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Mixed $\mathcal{L}^\infty/\mathcal{H}_\infty$ Suboptimal Controllers for SISO Continuous–Time Systems

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Abstract

A successful controller design paradigm must take into account both model uncertainty and performance specifications. Model uncertainty can be addressed using the $\mathcal{H}_\infty$ robust control framework. However, this framework cannot accommodate the realistic case where in addition to robustness considerations, the system is subject to time–domain performance specifications. In this paper we present a design procedure for suboptimal $\mathcal{L}^\infty/\mathcal{H}_\infty$ controllers. These controllers allow for minimising the maximum amplitude of the time–response due to a specified input, while, at the same time, addressing model uncertainty through bounds on the $\mathcal{H}_\infty$ norm of a relevant transfer function. The main result of the paper shows that suboptimal rational controllers can be obtained by solving a finite dimensional convex constrained optimisation problem and a standard unconstrained $\mathcal{H}_\infty$ problem.

I. Introduction

A large number of control problems involve designing a controller capable of achieving acceptable performance under system uncertainty and design constraints. However, in spite of its practical importance, this problem remains, to a large extent, open. During the last decade a large research effort led to procedures for designing robust controllers, capable of achieving desirable properties under various classes of plant uncertainties, while satisfying either time (in the case of l^1 theory [1]) or frequency–domain constraints ($\mathcal{H}_\infty$ theory [2]). However, these design procedures cannot accommodate directly the realistic case where the system must satisfy both time and frequency domain performance specifications. Recently, some progress has been made in this direction for the case of discrete–time systems [3–5]. However, these results do not have a counterpart for continuous–time systems. In principle it is possible to use a discrete–time controller, designed using the theory developed in [4–5], connected to the continuous–time plant through sample and hold devices (see [6–9] and references therein for a thorough discussion of the properties of sampled–data systems). However, due to intersampling ripple effects, satisfaction of time–domain constraints in the discretized system does not necessarily guarantee satisfaction of these constraints in the actual closed–loop sampled–data system. Moreover, the use of sample and hold elements usually entails a performance loss. For example, in [10] a discrete–time controller was designed for a robust control benchmark problem. Although this controller meets the performance specifications, comparison with other designs [11] clearly displays the loss of performance due to the sampled–data implementation.

In this paper we propose a method to design rational suboptimal $\mathcal{L}^\infty/\mathcal{H}_\infty$ controllers for continuous–time systems. These controllers minimize the worst–case time–domain response due to a specified input, while, at the same time, satisfying an $\mathcal{H}_\infty$ constraint on a relevant transfer function. The proposed method is based upon solving an auxiliary discrete–time $l_\infty/\mathcal{H}_\infty$ problem [5], obtained using the simple transformation $z = 1+\tau s$, and then transforming back the resulting controller to the s domain. Thus it only entails solving a *finite dimensional* convex constrained optimization problem and an *unconstrained* $\mathcal{H}_\infty$ problem. The main results of the paper show that: i) the performance of the resulting closed–loop continuous–time system is bounded above (both in the frequency and time domains) by the performance of the auxiliary discrete–time system used in the design; and ii) optimal performance is recovered as the parameter $\tau \to 0$.

The paper is organized as follows: In section II we introduce the notation to be used and we give a formal definition to the mixed $\mathcal{L}^\infty/\mathcal{H}_\infty$ control problem. Section III contains the bulk of the theoretical results. Here we introduce the discrete time Euler approximating system (EAS) and we show that the peak values of the time and frequency responses of the EAS are upper bounds of the corresponding continuous–time quantities. As an immediate consequence, it follows that suboptimal $\mathcal{L}^\infty/\mathcal{H}_\infty$ controllers with guaranteed cost can be designed by applying $l_\infty/\mathcal{H}_\infty$ theory to the EAS. Moreover, we also show that the optimal cost can be approximated arbitrarily close by taking τ small enough. In section IV we present a simple design example and we compare our controller to the unconstrained optimal $\mathcal{H}_\infty$ controller. Finally, in section V, we summarize our results and we indicate directions for future research.

II. Problem Formulation

2.1 Notation

By $\mathcal{L}_\infty(j\mathcal{R})$ we denote the Lebesgue space of complex valued transfer functions which are essentially bounded on the imaginary axis with norm $\|T(s)\|_{\mathcal{H}_\infty} \triangleq \sup_\omega |T(j\omega)|$. $\mathcal{H}_\infty(j\mathcal{R})$ ($\mathcal{H}_\infty(j\mathcal{R})^-$) denotes the set of stable (antistable) complex functions $G(s) \in \mathcal{L}_\infty(j\mathcal{R})$, i.e analytic in $\Re(s) \geq 0$ ($\Re(s) \leq 0$). Similarly, $\mathcal{L}_\infty(T)$ denotes the Lebesgue space of complex valued transfer functions which are essentially bounded on the unit circle with norm $\|T(z)\|_{\mathcal{H}_\infty} \triangleq \sup_\omega |T(e^{j\omega})|$, and $\mathcal{H}_\infty(T)$ ($\mathcal{H}_\infty^-(T)$) denotes the set of stable (antistable) complex functions $G(z) \in \mathcal{L}_\infty(T)$, i.e analytic in $|z| \geq 1$ ($|z| \leq 1$). $\mathcal{L}^\infty(R_+)$ denotes the space of measurable functions $f(t)$ equipped with the norm: $\|f\|_{\mathcal{L}^\infty} = ess.\sup_{R_+} |f(t)|$. $\mathcal{L}^1(R_+)$ denotes the space of measurable functions $f(t)$ equipped with the norm: $\|f\|_1 = \int_0^\infty |f(t)|dt < \infty$. The prefix $\mathcal{R}$ will be used to denote subspaces consisting of rational transfer functions. For simplicity the distinction between the different spaces $\mathcal{L}^\infty$ will be omitted in instances where it is clear from the context.

l_∞ denotes the space of bounded real sequences $h = \{h_o, h_1, \ldots\}$ equipped with the norm $\|h\|_{l_\infty} \triangleq \sup_k |h_k|$; similarly l^1 denotes the space of real sequences, equipped with the norm $\|h\|_1 = \sum_{k=0}^\infty |h_k| < \infty$. Given a sequence $h \in l_1$ (a function $h(t) \in \mathcal{L}^1$) we will denote its z–transform (Laplace transform) by $H(z)$ ($H(s)$). It is well known that $h \in l_1$ if and only if $H(z) \in \mathcal{RH}_\infty(T)$ and that $f(t) \in \mathcal{L}^1$ if and only if $F(s) \in \mathcal{RH}_\infty(j\mathcal{R})$. Given a sequence $h \in l_1$, by a slight

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abuse of notation, we will denote as $\|H(z)\|_{l_\infty} \triangleq \sup_k |h_k|$. Similarly, given $f(t) \in \mathcal{L}^1$, $\|F(s)\|_{\mathcal{L}^\infty} \triangleq ess.\sup_{R_+} |f(t)|$.

Given two transfer matrices $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ and Q with appropriate dimensions, the *lower linear fractional transformation* is defined as: $\mathcal{F}_l(T,Q) \triangleq T_{11} + T_{12}Q(I - T_{22}Q)^{-1}T_{21}$. For a discrete-time transfer matrix $G(z)$, we define its conjugate as $G^{\sim} \triangleq G'(\frac{1}{z})$, were $'$ denotes transpose. Similarly, $G^{\sim}(s) = G'(-s)$. Finally, throughout the paper we will use packed notation to represent state-space realizations, i.e.

$$G(s) = C(sI - A)^{-1}B + D \triangleq \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

2.2 Statement of the Problem

Consider the system represented by the block diagram 1, where S represents the system to be controlled; the scalar signals w, θ and u represent an exogenous disturbance, a ***known, fixed*** signal, and the control action respectively; and where ζ, ξ and y represent the outputs subject to frequency domain performance constraints, the output due to the input signal θ, and the measurements respectively. Then, the basic problem that we address in this paper is the following:

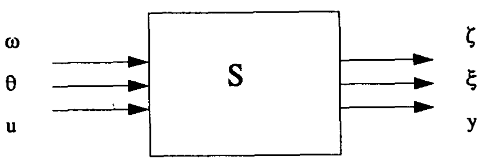

Figure 1. The Generalized Plant

- **Mixed $(\mathcal{L}^\infty/\mathcal{H}_\infty)$ Control Problem:**

Given the nominal system (S), with frequency-domain specifications of the form:

$$\|W(s)T_{\zeta w}(s)\|_{\mathcal{H}_\infty} \le \gamma \qquad (\mathcal{F})$$

where $W(s)$ is a suitable weighting function, find an internally stabilizing rational controller $u(s) = K(s)y(s)$ such that the maximum amplitude of the regulated output $\xi(t)$ due to θ is minimized subject to the specifications $(\mathcal{F})$

2.3 Problem Transformation

In this section we use the Youla parametrization to cast the problem into a convex optimization form. Assume that the system S has the following state-space realization (where without loss of generality we assume that all weighting factors have been absorbed into the plant):

$$\left(\begin{array}{c|ccc} A & B_{1f} & B_{1t} & B_2 \\ \hline C_f & D_{ff} & D_{ft} & D_{f2} \\ C_t & D_{tf} & D_{tt} & D_{t2} \\ C_2 & D_{2f} & D_{2t} & D_{22} \end{array}\right) \qquad (S)$$

where D_{f2} has full column rank, D_{2f} has full row rank, and where the pairs (A, B_2) and (C_2, A) are stabilizable and detectable respectively. It is well known (see for instance [12]) that the set of all internally stabilizing rational controllers can be parametrized in terms of a free parameter $Q \in \mathcal{RH}_\infty$ as:

$$K = \mathcal{F}_l(J, Q) \qquad (1)$$

where J has the following state-space realization:

$$\left(\begin{array}{c|cc} A + B_2F + LC_2 + LD_{22}F & -L & B_2 + LD_{22} \\ \hline F & 0 & I \\ -(C_2 + D_{22}F) & I & -D_{22} \end{array}\right) \qquad (J)$$

where F and L are selected such that $A + B_2F$ and $A + LC_2$ are stable. By using this parametrization, the closed-loop transfer functions $T_{\zeta w}$ and $T_{\xi\theta}$ can be written as:

$$\begin{aligned} T_{\zeta w} &= \mathcal{F}_l(T_f, Q) = T_{11} + T_{12}QT_{21} \\ T_{\xi\theta} &= \mathcal{F}_l(T_\theta, Q) = T_{11}^\theta + T_{12}^\theta Q T_{21}^\theta \end{aligned} \qquad (2)$$

where $T_i, T_i^\theta \in \mathcal{RH}_\infty$ †

Moreover [12], it is possible to select F and L in such a way that $T_{12}(s)$ and $T_{21}(s)$ are inner and co-inner respectively (i.e. $T_{12}^{\sim}T_{12} = I$, $T_{21}T_{21}^{\sim} = I$).

- **Remark 1:** For the SISO case, equation (2) reduces to:

$$\begin{aligned} T_{\zeta w}(s) &= T_1(s) + T_2(s)Q(s) \\ T_{\xi\theta}(s) &= T_1^\theta(s) + T_2^\theta(s)Q(s) \end{aligned} \qquad (4)$$

where T_i, T_i^θ, Q are stable transfer functions and where T_2 is inner.

By using this parametrization the mixed optimization problem can be now precisely stated as solving:

$$\mu^o = \inf_{Q \in \mathcal{RH}_\infty} \|\Xi(s)\|_{\mathcal{L}^\infty} \qquad (\mathcal{L}^\infty/\mathcal{H}_\infty)$$

subject to:

$$\|T_1(s) + T_2(s)Q(s)\|_{\mathcal{H}_\infty} = \|T_1^{\sim}(s)T_2(s) + Q\|_{\mathcal{H}_\infty} \triangleq \|R(s) + Q(s)\|_{\mathcal{H}_\infty} \le \gamma$$

where

$$\Xi(s) = T_{\xi\theta}(s)\Theta(s) = (T_1^\theta(s) + T_2^\theta(s)Q(s))\Theta(s)$$

and where $\Theta \in \mathcal{RL}^1$ ‡ is a known, fixed signal.

III. Problem Solution

In this section we present a method for finding suboptimal rational $\mathcal{L}^\infty/\mathcal{H}_\infty$ controllers, based upon the use of discrete-time $l_\infty/\mathcal{H}_\infty$ theory. The main result of this section shows that suboptimal controllers, with cost arbitrarily close to the optimum, can be found by solving a finite-dimensional convex constrained optimization problem and an unconstrained $\mathcal{H}_\infty$ problem.

† see for instance [5] for a state-space realization of T_f and T_θ.

‡ this restriction on θ can be relaxed to include steps functions, by absorbing the pole at $s = 0$ into the plant, thus forcing a controller with integral action

3.1 Definitions

• **Def. 1:** Consider the continuous time system (S). Its Euler Approximating System (EAS) is defined as the following discrete time system:

$$\left(\begin{array}{c|ccc} I+\tau A & \tau B_{1f} & \tau B_{1t} & \tau B_2 \\ \hline C_f & D_{ff} & D_{ft} & D_{f2} \\ C_t & D_{tf} & D_{tt} & D_{t2} \\ C_2 & D_{2f} & D_{2t} & D_{22} \end{array}\right) \qquad (EAS)$$

where $\tau > 0$.

• **Def. 2:** Consider the following system:

$$\dot{x}(t) = Ax(t) + Bv(t) \tag{5}$$

where $x \in R^n$ and $v(t) \in \Omega \subset R$. A set $\Sigma \subset R^n$ is a *positively invariant set* [13] of (5) if for any initial condition $x_o \in \Sigma$ and for any $v(t)$ the corresponding trajectory $x(t, x_o, v(t)) \in \Sigma$ for all t. A similar definition holds for the case of discrete–time systems.

3.2 Properties of the Euler Approximating System

In this section we introduce some properties of the EAS. The main result of this section shows that the peak values of the impulse and frequency responses of the Euler Approximating system are upper bounds of the corresponding continuous–time quantities. To establish this result we start by showing that the l_∞ norm of the impulse response of the EAS is an upper bound of the $\mathcal{L}^\infty$ norm of the impulse response of the continuous–time system (Theorem 1). Moreover, this upper bound is non–increasing with τ and converges to the exact value as $\tau \to 0$ (Lemma 2).

• **Lemma 1:** Consider the system:

$$\begin{aligned} \dot{x} &= Ax + Bv \\ \zeta &= Cx + Dv \end{aligned} \tag{6}$$

If the corresponding EAS:

$$\begin{aligned} x_{k+1} &= (I + \tau A)x_k + \tau B v_k \\ \zeta_k &= Cx_k + Dv_k \end{aligned} \tag{7}$$

is asymptotically stable, then (6) is also asymptotically stable. Conversely, if (6) is asymptotically stable, there exists $\tau_{max} > 0$ such that for all $0 < \tau \le \tau_{max}$ the EAS (7) is asymptotically stable.

Proof: Denote by Λ the set of eigenvalues of A and define $\tau_{max} \triangleq \min_{\lambda \in \Lambda} 2[\frac{-\Re(\lambda)}{|\lambda|^2}]$. The proof follows by noting that the eigenvalues of $(I + \tau A)$ are inside the unit disk if and only if (6) is stable and $0 < \tau < \tau_{max}$ ⋄.

• **Theorem 1:** Consider the stable, strictly proper system:

$$\begin{aligned} \dot{x} &= Ax + Bv \\ \zeta &= Cx \end{aligned} \tag{8}$$

and its corresponding EAS:

$$\begin{aligned} x_{k+1} &= (I + \tau A)x_k + \tau B v_k \\ \zeta_k^{EAS} &= Cx_k \end{aligned} \tag{9}$$

Let $\xi(t)$ denote the impulse response of (8) and $\xi(k,\tau)^{EAS}$ the response of (9) to the input $\{\tau^{-1}, 0, 0, ...\}$, with zero initial conditions. Then we have that:

$$\|\xi(t)\|_{\mathcal{L}^\infty} \le \|\xi(k,\tau)^{EAS}\|_{l_\infty}, \text{ for all } 0 < \tau < \tau_{max}$$

Proof: To simplify the expressions, take $k = -1$ as initial time for the EAS, so that $\xi(k,\tau)^{EAS} = Cz(k,\tau)$ and $\xi(t) = Cx(t)$, where $x(t)$ and $z(k,\tau)$ are the free state response of (8) and (9) respectively, taking the vector B as initial condition. Denote by n the dimension of A. We assume (without loss of generality) that (A, B) is a reachable pair. This is both a necessary and sufficient condition for the EAS to be reachable for all $\tau \ge 0$. The reachability of $(I + \tau A, \tau B)$ implies that the sequence $z(k,\tau)$ spans R^n. Denote by $\Sigma(\tau)$ the convex hull of the set of points $\pm z(i,\tau)$. Since $z(i,\tau), i = 0, 1, ...,$ span R^n, the set $\Sigma(\tau)$ is convex and contains the origin in its interior. If $\tau < \tau_{max}$, the EAS is stable, and so there exists k (which depends on τ) such that $z(i,\tau) \in int[\Sigma(\tau)]$ and $-z(i,\tau) \in int[\Sigma(\tau)]$, for $i \ge k$. This means that $\Sigma(\tau)$ is generated as a convex combination of finitely many points, namely is a polytope. By construction, any vertex v of $\Sigma(\tau)$ is equal to $z(i,\tau)$ for some $i < k$, so we have $[I + \tau A]v = [I + \tau A]z(i,\tau) = z(i+1,\tau) \in \Sigma(\tau)$. This implies that $\Sigma(\tau)$ is a positively invariant set for both the EAS and the system $\dot{x} = Ax$ [14]. Since $x(0) = B = z(0) \in \Sigma(\tau)$, it follows that the state impulse response $x(t)$ of $(A, B) \in \Sigma(\tau)$. Define the set

$$P(\rho) = \left\{x \in R^n : |Cx| \le \rho, \rho > 0\right\} \tag{10}$$

then

$$\begin{aligned} \|\xi(k,\tau)^{EAS}\|_{l_\infty} &= \sup_{k \ge 0} |Cz(k,\tau)| \\ &= \inf\left\{\rho \ge 0 : z(k,\tau) \in P(\rho), \text{ for all } k \ge 0\right\} \end{aligned} \tag{11}$$

Therefore, the points $\pm z(i,\tau), i \ge 0$, are in the convex set $P(\|\xi(k,\tau)^{EAS}\|_{l_\infty})$ and, since $\Sigma(\tau)$ is a polytope, $\Sigma(\tau) \subseteq P(\|\xi(k,\tau)^{EAS}\|_{l_\infty})$. On the other hand, $x(t) \in \Sigma(\tau) \subseteq P(\|\xi(k,\tau)^{EAS}\|_{l_\infty})$, and then $\sup_{t>0} |Cx(t)| = \|\xi\|_{\mathcal{L}^\infty} \le \|\xi(k,\tau)^{EAS}\|_{l_\infty}$ ⋄.

• **Lemma 2:** Consider a strictly decreasing sequence $\tau_{max} > \tau_i \to 0$, and let $\mu_i = \|\xi(k,\tau_i)^{EAS}\|_{l_\infty}$. Then the sequence μ_i is non–increasing and such that $\mu_i \to \mu^o \triangleq \|\xi\|_{\mathcal{L}^\infty}$, where ξ is the impulse response of the continuous time system.

Proof: See the Appendix.

Next we show that the $\|.\|_{\mathcal{H}_\infty}$ norm of the transfer function of the EAS provides an upper bound of the $\|.\|_{\mathcal{H}_\infty}$ norm of the transfer function of the continuous–time system.

• **Lemma 3:** Assume that (6) is asymptotically stable and consider a strictly decreasing sequence $\tau_{max} > \tau_i \to 0$. Let $T_{\zeta v}(s)$ denote the transfer function of (6) and $T_{\zeta v}^{EAS}(z,\tau_i)$ the transfer function of the EAS corresponding to τ_i. Then:

$$\begin{gathered} \sup_\omega |T_{\zeta w}(j\omega)| = \|T_{\zeta v}(s)\|_{\mathcal{H}_\infty} \le \|T_{\zeta v}^{EAS}(z,\tau_i)\|_{\mathcal{H}_\infty} = \sup_\omega |T_{\zeta v}^{EAS}(e^{j\omega},\tau_i)| \; \forall i \\ \|T_{\zeta v}^{EAS}(z,\tau_i)\|_{\mathcal{H}_\infty} \le \|T_{\zeta v}^{EAS}(z,\tau_j)\|_{\mathcal{H}_\infty} \; i > j \\ \lim_{\tau_i \to 0} \|T_{\zeta v}^{EAS}(z,\tau_i)\|_{\mathcal{H}_\infty} = \|T_{\zeta v}(s)\|_{\mathcal{H}_\infty} \end{gathered} \tag{12}$$

Proof: The proof, omitted for space reasons, follows from using the Maximum Modulus Theorem.

Combining the results of Theorem 1 and Lemmas 2 and 3 yields the main result of this section:

• **Theorem 2:** Assume that $\inf_{Q\in\mathcal{RH}_\infty} \|T_{\zeta w}(s)\|_{\mathcal{H}_\infty} = \gamma_o < \gamma$. Consider a strictly decreasing sequence $\tau_{max} > \tau_i \to 0$, and the corresponding EAS(τ_i). Let

$$\begin{aligned} \mu_i &= \inf_{\substack{Q\in\mathcal{RH}_\infty \\ \|T_{\zeta w}\|_{\mathcal{H}_\infty}\le\gamma}} \|\xi(k,\tau_i)^{EAS}\|_{l_\infty} \\ \mu_o &= \inf_{\substack{Q\in\mathcal{RH}_\infty \\ \|T_{\zeta w}\|_{\mathcal{H}_\infty}\le\gamma}} \|\xi(t)\|_{\mathcal{L}^\infty} \end{aligned} \tag{13}$$

where $\xi(k,\tau_i)^{EAS}, \xi(t)$ denote the impulse responses of the closed-loop EAS and continuous-time systems, respectively. Then the sequence μ_i is non-increasing and such that $\mu_i \to \mu^o$.

Proof: Given a controller $K(z,\tau_i)$ that internally stabilizes EAS(τ_i), let $S_{cl}(K,z,\tau_i)$ denote the corresponding closed-loop system, and $\xi(K,k,\tau_i)$ and $T_{\zeta w}(K,z,\tau_i)$ its impulse and frequency responses respectively. Assume that $K(z,\tau_i)$ is such that $\|T_{\zeta w}(K,z,\tau_i)\|_{\mathcal{H}_\infty} \le \gamma$. Given any $j > i$, consider the controller $\hat{K}(z)$ obtained from K_i using the change of variable $z \to (1 + \frac{\tau_i(z-1)}{\tau_j})$ and the corresponding closed-loop system $S_{cl}(\hat{K},z,\tau_j)$. Since $j > i$, it follows from Lemma 1 that $S_{cl}(\hat{K},z,\tau_j)$ is internally stable. Moreover, from Lemma 3 we have that:

$$\|T_{\zeta w}(\hat{K},z,\tau_j)\|_{\mathcal{H}_\infty} \le \|T_{\zeta w}(K,z,\tau_i)\|_{\mathcal{H}_\infty} \le \gamma \tag{14}$$

Hence, $\hat{K}$ is a feasible controller for EAS(τ_j). From Lemma 2 we have that:

$$\|\xi(\hat{K},k,\tau_j)\|_{l_\infty} \le \|\xi(K,k,\tau_i)\|_{l_\infty} \tag{15}$$

It follows then that

$$\mu_j = \inf_K \|\xi(K,k,\tau_j)\|_{l_\infty} \le \mu_i = \inf_K \|\xi(K,k,\tau_i)\|_{l_\infty}, \text{ for } j > i \tag{16}$$

Since μ_i is a non-increasing sequence, bounded below by μ_o, it has a limit $\hat{\mu} \ge \mu_o$. We will show that $\hat{\mu} = \mu_o$ by contradiction. Assume that $\hat{\mu} > \mu_o$ and define $\epsilon \triangleq \hat{\mu} - \mu_o$. Since $\inf_{Q\in\mathcal{RH}_\infty} \|T_1(s) + T_2(s)Q(s)\|_{\mathcal{H}_\infty} < \gamma$, there exists $Q_1 \in \mathcal{RH}_\infty$ such that $\|T_1(s) + T_2(s)Q_1(s)\|_{\mathcal{H}_\infty} = \gamma_1 < \gamma$. From the definition of μ_o it follows that there exists $Q_o \in \mathcal{RH}_\infty$ such that $\|T_1(s) + T_2(s)Q_o(s)\|_{\mathcal{H}_\infty} \le \gamma$ and $\|\xi(t)\|_{\mathcal{L}^\infty} \le \mu_o + \frac{\epsilon}{8}$. Let $\hat{Q} \triangleq Q_o + \eta(Q_1 - Q_o)$. It follows that:

$$\begin{aligned} \|T_1^\theta(s) + T_2^\theta(s)\hat{Q}(s)\|_{\mathcal{L}^\infty} &\le \mu^o + \frac{\epsilon}{8} + \eta\|T_2^\theta(s)(Q_1(s) - Q_o(s))\|_{\mathcal{L}^\infty} \\ \|T_1(s) + T_2(s)\hat{Q}(s)\|_{\mathcal{H}_\infty} &\le \eta\|T_1(s) + T_2(s)Q_1(s)\|_{\mathcal{H}_\infty} \\ &\quad + (1-\eta)\|T_1(s) + T_2(s)Q_o(s)\|_{\mathcal{H}_\infty} < \gamma \end{aligned} \tag{17}$$

Hence, by taking η small enough we have that the controller $K = \mathcal{F}_l(J,\hat{Q})$ yields $\|\xi(t)\|_{\mathcal{L}^\infty} \le \mu_o + \frac{1}{4}\epsilon$ and $\|T_{\zeta w}(s)\|_{\mathcal{H}_\infty} < \gamma$. It follows, from Lemmas 2 and 3, that for τ small enough we have:

$$\begin{aligned} \mu(\tilde{K}) &\triangleq \|\xi(\tilde{K},k,\tau)\|_{l_\infty} \le \mu_o + \frac{1}{2}\epsilon \\ \|T_{\zeta w}&(\tilde{K},z,\tau)\|_{\mathcal{H}_\infty} \le \gamma \end{aligned} \tag{18}$$

Where $\tilde{K}(z) \triangleq K(s)|_{z=1+\tau s}$. Hence $\mu(\tilde{K}) < \hat{\mu}$ which contradicts the definition of $\hat{\mu}$ ◇.

3.3 The SISO Mixed $l_\infty/\mathcal{H}_\infty$ Problem

Theorem 2 shows that the $\mathcal{L}^\infty/\mathcal{H}_\infty$ problem can be solved by solving a sequence of discrete-time $l_\infty/\mathcal{H}_\infty$ problems. In this section we briefly review the main result of $l_\infty/\mathcal{H}_\infty$ theory [5]. The discrete-time mixed $l_\infty/\mathcal{H}_\infty$ problem [5] is defined as: Given $R(z) \in \mathcal{RH}_\infty^{-}(T), T_1^\theta(z), T_2^\theta(z) \in \mathcal{RH}_\infty(T)$, find:

$$\mu^o = \inf_{Q\in\mathcal{RH}_\infty} \|\Xi\|_{l_\infty} \tag{$l_\infty/\mathcal{H}_\infty$}$$

subject to:

$$\|R(z) + Q(z)\|_{\mathcal{H}_\infty} \le \gamma$$

where

$$\Xi(z) = (T_1^\theta(z) + T_2^\theta(z)Q(z))\Theta(z)$$

and where $\Theta \in \mathcal{R}l_1$ is a known, fixed signal. Problem $l_\infty/\mathcal{H}_\infty$ problem can be though of as an optimization problem inside the origin centered γ^{-1}–ball. However, the γ^{-1}–ball is not compact in $\mathcal{H}_\infty$. Thus a minimizing solution may not exists. This difficulty is circumvented by introducing the following *modified* problem: Let $\mathcal{H}_{\infty,\delta} = \{Q(z) \in \mathcal{H}_\infty : Q(z) \text{ analytic in} |z| \ge \delta\}$. Then, given $R(z) \in \mathcal{H}_{\infty,\delta}{}^{-}$, and $\Theta(z) \in \mathcal{H}_{\infty,\delta}$, find:

$$\mu_o^\delta = \min_{Q\in\mathcal{H}_{\infty,\delta}} \|\xi\|_{l_\infty} \tag{$l_\infty/\mathcal{H}_{\infty,\delta}$}$$

subject to: $\|R(z) + Q(z)\|_{\mathcal{H}_{\infty,\delta}} \le \gamma$, where $\delta < 1$ and $\|Q\|_{\mathcal{H}_{\infty,\delta}} \triangleq \sup_{|z|=\delta} |Q(z)|$.

Next, we recall the main result of [5], showing that if $l_\infty/\mathcal{H}_{\infty,\delta}$ is feasible, it always admits a minimizing solution. Moreover, this solution is rational (i.e $Q \in \mathcal{RH}_\infty$), requires considering only a finite number of elements of the sequence $\{\xi_k\}$, and can be exactly obtained by solving a convex finite–dimensional optimization problem and a standard $\mathcal{H}_\infty$ problem.

• **Theorem 3:** Let $\mu^* \triangleq \inf_{Q\in\mathcal{RH}_\infty} \|\xi\|_{l_\infty}$ † Then, $Q^o = Q_F^o + z^{-N}Q_R^o$, where $Q_F^o = \sum_{i=0}^{N-1} q_i z^{-i}$, solves the mixed $l_\infty/\mathcal{H}_{\infty,\delta}$ problem **iff** $\underline{q}^o = (q_o \ldots q_{N-1})'$ solves the following finite dimensional convex optimization problem:

$$\underline{q}^o = \operatorname*{argmin}_{\substack{\underline{q}\in R^N \\ \|Q\|_2\le\gamma}} \left\{ \max_{0\le k\le N-1} |(\underline{t}_1 + \mathcal{T}\underline{q})_k| \right\} \tag{19}$$

and Q_R^o solves the unconstrained approximation problem

$$Q_R^o(z) = \operatorname*{argmin}_{Q_R\in\mathcal{H}_{\infty,\delta}} \|R(z) + Q_F^o + z^{-N}Q_R(z)\|_{\mathcal{H}_{\infty,\delta}} \tag{20}$$

where:

$$\begin{aligned} \underline{t}_1 &\triangleq (t_{1_o} \ \ \ldots \ \ t_{1N-1})' \\ \mathcal{T} &= \begin{pmatrix} t_{2_o}^r & 0 & \ldots & 0 \\ t_{2_1}^r & t_{2_0}^r & \ldots & 0 \\ \vdots & & \ddots & \\ t_{2N-1}^r & \ldots & & t_{2_o}^r \end{pmatrix} \\ \underline{q} &\triangleq (q_o \ \ \ldots \ \ q_{N-1})' \end{aligned} \tag{21}$$

t_{ik} denotes the k^{th} element of the impulse response of $T_i^\theta(z)\Theta(z)$ (i.e. $T_i^\theta(z)\Theta(z) = \sum_0^\infty t_{ik} z^{-k}$); N is selected such that:

$$\delta^N \le \frac{\mu^*}{\left(\|T_1^\theta(z)\|_{\mathcal{H}_{\infty,\delta}} + \|T_2^\theta(z)\|_{\mathcal{H}_{\infty,\delta}}\left(\gamma + \|R(z)\|_{\mathcal{H}_{\infty,\delta}}\right)\right)\|\Theta(z)\|_{\mathcal{H}_{\infty,\delta}}}$$

and:

$$G = R^{-} \triangleq \left(\begin{array}{c|c} A_G & b_G \\ \hline c_G & d_G \end{array}\right)$$

$$Q = \begin{pmatrix} yA_G^N x & yA_G^{N-1}b_G & \ldots & \ldots & yA_Gb_G & yb \\ c_GA_G^{N-1}x & c_GA_G^{N-2}b & \ldots & \ldots & c_Gb_G & d_G + q_0 \\ c_GA_G^{N-2}x & c_GA_G^{N-3}b & \ldots & \ldots & d_G + q_0 & q_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_GA_Gx & c_Gb_G & d_G + q_0 & \ldots & q_{N-3} & q_{N-2} \\ c_Gx & d_G + q_0 & q_1 & \ldots & q_{N-2} & q_{N-1} \end{pmatrix} \tag{22}$$

† μ^*, the unconstrained l_∞ optimum, can be found by solving a linear programming problem [15]

where $X > 0$ and $Y > 0$ are the discrete controllability and observability grammians of G and where x and y are the positive square roots of X and Y respectively.

Proof: The proof follows from combining Lemma 1 in [5] with the corollary to Theorem 3 in [4].

• **Remark 2:** By using the transformation $z = \delta\hat{z}$ we have that:

$$\|R(z) + Q_F^o(z) + z^{-N}Q_R(z)\|_{\mathcal{H}_{\infty,\delta}} = \|R(\delta\hat{z}) + Q_F^o(\delta\hat{z}) + \delta^{-N}\hat{z}^{-N}Q_R(\delta\hat{z})\|_{\mathcal{H}_\infty} \triangleq \|\hat{R}(\hat{z}) + \hat{Q}_F^o(\hat{z}) + \hat{z}^{-N}\hat{Q}_R(\hat{z})\|_{\mathcal{H}_\infty} = \|\hat{z}^N(\hat{R}(\hat{z}) + \hat{Q}_F^o(\hat{z})) + Q_R(\hat{z})\|_{\mathcal{H}_\infty}$$

where we used the fact that $\hat{z}^N$ is inner in $\mathcal{H}_\infty$. It follows that the approximation problem (20) is equivalent to the following unconstrained Nehari approximation problem:

$$\hat{Q}_R^o = \underset{Q_R \in \mathcal{RH}_\infty}{\operatorname{argmin}} \|\hat{z}^N(\hat{R} + \hat{Q}_F^o) + Q_R\|_{\mathcal{H}_\infty} \tag{23}$$

Hence, the mixed optimization problem can be solved by using the following algorithm: i) Use the transformation $z = \delta\hat{z}$ to map the δ–disk to the unit disk; ii) solve the convex finite dimensional optimization (19); iii) solve the unconstrained Nehari approximation problem (23); iv) use the transformation $\hat{z} = \delta^{-1}z$ to obtain the controller and the closed–loop system.

3.4 Proposed Design Method

From the definition of the EAS it is easily seen that the closed–loop transfer function obtained by applying the rational controller $K(s)$ to (S) is the same as the closed–loop transfer function obtained by applying the controller $K(\frac{z-1}{\tau})$ to the EAS, up to the complex transformation $z = \tau s + 1$. Therefore, if a rational compensator $K(z)$ yielding an $l_\infty/\mathcal{H}_\infty$ cost μ is found for the EAS, then $K(\tau s + 1)$ internally stabilizes (S) and yields an $\mathcal{L}^\infty/\mathcal{H}_\infty$ cost $\mu_c \leq \mu_d$. It follows that a rational compensator can be synthesized using the EAS with a suitably small τ. These observations are formalized in the following lemma:

• **Lemma 4:** Consider the mixed $\mathcal{L}^\infty/\mathcal{H}_\infty$ control problem for SISO continuous time–systems. A suboptimal rational solution can be obtained by solving a discrete–time mixed $l_\infty/\mathcal{H}_{\infty,\delta}$ control problem for the corresponding EAS, with $\delta = 1 - \tau^2$. Moreover, if $K(z)$ denotes the $l_\infty/\mathcal{H}_\infty$ controller for the EAS, the suboptimal $\mathcal{L}^\infty/\mathcal{H}_\infty$ controller is given by $K(\tau s + 1)$.

Proof: Since $\theta(t) \in \mathcal{RL}^1$, it can be modelled as the impulse response of a stable, strictly proper system, $G_\theta(s)$. Therefore, without loss of generality, we can assume (by absorbing G_θ into (S)) that the input to the system (S) is an impulse. Consider now the system (S) and its corresponding EAS. From Lemmas 1,2 and 3 it follows that if a compensator $K(z)$ yields a $l_\infty/\mathcal{H}_\infty$ cost μ_d for the EAS, the compensator $K(s)|_{z=1+\tau s}$ internally stabilizes (S) and yields an $\mathcal{L}^\infty/\mathcal{H}_\infty$ cost $\mu_c \leq \mu$. Assume that $\tau < 1$ and let $\delta = 1 - \tau^2$. Since the solution to the modified problem $l_\infty/\mathcal{H}_{\infty,\delta}$ is analytic outside the disk of radius δ, it follows from the maximum modulus theorem [16] that any solution to $l_\infty/\mathcal{H}_{\infty,\delta}$ is an upper bound to $l_\infty/\mathcal{H}_\infty$. Let $K_\tau(z)$ be the solution to $l_\infty/\mathcal{H}_{\infty,\delta}$ obtained using Theorem 3 and let μ_τ be the corresponding l_∞ cost. It follows that $\|\xi(t)\|_{\mathcal{L}^\infty} \leq \mu_d \leq \mu_\tau \diamond$.

Finally, we show that by taking $\tau \to 0$, the proposed design method yields controllers with cost arbitrarily close to the optimal $\mathcal{L}^\infty/\mathcal{H}_\infty$ cost.

• **Theorem 4:** Let $\tau_{max} > \tau_i \to 0$ be a strictly decreasing sequence. Denote by K_i the controller obtained using the design procedure of Lemma 4 with $\tau = \tau_i$; by $S_{cl}(K_i)$ the corresponding closed loop system; and by $\xi_i(t)$ its impulse response. Then the sequence $\mu_i \triangleq \|\xi_i(t)\|_{\mathcal{L}^\infty}$ is non–increasing and such that $\lim_{i\to\infty} \mu_i = \mu_o$.

Proof: The proof, omitted for space reasons, follows along the lines of the proof of Theorem 2.

IV. A Simple Example

Consider the problem of minimizing the step response tracking error for the non–minimum phase plant:

$$G(s) = \frac{s-1}{s-2}$$

subject to the robust stability condition $\|T_{cl}\|_\infty \leq \gamma$, where T_{cl} denotes the closed–loop complementary sensitivity and γ^{-1} is the desired robustness level. In order to achieve zero steady–state error, an additional pole at $s = 0$ is added to the controller. Table 1 shows a comparison of $\|e\|_{\mathcal{L}^\infty}$ and $\|T_{\zeta w}\|_{\mathcal{H}_\infty}$ for an $\mathcal{H}_\infty$ and a mixed $\mathcal{L}^\infty/\mathcal{H}_\infty$ design. Since the plant has a pole at $s = 0$, $\mathcal{H}_\infty$ is not directly applicable. This difficulty was solved by shifting the $j\omega$–axis using a bilinear transformation [17], selected to yield settling time $t_s \sim 100$ seconds, and then designing a controller with $\gamma = 5$. This results in a second order controller yielding an actual $\|T_{cl}\|_{\mathcal{H}_\infty} = 4.47$ (since, as shown in [17] this procedure yields an upper bound on the true $\mathcal{H}_\infty$–norm.) The mixed $\mathcal{L}^\infty/\mathcal{H}_\infty$ design corresponds to the value of $\tau = 0.08$ (selected to yield approximately the same settling time). This controller yields a 38% reduction of the peak tracking error at the cost of a 5% increase in $\|T_{cl}\|_\infty$. Although in principle the proposed design procedure required a controller with order 100, we were able to reduce it to order 10, with virtually no performance loss.

	$\|T_{\zeta w}\|_{\mathcal{H}_\infty}$	$\|e\|_{\mathcal{L}^\infty}$
$\mathcal{H}_\infty$	4.47	4.82
$\mathcal{L}^\infty/\mathcal{H}_\infty$	4.72	3.01

Table 1. $\|T_{\zeta w}\|_{\mathcal{H}_\infty}$ vs $\|e\|_{l_\infty}$ for the Example

V. Discussion and Conclusions

In this paper we address the problem of finding internally stabilizing controllers that minimize the peak amplitude of the output to a fixed given input, subject to robustness constraints given in the form of an $\mathcal{H}_\infty$ constraint upon the norm of a relevant transfer function. This problem is of importance for example for tracking applications, or cases where either the control action or some outputs are subject to hard bounds.

The main result of the paper shows that the resulting convex optimization problem can be decoupled into a finite dimensional, albeit non–differentiable, constrained optimization and an unconstrained Nehari approximation problem. This is a notorious departure from previous approaches to solving this types of problems [18–19], where several approximations where required in order to obtain a tractable mathematical problem.

Although here we considered only the simpler case of a SISO system, the theoretical results showing that the impulse and frequency response of the EAS are upper bounds of the corresponding continuous–time quantities are also valid in the MIMO case. Hence the proposed design procedure can be applied to MIMO systems by using an embedding procedure to deal with the $\mathcal{H}_\infty$ constraint, as proposed in [20].

The example of section 4 highlights the strengths of our approach, and also suggest future research topics. Namely, the method allows for dealing *explicitly and exactly* with time–domain specifications, eliminating multiple (and non necessarily convergent) trial and error type iterations. This will usually result in an improved and less costly design. However, in its present status, the method usually produces very complex controllers, necessitating some type of model reduction. Application of some well established methods in order reduction (noteworthy, weighted balanced truncation) usually succeed in producing controllers of manageable order. Furthermore, consistent numerical experience suggests that this order reduction can be accomplished with virtually no performance degradation. Research is currently under way addressing this issue and pursuing the extension of the formalism to allow more control on the shape of the error response.

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Appendix: Proof of Lemma 2

We first show that if $\tau_1 < \tau_2$, then:

$$S(\tau_1) \subseteq S(\tau_2)$$

This is an immediate consequence of the facts that $S(\tau_2)$ is a positively invariant set for the EAS, and $z(k+1,\tau) = [I + \tau A]z(k,\tau)$. Indeed, if v is a vertex of $S(\tau_2)$, then $w = [I + \tau_2 A]v \in S(\tau_2)$ and, for $\tau < \tau_2$, the point $[I + \tau A]v$ belongs to the interior of the segment vw. Hence $[I + \tau A]v \in S(\tau_2)$. It follows [14] that $S(\tau_2)$ is positively invariant for the EAS with such a τ. Thus, for $\tau_1 < \tau_2$, $S(\tau_1) \subseteq S(\tau_2)$. From (10) and (11), it follows that $\|\xi^{EAS}_{\tau_1}\|_{l_\infty} \leq \|\xi^{EAS}_{\tau_2}\|_{l_\infty}$. Let now $\delta > 0$ be given. Since A is a stable matrix, the free response of $\dot{x} = Ax(t)$, with initial condition B is such that there exists t_1 such that $\|x(t)\| \leq \delta/2$, for $t \geq t_1$. Take now m such that $\tau_m = \tau_1/m < \tau_{max}$ and define $\tau_h = \tau_1/h, h \geq m$. Since the trajectory $x(t)$ is continuously differentiable, the solution of the EAS, $z(k+1,\tau) = [I + \tau_h A]z(k,\tau_h)$ uniformly converges on the finite horizon $[0, t_1]$ to the sampled values of $x(t)$ [21]. Hence there exists $H \geq m$, such that

$$\|z(k,\tau_h) - x(k\tau_h)\| \leq \delta/2, k = 1,..,h, \text{ for all } h \geq H \qquad (A1)$$

Therefore, for any arbitrary δ, there exists τ_1 and H such that for $h \geq H$

$$\|z(h,\tau_h)\| = \|z(h,\tau_h) - x(t_1) + x(t_1)\| \leq \|z(h,\tau_h) - x(t_1)\| + \|x(t_1)\| \leq \delta \quad (A2)$$

Take now μ such that $\mu S(\tau_m) \subseteq P(\|\xi\|_{\mathcal{L}^\infty})$. Since $\mu S(\tau_m)$ contains the origin in its interior, we can choose t_1 and $H \geq m$ such that for $h \geq H$, $z(h,\tau_h) \in \mu S(\tau_m)$. Moreover, since the set $S(\tau_m)$ is positively invariant for the system $z(k+1,\tau_h) = [I + \tau_h A]z(k,\tau_h), h \geq m$, by linearity $\mu S(\tau_m)$ has the same property. So for $k \geq h, z(k,\tau_h) \in \mu S(\tau_m) \subseteq P(\|\xi\|_{\mathcal{L}^\infty})$, and thus $\sup_{k \geq h} |Cz(k,\tau_h)| \leq \|\xi\|_{\mathcal{L}^\infty}$. To complete the proof, we have to show that that for each $\epsilon > 0$, we can take $h \geq H$ in such a way that the property $|Cz(k,\tau_h)| \leq \|\xi\|_{\mathcal{L}^\infty} + \epsilon$, holds over the finite horizon $0 \leq k \leq h$. We have that

$$\begin{aligned} |Cz(k,\tau_h)| &= |Cz(k,\tau_h) - Cx(k\tau_h) + Cx(k\tau_h)| \\ &\leq |Cz(k,\tau_h) - Cx(k\tau_h)| + |Cx(k\tau_h)| \\ &\leq \alpha\|z(k,\tau_h) - x(k\tau_h)\| + \|\xi\|_{\mathcal{L}^\infty} \end{aligned} \qquad (A3)$$

for $0 \leq k \leq h$, where α is a positive constant depending only on C and on the particular norm selected for R^n. According to $(A1)$, there exists h such that $\|z(k,\tau_h) - x(k\tau_h)\| \leq \epsilon/\alpha, 1 \leq k \leq h$. Therefore $|Cz(k,\tau_h)| \leq \|\xi\| + \epsilon, 0 \leq k \leq h$. This implies that $\|\xi^{EAS}_{\tau_h}\|_{l_\infty} \leq \|\xi\|_{\mathcal{L}^\infty} + \epsilon$. The proof is completed by recalling that, from Theorem 1, $\|\xi^{EAS}_{\tau}\|_{l_\infty}$ is bounded below by $\|\xi\|_{\mathcal{L}^\infty}$ and then $\|\xi\|_{\mathcal{L}^\infty} \leq \|\xi^{EAS}_{\tau_h}\|_{l_\infty} \leq \|\xi\|_{\mathcal{L}^\infty} + \epsilon$, for $0 < \tau < \tau_h$ ◦.