleq T_{12a}^{\sim} T_{11} T_{21a}^{\sim} \in \mathcal{RH}_\infty$.

III. Preliminaries

3.1 Definitions

• **Def. 1:** Consider the continuous time system (S). Its Euler Approximating System (EAS) is defined as the following discrete time system:

$$\left(\begin{array}{c|ccc} I + \tau A & \tau B_1 & \tau B_2 & \tau B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ C_3 & D_{31} & D_{32} & D_{33} \end{array}\right) \qquad (EAS)$$

where $\tau > 0$. In the sequel, given any transfer function $T_{ij}(s)$ we define as $T^{EAS}_{ij}(z, \tau) \triangleq T_{ij}(\frac{z-1}{\tau})$.

3.2 Properties of the Euler Approximating System

In this section we recall some properties of the EAS. The main result of this section shows that the $\mathcal{H}_\infty$ and the $\mathcal{L}^\infty$ norm of the Euler Approximating system are upper bounds of the corresponding continuous-time norms. Moreover, this upper bound is non-increasing with τ and converges to the exact value as $\tau \to 0$.

• **Theorem 1:** Given a strictly decreasing sequence $\tau_i \to 0$, consider the system:

$$\begin{aligned} \dot{x} &= Ax + Bv \\ \zeta &= Cx + Dv \end{aligned} \tag{8}$$

and its corresponding EAS(τ_i):

$$\begin{aligned} x_{k+1} &= (I + \tau_i A)x_k + \tau_i B v_k \\ \zeta_k^{EAS} &= C x_k + D v_k \end{aligned} \tag{9}$$

Then:

$$\begin{aligned} \|T_{\zeta v}(s)\|_{\mathcal{L}^\infty} &\leq \|T_{\zeta v}^{EAS}(z,\tau_i)\|_{l^\infty} \forall\, i \\ \|T_{\zeta v}^{EAS}(z,\tau_i)\|_{l^\infty} &\leq \|T_{\zeta v}^{EAS}(z,\tau_j)\|_{l^\infty}\ i > j \\ \lim_{\tau_i \to 0} \|T_{\zeta v}^{EAS}(z,\tau_i)\|_{l^\infty} &= \|T_{\zeta v}(s)\|_{\mathcal{L}^\infty} \end{aligned} \tag{11}$$

and

$$\begin{aligned} \|T_{\zeta v}(s)\|_{\mathcal{H}_\infty} &\leq \|T_{\zeta v}^{EAS}(z,\tau_i)\|_{\mathcal{H}_\infty} \forall\, i \\ \|T_{\zeta v}^{EAS}(z,\tau_i)\|_{\mathcal{H}_\infty} &\leq \|T_{\zeta v}^{EAS}(z,\tau_j)\|_{\mathcal{H}_\infty}\ i > j \\ \lim_{\tau_i \to 0} \|T_{\zeta v}^{EAS}(z,\tau_i)\|_{\mathcal{H}_\infty} &= \|T_{\zeta v}(s)\|_{\mathcal{H}_\infty} \end{aligned} \tag{11}$$

• **Theorem 2:** Assume that $\inf_{Q \in \mathcal{RH}_\infty} \|T_{\zeta_f w_f}(s)\|_{\mathcal{H}_\infty} = \gamma_o < \gamma$. Consider a strictly decreasing sequence $\tau_{max} > \tau_i \to 0$, and the corresponding EAS(τ_i). Let

$$\begin{aligned} \nu_i &= \inf_{\substack{Q \in \mathcal{RH}_\infty \\ \|T_{\zeta_f w_f}\|_{\mathcal{H}_\infty} \leq \gamma}} \|T_{\xi_t w_t}^{EAS}(z,\tau_i)\|_{l^\infty} \\ \nu^o &= \inf_{\substack{Q \in \mathcal{RH}_\infty \\ \|T_{\xi_f w_f}\|_{\mathcal{H}_\infty} \leq \gamma}} \|T_{\xi_t w_t}(t)\|_{\mathcal{L}^\infty} \end{aligned} \tag{12}$$

Then the sequence ν_i is non-increasing and such that $\nu_i \to \nu^o$.

IV. Problem Solution

In this section we present a method for finding suboptimal rational solutions to problem 1, based upon the use of an auxiliary discrete-time problem. Note that, from Theorem 2, it follows that the $(\mathcal{L}^\infty/\mathcal{H}_\infty)$ problem can be solved by solving a sequence of discrete-time $(l^\infty/\mathcal{H}_\infty)$ problems, each one having the form:

$$\nu^o = \inf_{\substack{Q \in \mathcal{RH}_\infty(T) \\ \|T_{11} + T_{12} Q T_{21}\|_{\mathcal{H}_\infty} \leq \gamma}} \|V_{11} + V_{12} Q V_{21}\|_{l^\infty} \qquad (l^\infty/\mathcal{H}_\infty)$$

The solution of this problem is discussed next.

4.1 A Suboptimal Solution to Discrete Time MIMO 4-Blocks $l_\infty/\mathcal{H}_\infty$ Control Problems

In this section we generalize the results of [7] to general MIMO systems. As in the SISO case, the main result shows that the mixed $l^\infty/\mathcal{H}_\infty$ problem can be solved by solving a finite-dimensional convex optimization problem and an unconstrained $\mathcal{H}_\infty$ problem. This result will be established by showing that: i) $(l^\infty/\mathcal{H}_\infty)$ can be solved by considering a sequence of modified problems; ii) a solution to each modified problem can be found by solving a truncated problem; and iii) this truncated problem can be decoupled into a finite-dimensional convex optimization and an unconstrained $\mathcal{H}_\infty$ problem.

Let $\delta < 1$, and define the space:

$$\mathcal{H}_{\infty,\delta} \triangleq \{Q(z) \in \mathcal{H}_\infty : Q(z) \text{ analytic in} |z| \geq \delta\}$$

equipped with the norm $\|Q\|_{\mathcal{H}_{\infty,\delta}} \triangleq \sup_{|z|=\delta} \bar{\sigma}(Q(z))$. Then, given $V_{ij}(z), T_{ij}(z) \in \mathcal{RH}_{\mathcal{H}_{\infty,\delta}}$, consider the following modified problem:

Problem 2: Find

$$\nu_\delta^o = \inf_{Q \in \mathcal{RH}_{\infty,\delta}} \|\{V_{11} + V_{12} Q V_{21}\}\|_{l^\infty} \qquad (l^\infty/\mathcal{H}_{\infty,\delta})$$

subject to:

$$\|R + \begin{bmatrix} Q(z) & 0 \\ 0 & 0 \end{bmatrix}\|_{\mathcal{H}_{\infty,\delta}} \leq \gamma$$

• **Lemma 1:** Consider an increasing sequence $\delta_i \to 1$. Then $\nu_{\delta_i}^o \leq \nu_{\delta_j}^o$, $i \geq j$ and $\nu_{\delta_i}^o \to \nu^o$.

• **Lemma 2:** Let V_k denote the coefficients of the impulse response of $T_{\zeta_t w_t} = V_{11} + V_{12} Q V_{21}$. Then, if $Q \in \mathcal{RH}_{\infty,\delta}$ satisfies the constraint $\|R(z) + \begin{bmatrix} Q(z) & 0 \\ 0 & 0 \end{bmatrix}\|_{\mathcal{H}_{\infty,\delta}} \leq \gamma$, it also satisfies $\|V_k\|_1 \leq M\delta^k$ where M is a number independent of Q.

• **Corollary 1:** Let $\nu^* \triangleq \inf_{Q \in \mathcal{RH}_\infty} \|V_{11} + V_{12} Q V_{21}\|_{l^\infty}$ and select $N \geq \frac{\log \nu^* - \log M}{\log \delta}$. Then, Problem 2 is equivalent to the following semi-infinite convex optimization problem:

$$\inf_{Q_i \in R^{m \cdot N m_y}} \|\underline{v}_1 + \mathcal{V}_{12} \underline{Q} \mathcal{V}_{21}\|_{l^\infty}$$

subject to:

$$\left\| R(z) + \begin{bmatrix} Q(z) & 0 \\ 0 & 0 \end{bmatrix} \right\|_{\mathcal{H}_{\infty,\delta}} \leq \gamma$$

where $\underline{v}$, $\mathcal{V}$ and $\underline{Q}$ are defined as

$$\mathcal{V}_{12} = \begin{bmatrix} V_{12_o} & 0 & \dots & 0 \\ V_{12_1} & V_{12_o} & \dots & 0 \\ \vdots & & \ddots & \\ V_{12_{N-1}} & \dots & & V_{12_o} \end{bmatrix} \quad \mathcal{V}_{21} = \begin{bmatrix} V_{21_o} \\ V_{21_1} \\ \vdots \\ V_{21_{N-1}} \end{bmatrix}$$

$$\underline{v}_1 = \begin{bmatrix} V_{11_o}' & \dots & V_{11_{N-1}}' \end{bmatrix}' \quad \underline{Q} = \begin{bmatrix} Q_o & 0 & \dots & 0 \\ Q_1 & Q_o & \dots & 0 \\ \vdots & & \ddots & \\ Q_{N-1} & \dots & & Q_o \end{bmatrix} \tag{20}$$

and where Q_k, V_{ij_k} denote the k^{th} element of the impulse response of $Q(z), V_{ij}(z)$ respectively.

In the next section we will show that this problem can be further decoupled into a finite dimensional constrained optimization and an unconstrained $\mathcal{H}_\infty$ problem.

4.2 Handling the $\mathcal{H}_\infty$ Constraint

In the previous section we showed that the discrete time constrained robust control problems can be solved by solving a sequence of truncated problems. In this section we show that each of these problems can be *exactly* solved by solving a finite dimensional convex optimization problem and an unconstrained $\mathcal{H}_\infty$ problem. To establish this result we recall first a result from [17] establishing a necessary and sufficient condition for the feasibility of the $\mathcal{H}_\infty$ constraint when the first N parameters in the expansion $Q(z) = Q_0 + Q_1 z^{-1} + \cdots + Q_{n-1} z^{-(n-1)} + \cdots$ are fixed. Define $G \triangleq T_{21a} T_{11}^{\sim} T_{12a} = R^{\sim}$ and assume that it has the following state-space realization:

$$G = \left(\begin{array}{c|cc} \hat{A} & B_a & B_b \\ \hline C_a & D_{aa} & D_{ab} \\ C_b & D_{ba} & D_{bb} \end{array} \right)$$

Finally, consider the following Riccati equations:

$$\begin{aligned} \hat{X} &= \hat{A}\hat{X}\hat{A}' + \gamma^{-2} B_e B_e' + \\ &\left(\hat{A}\hat{X}C_a' + \gamma^{-2} B_e D_{er}'\right) \left(I - \gamma^{-2} D_{er} Der' - C_1 \hat{X} C_1'\right)^{-1} \left(C_a \hat{X} \hat{A}' + \gamma^{-2} D_{er} B_e'\right) \\ \hat{Y} &= \left(\hat{A}'\hat{Y} B_a + C_e' D_{ec}\right) \left(I - D_{ec}' Dec - B_1' \hat{Y} B_1\right)^{-1} \left(B_a' \hat{Y} \hat{A} + D_{ec}' C_a\right) \\ &+ \hat{A}'\hat{Y}\hat{A} + C_e' C_e \end{aligned} \tag{21}$$

where:

$$B_e = [\, B_a \quad B_b \,] \; C_e = \begin{bmatrix} C_a \\ C_b \end{bmatrix}$$

$$D_{er} = [\, D_{aa} \quad D_{ab} \,] \; D_{ec} = \begin{bmatrix} D_{aa} \\ D_{ba} \end{bmatrix}$$

From [17], there exist a Q satisfying the *strict* $\mathcal{H}_\infty$ constraint if and only if there exist positive-definite solutions $\hat{X}$ and $\hat{Y}$ to these Riccati equations such that $\rho(\hat{X}\hat{Y}) < \gamma$. For ease of notation, let $x \triangleq \hat{X}^{1/2}$, $y \triangleq \hat{Y}^{1/2}$.

• **Theorem 3:** Let $Q^n(z) = \sum_{i=0}^{n-1} Q_i z^{-i}$. Then there exists a $Q_{tail}^n(z) \in \mathcal{H}_\infty$ such that

$$\left\| \begin{bmatrix} R_{11} - \sum_{i=0}^{n-1} Q_i' z^i - z^n Q_{tail}^n(z) & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_{\mathcal{H}_\infty} \leq \gamma$$

if and only if $\overline{\sigma}(\mathcal{Q}(\mathbf{Q_n})) \leq \gamma$, where:

$$\mathcal{Q}(\mathbf{Q_n}) = \begin{bmatrix} y\hat{A}^n x & y\hat{A}^{n-1}B_a & \cdots & y\hat{A}B_a & yB_a & y\hat{A}^{n-1}B_b & y\hat{A}^{n-2}B_b & \cdots & y\hat{A}B_b & yB_b \\ C_a\hat{A}^{n-1}x & C_a\hat{A}^{n-2}B_a & \cdots & C_aB_a & D_{aa} & C_a\hat{A}^{n-2}B_b & C_a\hat{A}^{n-3}B_b & \cdots & C_aB_b & D_{ab} \\ C_a\hat{A}^{n-2}x & C_a\hat{A}^{n-3}B_a & \cdots & D_{aa} & 0 & C_a\hat{A}^{n-3}B_b & C_a\hat{A}^{n-4}B_b & \cdots & D_{ab} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_a x & D_{aa} & 0 & \cdots & 0 & D_{ab} & 0 & 0 & \cdots & 0 \\ C_b\hat{A}^{n-1}x & C_b\hat{A}^{n-2}B_a & \cdots & C_bB_a & D_{ba} & C_b\hat{A}^{n-2}B_b & C_b\hat{A}^{n-3}B_b & \cdots & C_bB_b & -Q_0^t \\ C_b\hat{A}^{n-2}x & C_b\hat{A}^{n-3}B_a & \cdots & D_{ba} & 0 & C_b\hat{A}^{n-3}B_b & C_b\hat{A}^{n-4}B_b & \cdots & -Q_0^t & -Q_1^t \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_b x & D_{ba} & 0 & \cdots & 0 & -Q_0^t & -Q_1^t & -Q_2^t & \cdots & -Q_{(n-1)}^t \end{bmatrix}$$

Remark 4: This result can be easily extended to the case $Q \in \mathcal{RH}_{\infty,\delta}$, $\|R + \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}\|_{\mathcal{H}_{\infty,\delta}} \leq \gamma$ by using the change of variable $z = \delta\hat{z}$

Combining Corollary 1 and Theorem 3, recalling that $\overline{\sigma}(\mathcal{Q}(\mathbf{Q_n})) = \|Q_n\|_2$ is a convex function of Q_n, yields the main result of this section:

• **Theorem 4:** A solution to the mixed $l^\infty/\mathcal{H}_{\infty,\delta}$ control problem is given by $Q^o = Q_F^o + z^{-N} Q_R^o$ where $Q_F^o = \sum_{i=0}^{N-1} Q_i z^{-i}$, $\underline{Q}^o = [Q_o \ldots Q_{N-1}]$ solves the following finite dimensional convex optimization problem:

$$\underline{Q}^o = \operatorname*{argmin}_{\substack{\underline{Q} \in R^{m \cdot N m_y} \\ \|\mathcal{Q}\|_2 \leq \gamma}} \|\underline{v}_1 + \mathcal{V}_{12} \underline{Q} \mathcal{V}_{21}\|_{l^\infty}$$

and $Q \in R^{p \cdot N m_y}$ solves the approximation problem

$$Q_R^o(z) = \operatorname*{argmin}_{Q_R \in \mathcal{H}_{\infty,\delta}} \|T_{11}(z) + T_{12} Q_F^o T_{21}(z) + z^{-N} T_{12} Q_R(z) T_{21}(z)\|_{\mathcal{H}_{\infty,\delta}} \tag{22}$$

where $N(\delta)$ is selected according to Corollary 1, $\underline{v}_1$ and $\mathcal{V}_{ij}$ are defined in (20), and $\mathcal{Q}$ is defined in Theorem 3.

4.3 Proposed Design Method

From the definition of the EAS it is easily seen that the closed-loop transfer function obtained by applying the rational controller $K(s)$ to (S) is the same as the closed-loop transfer function obtained by applying the controller $K(\frac{z-1}{\tau})$ to the EAS, up to the complex transformation $z = \tau s + 1$. Therefore, if a rational compensator $K(z)$ yielding an $l^\infty/\mathcal{H}_\infty$ cost ν_d is found for the EAS, then $K(\tau s + 1)$ internally stabilizes (S) and yields an $\mathcal{L}^\infty/\mathcal{H}_\infty$ cost $\nu_c \leq \nu_d$. It follows that a rational compensator can be synthesized using the EAS with a suitably small τ. These observations are formalized in the following lemma:

• **Lemma 3:** Consider the mixed $\mathcal{L}^\infty/\mathcal{H}_\infty$ problem. A suboptimal rational solution can be obtained by solving a discrete-time constrained control problem for the corresponding EAS, with $\delta = 1 - \tau^2$. Moreover, if $K(z)$ denotes the controller for the EAS, the suboptimal continuous-time controller is given by $K(\tau s + 1)$.

Finally, we show that by taking $\tau \to 0$, the proposed design method yields controllers with cost arbitrarily close to the optimal $\mathcal{L}^\infty/\mathcal{H}_\infty$ cost.

• **Theorem 5:** Let $\tau_{max} > \tau_i \to 0$ be a strictly decreasing sequence. Denote by K_i the controller obtained using the design procedure of Lemma 4 with $\tau = \tau_i$ and by $T_{z,w_i}(s, K_i)$ the corresponding closed loop transfer function. Then the sequence $\nu_i \triangleq \|T_{z,w_i}(s, K_i)\|_{l^\infty}$ is non-increasing and such that $\lim_{i \to \infty} \nu_i = \nu^o$.

V. A Design Example

Consider the system shown in Figure 2, consisting of two unity masses coupled by a spring with constant $0.5 \le k \le 2$ but otherwise unknown. A control force acts on body 1 and the position of body 2 is measured, resulting in a non-colocated sensor actuator problem. This system has been used as a benchmark during the last few years at the American Control Conference [19-21] to highlight the issues and trade-offs involved in robust control design. Assume that it is desired to design an internally stabilizing controller subject to the following specifications: i) the closed-loop system must be stable for all possible values of the uncertain parameter $k \in [0.5, 2]$; ii) An impulse disturbance $w = \delta(t)$ acting on m_2 should be rejected, with a control action $|u(t)| \le 1$; and iii) for the same disturbance the displacement y of m_2 should have a settling time of about 15 seconds.

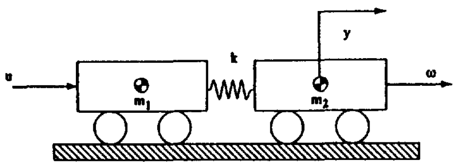

Figure 2: The ACC Robust Control Benchmark Problem.

In order to fit the problem into the $\mathcal{H}_\infty$ framework, the uncertain spring constant k is modeled as $k = k_o + \Delta$ (with $k_o = 1.25$ and $\|\Delta\| \le 0.75$) and, following a standard procedure [15], Δ is "pulled out" of the system. The problem can be stated now as the problem of minimizing the peak control effort $\|u\|_{\mathcal{L}^\infty}$ over the set of all internally stabilizing controllers, subject to the settling time and $\|T_{\zeta v}\|_{\mathcal{H}_\infty} \le \frac{4}{3}$ constraints.

Figure 3 (a) shows the control action following an impulse disturbance on m_2, for an $\mathcal{H}_\infty$ controller, a controller designed using μ-synthesis and a mixed $\mathcal{L}^\infty/\mathcal{H}_\infty$ controller, all of them satisfying the robustness and settling time specifications (the latter were enforced by exponentially weighting the output y). Note that the plain $\mathcal{H}_\infty$ controller requires a *clearly unrealistically large* peak control action. The time-domain behavior of the closed-loop system can be improved by considering a weighted $\mathcal{H}_\infty$ design that penalizes the control action. Figure 3 (b) shows the control action for a controller designed using μ-synthesis [22], with the control action weighted with a high-pass filter (to avoid high frequency control activity). Although this controller has substantially better time-domain behavior than the plain $\mathcal{H}_\infty$ controller, the control action must be further reduced in order to meet the design specifications. Moreover, the problem has been shifted now from designing a controller to finding appropriate weights. Although a proper weight selection will finally yield a controller satisfying the specifications (see [23]), this process requires considerably design skills and multiple trial and error iterations without guarantee of success. Indeed, it is worth stressing that the main motivation behind our design framework was to introduce some flexibility into $\mathcal{H}_\infty$ design by treating a *time domain* specification exactly, i.e., without resorting to approximations or weigh-selection.

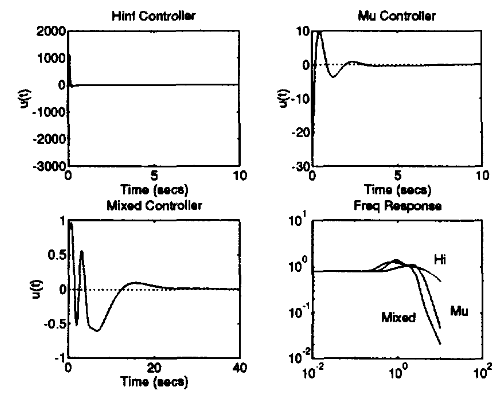

Figure 3. Control Action for $\mathcal{H}_\infty$ (a), μ (b) and mixed $\mathcal{L}^\infty/\mathcal{H}_\infty$ controllers

Finally, Figure 3 (c) shows the impulse response corresponding to a controller designed using mixed $\mathcal{L}^\infty/\mathcal{H}_\infty$ theory, satisfying both the performance and the robustness specifications. This controller has order roughly equal to 2N, where N is determined using Corollary 1. In this case, due to the existence of plant zeros on the stability boundary, $N \sim 150$, thus resulting in a 300 order controller. However, using weighted balanced truncations, we were able to reduce this controller to 4^{th} order, with virtually no performance loss.

VI. Conclusions and Discussion

In this paper we address the problem of finding internally stabilizing controllers that minimize the peak amplitude of the output due to inputs belonging to a given set $\mathcal{D}_i$, subject to robustness constraints given in the form of an $\mathcal{H}_\infty$ constraint upon the norm of a relevant transfer function. This problem is of importance for applications where either the control action or some outputs are subject to hard bounds. It can be thought as the problem of designing a controller capable of guaranteeing an adequate robustness level agains dynamic uncertainty while using the extra available degrees of freedom to optimize a time-domain performance.

The main result of the paper shows that the resulting convex optimization problem can be decoupled into a finite dimensional, albeit non-differentiable, constrained optimization and an unconstrained Nehari approximation problem. This is a notorious departure from previous approaches to solving this types of problems [24-25], where several approximations where required in order to obtain a tractable mathematical problem. Moreover, some recent work [26] shows that these approximations may fail to converge to a solution.

The example of section 4 highlights the strengths of our approach, and also suggest future research topics. Namely, the method allows for dealing *explicitly and exactly* with time-domain specifications, eliminating multiple (and non necessarily convergent) trial and error type iterations. This will usually result in an improved and less costly design. However, in its present status, the method usually produces very complex controllers, necessitating some type of model reduction, a disadvantage also shared by some popular design methods such as μ-synthesis [22] or l_1 optimal control theory [27]. Application of some well established methods in order reduction (noteworthy, weighted balanced truncation) usually succeed in producing controllers of manageable order. Furthermore, consistent numerical experience suggests that this order reduction can be accomplished with virtually no performance degradation. Research is currently under way addressing this issue.

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