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Mixed $l_1/\mathcal{H}_\infty$ Controllers for SISO Discrete Time Systems

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Abstract

A successful controller design paradigm must take into account both model uncertainty and design specifications. Model uncertainty can be addressed using either $\mathcal{H}_\infty$ or l_1 robust control theory, depending upon the uncertainty characterization. However, these frameworks cannot accommodate the realistic case where the design specifications include both time and frequency domain constraints. In this paper we address these problems using a mixed $l_1/\mathcal{H}_\infty$ approach. This approach allows for minimizing the worst-case peak output due to a persistent disturbance, while, at the same time, satisfying an $\mathcal{H}_\infty$-norm constraint upon some closed-loop transfer function of interest. The main result of the paper shows that mixed $l_1/\mathcal{H}_\infty$ optimal controllers can be obtained by solving a sequence of problems, each one consisting of a finite-dimensional convex optimization and a standard, unconstrained $\mathcal{H}_\infty$ problem.

I. Introduction

A large number of control problems involve designing a controller capable of stabilizing a given linear time invariant system while minimizing the worst case response to some exogenous disturbances. This problem is relevant for instance for disturbance rejection, tracking and robustness to model uncertainty (see [11] and references therein). When the exogenous disturbances are modeled as bounded energy signals and performance is measured in terms of the energy of the output, this problem leads to the well known $\mathcal{H}_\infty$ theory. Since its introduction, the original formulation of Zames [13] has been substantially simplified, resulting in efficient computational schemes for finding solutions. The $\mathcal{H}_\infty$ framework, combined with μ-analysis [4] has been successfully applied to a number of hard practical control problems (see for instance [7]). However, in spite of this success, it is clear that plain $\mathcal{H}_\infty$ control can only address a subset of the common performance requirements since, being a frequency domain method, it can not address time domain specifications. Recently, methodologies incorporating some classes of time domain constraints into the $\mathcal{H}_\infty$ formalism have been developed [8–10]. However, in its present form these techniques allow only for shaping the response to a given, fixed input.

The case where the signals involved are persistent bounded signals leads to the l_1 optimal control theory, formulated and further explored by Vidyasagar [11–12] and solved by Dahleh and Pearson both in the discrete [2] and continuous time [3] cases. These methods are attractive since they allow for an explicit solution to the robust performance problem [6]. However, they cannot accommodate some common classes of frequency domain specifications (such as $\mathcal{H}_2$ or $\mathcal{H}_\infty$ bounds).

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In this paper we propose a method for designing mixed $l_1/\mathcal{H}_\infty$ controllers. These controllers allow for minimizing the l_1 norm of the closed-loop transfer function between an input-ouput pair of signals, while at the same time satisfying an $\mathcal{H}_\infty$ norm constraint upon the transfer function between a different pair of signals. Our approach resembles that of Boyd et. al. [1] in the sense that we use the Youla parametrization to cast the problem into a semi-infinite convex optimization form. However, in a significant departure from [1], where several approximations where used in order to obtain a tractable mathematical problem, we use the special structure of the problem to find a *global* solution. The main result of the paper shows that this solution can be found by solving a sequence of modified problems, each one entailing solving of a *finite dimensional* convex, constrained optimization problem and an unconstrained $\mathcal{H}_\infty$ problem. Moreover, the proposed solution method yields, at each stage, a feasible controller (in the sense of satisfying the $\mathcal{H}_\infty$ constraint) that provides an upper bound on the optimal l_1 cost.

The paper is organized as follows: In section II we introduce the notation to be used and some preliminary results. Section III contains the proposed solution method. In section IV we present a simple design example. Finally, in section V, we summarize our results and we indicate directions for future research.

II. Preliminaries

2.1 Notation

l_∞ denotes the space of bounded real sequences $q = \{q_k\}$ equipped with the norm $\|q\|_{l_\infty} \triangleq \sup_k |q_k|$. l_1 denotes the space of real sequences, equipped with the norm $\|q\|_1 = \sum_{k=0}^{\infty} |q_k| < \infty$. $\mathcal{L}_\infty$ denotes the Lebesgue space of complex valued transfer functions which are essentially bounded on the unit circle with norm $\|T(z)\|_\infty \triangleq \sup_{|z|=1} |T(z)|$. $\mathcal{H}_\infty$ ($\mathcal{H}_\infty^-$) denotes the set of stable (antistable) complex functions $G(z) \in \mathcal{L}_\infty$, i.e. analytic in $|z| \geq 1$ ($|z| \leq 1$). The prefix $\mathcal{R}$ denotes real rational transfer matrices. Given $R \in \mathcal{L}_\infty$, $\Gamma_H(R)$ denotes its maximum Hankel singular value. Given a sequence $q \in l_1$ we will denote its z-transform by $Q(z)$. It is a standard result that $q \in l_1$ iff $Q(z) \in \mathcal{RH}$. Throughout the paper we will use packed notation to represent state-space realizations, i.e.

$$G(z) = C(zI - A)^{-1}B + D \triangleq \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

Given two transfer matrices $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ and Q with appropriate dimensions, the lower *linear fractional transformation* is defined as:

$$\mathcal{F}_l(T,Q) \triangleq T_{11} + T_{12}Q(I - T_{22}Q)^{-1}T_{21}$$

Finally, for a transfer matrix $G(z)$, $G^{\sim} \triangleq G'(\frac{1}{z})$, where $'$ denotes transpose.

2.2 Statement of the Problem

Consider the system represented by the block diagram 1, where S represents the system to be controlled; the scalar signals w_∞ (a bounded energy signal), w_1 (a persistent l_∞ signal) and u represent exogenous disturbances and the control action respectively; and ζ_∞, ζ_1 and y represent the regulated outputs and the measurements respectively. Then, the mixed $l_1/\mathcal{H}_\infty$ control problem can be stated as: Given the nominal system (S), find an internally stabilizing controller

$$u(z) = K(z)y(z) \tag{C}$$

such that worst case peak amplitude of the performance output $\|\zeta_1\|_\infty$ due to signals inside the l_∞-unity ball is minimized, subject to the constraint $\|T_{\zeta_\infty w_\infty}\|_\infty \le \gamma$.

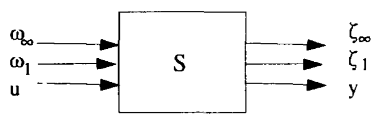

Figure 1. The Plant

2.3 Problem Transformation

Assume that the system S has the following state-space realization (where without loss of generality we assume that all weighting factors have been absorbed into the plant):

$$\left(\begin{array}{c|ccc} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ C_3 & D_{31} & D_{32} & D_{33} \end{array}\right) \tag{S}$$

where D_{13} has full column rank, D_{31} has full row rank, and where the pairs (A, B_3) and (C_3, A) are stabilizable and detectable respectively. It is well known (see for instance [14]) that the set of all internally stabilizing controllers can be parametrized in terms of a free parameter $Q \in \mathcal{H}_\infty$ as:

$$K = \mathcal{F}_l(J,Q) \tag{1}$$

where J has the following state-space realization:

$$\left(\begin{array}{c|cc} A + B_3F + LC_3 + LD_{33}F & -L & B_3 + LD_{33} \\ \hline F & 0 & I \\ -(C_3 + D_{33}F) & I & -D_{33} \end{array}\right) \tag{J}$$

and where F and L are selected such that $A + B_3F$ and $A + LC_3$ are stable. By using this parametrization, the scalar closed-loop transfer functions $T_{\zeta_\infty w_\infty}$ and $T_{\zeta_1 w_1}$ can be written as:

$$\begin{aligned} T_{\zeta_\infty w_\infty}(z) &= T_1^\infty(z) + T_2^\infty(z)Q(z) \\ T_{\zeta_1 w_1}(z) &= T_1(z) + T_2(z)Q(z) \end{aligned} \tag{2}$$

where T_i, T_i^∞, Q are stable transfer functions. Moreover (see [10, 14]), it is possible to select F and L in such a way that $T_2^\infty(z)$ is inner (i.e. $T_2^{\infty\sim}T_2^\infty = I$). By using this parametrization the mixed $l_1/\mathcal{H}_\infty$ problem can be now precisely stated as solving:

$$\mu^o = \inf_{Q \in \mathcal{RH}_\infty} \|T_{\zeta_1 w_1}\|_1 = \inf_{q \in l^1} \sum_{i=0}^{\infty} |t_i| \tag{$l_1/\mathcal{H}_\infty$}$$

subject to:

$$\|T_1^\infty(z) + T_2^\infty(z)Q(z)\|_\infty \le \gamma \tag{3}$$

where $\{t_i\}$ and $\{q_i\}$ are the coefficients of the impulse responses of $T_{\zeta_1 w_1}$ and Q respectively.

III. Problem Solution

In this section we show that the mixed $l_1/\mathcal{H}_\infty$ problem can be solved by solving a sequence of problems, each one requiring the solution of a finite dimensional convex optimization problem and an unconstrained $\mathcal{H}_\infty$ problem.

3.1 A Modified $l_1/\mathcal{H}_\infty$ Problem

Since all the solutions to a suboptimal Nehari extension problem of the form $\|R + Q\|_\infty \le \gamma$ can be parametrized in terms of a free parameter $W(z) \in \mathcal{RH}_\infty, \|W\|_\infty \le \gamma^{-1}$ problem $l_1/\mathcal{H}_\infty$ can be thought of as an optimization problem inside the origin centered γ^{-1}-ball. However, even though the space $\mathcal{H}_\infty$ is complete, it is easily seen that the γ-ball is not compact. Thus a minimizing solution may not exist. Motivated by this difficulty, we introduce the following *modified* mixed $l_1/\mathcal{H}_\infty$ problem. Let $\mathcal{H}_{\infty,\delta} = \{Q(z) \in \mathcal{H}_\infty : Q(z) \text{ analytic in} |z| \ge \delta\}$ and define the $l_1/\mathcal{H}_{\infty,\delta}$ problem as follows: Given $T_1(z), T_2(z), T_1^\infty(z), T_2^\infty(z) \in \mathcal{RH}_{\infty,\delta}$, find

$$\mu_\delta^o = \inf_{Q \in \mathcal{H}_{\infty,\delta}} \|T_{\zeta_1 w_1}\|_1 \tag{$l_1/\mathcal{H}_{\infty,\delta}$}$$

subject to:

$$\|T_1^\infty(z) + T_2^\infty(z)Q(z)\|_{\infty,\delta} \le \gamma$$

where $\delta < 1$ and $\|Q\|_{\infty,\delta} \triangleq \sup_{|z|=\delta} |Q(z)|$.

Remark 1: From the maximum modulus theorem, it follows that any solution Q to $l_1/\mathcal{H}_{\infty,\delta}$ is an admissible solution for $l_1/\mathcal{H}_\infty$. It follows that μ_δ^o is an upper bound for μ^o.

Remark 2: Problem $l_1/\mathcal{H}_{\infty,\delta}$ can be thought as solving problem $l_1/\mathcal{H}_\infty$ with the additional constraint that all the poles of the closed-loop system must be inside the disk of radius δ. A parametrization of all achievable closed-loop transfer functions, such that T, T^∞ satisfy this additional constraint can be obtained from (1) by simply changing the stability region from the unit-disk to the δ-disk using the transformation $z = \delta\hat{z}$ before performing the factorization. Furthermore, by combining this transformation with the inner–coinner factorization, the resulting $T_2^{\infty}(z)$ satisfies $T_2^{\infty}(\delta z)T_2^{\infty}(\frac{1}{\delta z}) = 1$.

Next we show that a suboptimal solution to $l_1/\mathcal{H}_\infty$, with cost arbitrarily close to the optimum, can be found by solving a sequence of truncated problems, each one requiring consideration of only a *finite* number of elements of the impulse response of $T_{\zeta_1 w_1}$. To establish this result we will show that: i) $l_1/\mathcal{H}_\infty$ can be solved by considering a sequence of modified problems $l_1/\mathcal{H}_{\infty,\delta}$. ii) Given $\epsilon > 0$, a suboptimal solution to $l_1/\mathcal{H}_{\infty,\delta}$ with cost no greater than $\mu_\delta^o + \epsilon$ can be found by solving a truncated problem.

- **Lemma 1:** Consider an increasing sequence $\delta_i \to 1$. Let μ^o and μ_i denote the solution to problems $l_1/\mathcal{H}_\infty$ and $l_1/\mathcal{H}_{\infty,\delta_i}$ respectively and assume that $\Gamma_H(T_2^{\infty\sim}T_1^{\infty}) < \gamma$. Then the sequence $\mu_i \to \mu^o$.

Proof: From the maximum modulus theorem it follows that the solution Q_i to $l_1/\mathcal{H}_{\infty,\delta_i}$ is a feasible solution for $l_1/\mathcal{H}_{\infty,\delta_{i+1}}$. Thus, the sequence μ_i is non–increasing, bounded below by the value of $\|T_{\zeta_1 w_1}\|_1$ obtained when using the optimal l_1 controller. It follows then that it has a limit $\mu \geq \mu^o$. We will show next that $\mu = \mu^o$. Assume by contradiction that $\mu^o < \mu$ and select $\mu^o < \hat{\mu} < \mu$. Let $R \triangleq T_2^{\infty\sim}T_1^{\infty}$. Since $\Gamma_H(R) < \gamma$, it follows that there exists $Q_1 \in \mathcal{RH}_\infty$ such that $\|R + Q_1\|_\infty = \Gamma_H(R) < \gamma$. From the definition of μ^o it follows that, given $\eta > 0$, there exists $Q_o \in \mathcal{RH}_\infty$, $\|R + Q_o\|_\infty \leq \gamma$, such that $\|T_{\zeta_1 w_1}(Q_o)\|_1 \leq \mu^o + \eta$. Let $\hat{Q} \triangleq Q_o + \epsilon(Q_1 - Q_o)$. It follows that:

$$\begin{aligned}\|T_{\zeta_1 w_1}(\hat{Q})\|_1 &\leq \mu^o + \eta + \epsilon\|T_2(Q_1 - Q_o)\|_1 \\ \|R + \hat{Q}\|_\infty &\leq \epsilon\|R + Q_1\|_\infty + (1-\epsilon)\|R + Q_o\|_\infty < \gamma\end{aligned}$$

Since $\hat{Q} \in \mathcal{RH}_\infty$ it follows that there exists $\delta_1 < 1$ such that $T_1^{\infty} + T_2^{\infty}\hat{Q}$ is analytic in $|z| \geq \delta_1$. Since $\|T_1^{\infty} + T_2^{\infty}\hat{Q}\|_\infty < \gamma$, it follows from continuity that there exists $\delta_2 < 1$ such that $\|T_1^{\infty} + T_2^{\infty}\hat{Q}\|_{\infty,\delta_2} \leq \gamma$. Therefore, by taking ϵ and η small enough and $\delta \triangleq \max\left\{\delta_1, \delta_2\right\} < 1$ we have that $\|T_1^{\infty} + T_2^{\infty}\hat{Q}\|_{\infty,\delta} \leq \gamma$ and $\|T_{\zeta_1 w_1}(\hat{Q})\|_1 < \hat{\mu}$. Hence for $\delta_i \geq \delta$, $\mu_i \leq \hat{\mu}$. However, this contradicts the fact that the sequence μ_i is non–increasing and that $\hat{\mu} < \mu = \lim_{\delta_i \to 1} \mu_i$ ◇.

Next we show show that, given $\epsilon > 0$, a suboptimal solution to $l_1/\mathcal{H}_{\infty,\delta}$, with cost μ_δ^ϵ such that $\mu_\delta^o \leq \mu_\delta^\epsilon \leq \mu_\delta^o + \epsilon$ can be found by solving a truncated problem.

- **Lemma 2:** Let $\epsilon > 0$ be given. Then, there exists $N(\epsilon, \delta)$ such that if $Q \in \mathcal{H}_{\infty,\delta}$ satisfies the constraint $\|R + Q\|_{\infty,\delta} \leq \gamma$ then it also satisfies $\sum_{i=N}^{\infty} |t_k| \leq \epsilon$, where t_k denote the coefficients of the impulse response of $T_{\zeta_1 w_1} = T_1 + T_2 Q$.

Proof: Since $Q \in \mathcal{H}_{\infty,\delta}$, $T_{\zeta_1 w_1}$ is analytic in $|z| \geq \delta$ and:

$$t_k = \frac{1}{2\pi j}\oint_{|z|=\delta} T_{\zeta_1 w_1}(z) z^{k-1} dz \tag{4}$$

Hence

$$\begin{aligned}|t_k| &\leq \|T_{\zeta_1 w_1}\|_{\infty,\delta}\delta^k \\ \sum_{i=N}^{\infty} |t_k| &\leq \frac{\|T_{\zeta_1 w_1}\|_{\infty,\delta}\delta^N}{1-\delta}\end{aligned} \tag{5}$$

If Q satisfies $\|R + Q\|_{\infty,\delta} \leq \gamma$, since $\|.\|_{\infty,\delta}$ is submultiplicative, we have:

$$\begin{aligned}\|T_{\zeta_1 w_1}(z)\|_{\infty,\delta} &\leq \|T_1\|_{\infty,\delta} + \|T_2\|_{\infty,\delta}\|Q\|_{\infty,\delta} \\ &\leq \|T_1\|_{\infty,\delta} + \|T_2\|_{\infty,\delta}(\gamma + \|R\|_{\infty,\delta}) \triangleq K\end{aligned} \tag{6}$$

The desired result follows by selecting $N \geq N_o = \frac{\log \epsilon(1-\delta) - \log K}{\log \delta}$ ◇

- **Lemma 3:** Consider the following optimization problem:

$$\min_{Q \in \mathcal{H}_{\infty,\delta}} \sum_{i=0}^{N-1} |t_i| = \|\underline{t}_1 + \tau \underline{q}\|_1 \qquad (l_1/\mathcal{H}^\epsilon_{\infty,\delta})$$

subject to:

$$\|R + Q\|_{\infty,\delta} \leq \gamma$$

where:

$$\begin{aligned}\underline{t}_1 &\triangleq (t_{1_o} \ \ \dots \ \ t_{1N-1})' \\ \tau &= \begin{pmatrix} t_{2_o} & 0 & \dots & 0 \\ t_{21} & t_{20} & \dots & 0 \\ \vdots & & \ddots & \\ t_{2N-1} & \dots & & t_{2_o} \end{pmatrix} \\ \underline{q}^o &\triangleq (q_o \ \ \dots \ \ q_{N-1})'\end{aligned} \tag{7}$$

and where q_k, t_{k_i} denote the k^{th} element of the impulse response of $Q(z), T_i(z)$ respectively. Let Q^* and $T^*_{\zeta_1 w_1}$ denote the optimal solution and define $\mu_\delta^\epsilon = \|T^*_{\zeta_1 w_1}\|_1$. Then $\mu_\delta^o \leq \mu_\delta^\epsilon \leq \mu_\delta^o + \epsilon$

Proof: $\mu_\delta^o \leq \mu_\delta^\epsilon$ is immediate from the definition of μ_δ^o. Denote by $T^\epsilon_{\zeta_1 w_1}$ and $T^\delta_{\zeta_1 w_1}$ the solution to problems $l_1/\mathcal{H}^\epsilon_{\infty,\delta}$ and $l_1/\mathcal{H}_{\infty,\delta}$ respectively and let t_i^ϵ, t_i^δ be the corresponding impulse responses. Then:

$$\begin{aligned}\mu_\delta^\epsilon = \|T^\epsilon_{\zeta_1 w_1}\|_1 &= \sum_{i=0}^{\infty} |t_i^\epsilon| \\ &= \sum_{i=0}^{N-1} |t_i^\epsilon| + \sum_{i=N}^{\infty} |t_i^\epsilon| \\ &\leq \sum_{i=0}^{N-1} |t_i^\delta| + \epsilon \leq \sum_{i=0}^{\infty} |t_i^\delta| + \epsilon = \mu_\delta^o + \epsilon \qquad \diamond\end{aligned}$$

By combining the results of Lemmas 1, 2 and 3, the following result is now apparent:

- **Lemma 4:** Consider an increasing sequence $\delta_i \to 1$. Let μ^o and $\mu^\epsilon_{\delta_i}$ denote the solution to problems $l_1/\mathcal{H}_\infty$ and $l_1/\mathcal{H}^\epsilon_{\infty,\delta_i}$ respectively. Then the sequence $\mu^\epsilon_{\delta_i}$ has an accumulation point $\hat{\mu}_\epsilon$ such that $\mu^o \leq \hat{\mu}_\epsilon \leq \mu^o + \epsilon$.

3.2 The $\mathcal{H}_\infty$ Performance Constraint

In the last section we showed that $l_1/\mathcal{H}_\infty$ can be solved by solving a sequence of truncated problems. In principle these problems have the form of a semi–infinite optimization problem, and can be approximately solved by discretizing the unit–circle and applying outer approximation methods (see [5]). In this section we show that each problem $l_1/\mathcal{H}^\epsilon_{\infty,\delta}$ can be *exactly* solved by solving a finite dimensional convex optimization problem and an unconstrained $\mathcal{H}_\infty$ problem. Moreover, since the solution to this $\mathcal{H}_\infty$ approximation problem is rational, it follows that the solution to $l_1/\mathcal{H}^\epsilon_{\infty,\delta}$ is also rational. To establish this result, we recall first a result on constrained Nehari approximation problems:

- **Theorem 1:** Let $R \in \mathcal{RH}_\infty^-$ and $Q_F = \sum_{i=0}^{N-1} q_i z^{-i}$ be given. Then there exist $Q_R \in \mathcal{RH}_\infty$, such that $\|R + Q_F + z^{-N} Q_R\|_\infty \leq \gamma$, **iff** $\|\mathcal{Q}\|_2 \leq \gamma$ where:

$$
\mathcal{Q} = \begin{pmatrix} yA_G^N x & yA_G^{N-1} b_G & \dots & \dots & yA_G b_G & yb \\ c_G A_G^{N-1} x & c_G A_G^{N-2} b & \dots & \dots & c_G b_G & d_G + q_0 \\ c_G A_G^{N-2} x & c_G A_G^{N-3} b & \dots & \dots & d_G + q_0 & q_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_G A_G x & c_G b_G & d_G + q_0 & \dots & q_{N-3} & q_{N-2} \\ c_G x & d_G + q_0 & q_1 & \dots & q_{N-2} & q_{N-1} \end{pmatrix}
$$

$$
G = R^{-} \triangleq \left(\begin{array}{c|c} A_G & b_G \\ \hline c_G & d_G \end{array} \right)
$$

$$
x = L_c^{\frac{1}{2}}
$$

$$
y = L_o^{\frac{1}{2}} \tag{8}
$$

and where L_c and L_o are the discrete controllability and observability grammians of G, i.e they satisfy the discrete time Lyapunov equations $L_c = A_G L_c A_G' + b_G b_G'$ and $L_o = A_G' L_o A_G + c_G' c_G$.

Proof: See Theorem 2 in [10] or the corollary to Theorem 3 in [8].

Combining Lemma 3 and Theorem 1 yields the main result of this section:

- **Theorem 2:** A suboptimal solution to $l_1/\mathcal{H}_{\infty,\delta}$, with cost $\mu_\delta \leq \mu_\delta^\epsilon \leq \mu_\delta + \epsilon$ is given by $Q^o = Q_F^o + z^{-N} Q_R^o$ where $Q_F^o = \sum_{i=0}^{N-1} q_i z^{-i}$, $\underline{q}^o = (q_o \dots q_{N-1})'$ solves the following finite dimensional convex optimization problem:

$$
\underline{q}^o = \operatorname*{argmin}_{\substack{\underline{q} \in R^N \\ \|\mathcal{Q}\|_2 \leq \gamma}} \|\underline{t}_1 + \tau \underline{q}\|_1 \tag{9}
$$

and Q_R^o solves the unconstrained approximation problem

$$
\begin{aligned} Q_R^o(z) &= \operatorname*{argmin}_{Q_R \in \mathcal{H}_{\infty,\delta}} \|R(z) + Q_F^o + z^{-N} Q_R(z)\|_{\infty,\delta} \\ &= \operatorname*{argmin}_{Q_R \in \mathcal{RH}_{\infty,\delta}} \|R(z) + Q_F^o + z^{-N} Q_R(z)\|_{\infty,\delta} \end{aligned} \tag{10}
$$

where $R = T_2^{\infty\sim} T_1^{\infty}$, $\underline{t}_1, \tau$ are defined in (7) and N is selected according to Lemma 2.

Remark 3: By using the transformation $z = \delta \hat{z}$ we have that:

$$
\begin{aligned} \|R(z) + Q_F^o(z) + z^{-N} Q_R(z)\|_{\infty,\delta} &= \|R(\delta\hat{z}) + Q_F^o(\delta\hat{z}) + \delta^{-N} \hat{z}^{-N} Q_R(\delta\hat{z})\|_\infty \\ &\triangleq \|\hat{R}(\hat{z}) + \hat{Q}_F^o(\hat{z}) + \hat{z}^{-N} \hat{Q}_R(\hat{z})\|_\infty \\ &= \|\hat{z}^N(\hat{R}(\hat{z}) + \hat{Q}_F^o(\hat{z})) + Q_R(\hat{z})\|_\infty \end{aligned}
$$

where we used the fact that $\hat{z}^N$ is inner in $\mathcal{H}_\infty$. It follows that the approximation problem (10) is equivalent to the following unconstrained Nehari approximation problem:

$$
\hat{Q}_R^o = \operatorname*{argmin}_{Q_R \in \mathcal{RH}_\infty} \|\hat{z}^N(\hat{R} + \hat{Q}_F^o) + Q_R\|_\infty \tag{11}
$$

3.3 Synthesis Algorithm

Combining Theorem 2 and Lemma 4, it follows that a suboptimal solution to $l_1/\mathcal{H}_\infty$, with cost arbitrarily close to the optimum, can be found using the following iterative algorithm.

0) *Data:* An increasing sequence $\delta_i \to 1, \epsilon > 0, \nu > 0$.

1) Solve the unconstrained l_1 problem (using the standard l_1 theory [2]). Compute $\|T_{\zeta_\infty w_\infty}\|_\infty$. If $\|T_{\zeta_\infty w_\infty}\|_\infty \leq \gamma$ stop, else set $i = 1$.

2) For each i, find a suboptimal solution to problem $l_1/\mathcal{H}_{\delta_i}$ proceeding as follows:

   2.1) Let $z = \delta_i \hat{z}$ and consider the system $S(\hat{z})$

   2.2) Obtain $T_i(\hat{z}), T_i^\infty(\hat{z})$ using the Youla parametrization (1).

   2.3) Compute N from Lemma 2.

   2.4) Find $\hat{Q}(\hat{z})$ using Theorem 2.

3) Let $Q = \hat{Q}(\frac{z}{\delta_i}), K = F_l(J, Q)$. Compute $\|T_{\zeta_\infty w_\infty}(z)\|_\infty$. If $\|T_{\zeta_\infty w_\infty}(z)\|_\infty \geq \gamma - \nu$ stop, else set $i = i + 1$ and go to 2.

Remark 4: At each stage the algorithm produces a feasible solution to $l_1/\mathcal{H}_\infty$, with cost μ_i which is an upper bound of the optimal cost μ^o.

IV. A Simple Example

Consider the plant used in [2]:

$$
P(z) = \frac{2z}{z^2 + 2}
$$

and assume that the objective is to design a compensator K to minimize $\|T\|_1 = \|PK(1+PK)^{-1}\|_1$ subject to the constraint $\|S\|_\infty = \|(1+PK)^{-1}\|_\infty \leq \gamma^\ddagger$. [‡] It is shown in [2] that the optimal l_1 controller is: $K_1 = \frac{-1}{z}$ yielding $\|T\|_1 = 2$ and $\|S\|_\infty = 3$. Table 1 shows a comparison of the optimal l_1, optimal $\mathcal{H}_\infty$ and a mixed $l_1/\mathcal{H}_\infty$ controller (corresponding to $\gamma = 2.4$). The corresponding impulse and frequency domain response are shown in Figure 2.

	$\|T\|_1$	$\|S\|_\infty$
l_1	2	3
$l_1/\mathcal{H}_\infty$	2.38	2.4
$\mathcal{H}_\infty$	3	2

Table 1. $\|T\|_1$ and $\|S\|_\infty$ for the example

The mixed $l_1/\mathcal{H}_\infty$ design requires a 44^{th} order controller, since it can be shown that, for $\epsilon = 0.001$, we need to consider only 20 elements of the impulse response. Noteworthy, using model reduction techniques, we were able to obtain a second order controller, with virtually no performance loss. The state–space realization of this non–minimum phase second order controller is given by:

$$
K(z) = \left(\begin{array}{cc|c} 0.3606 & -0.0353 & 1 \\ 1 & 0 & 0 \\ \hline -0.8704 & 0.3119 & -0.014 \end{array} \right)
$$

‡ The optimal $\mathcal{H}_\infty$ controller, $K(z) = \frac{-0.75}{z}$, yields $\|S\|_\infty = 2$. Thus γ should be ≥ 2

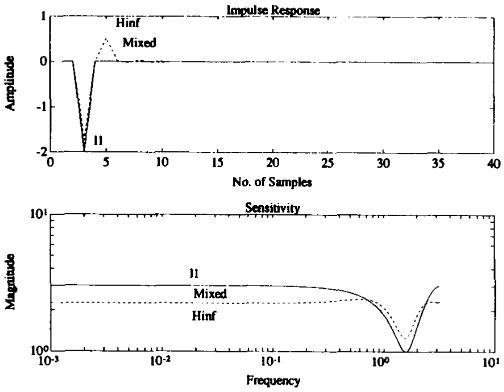

Figure 2. Impulse and Frequency Responses for the l_1, $l_1/\mathcal{H}_\infty$, and $\mathcal{H}_\infty$ Controllers

V. Conclusions

In this paper we present a method for designing discrete-time mixed $l_1/\mathcal{H}_\infty$ controllers. These controllers allow for minimizing the worst case output to persistent bounded excitations, while at the same time satisfying a constraint upon the $\mathcal{H}_\infty$ norm of the transfer function between a different pair of signals. Thus, they can be thought of as achieving robust stability subject to a nominal performance specification. Although here we considered only the simpler case of a SISO system, the proposed design procedure can be easily extended to MIMO systems by using an embedding procedure to deal with the $\mathcal{H}_\infty$ constraint, as proposed in [9].

Perhaps the most severe limitation of the proposed method is that may result in very large order controllers (roughly $2N$), necessitating some type of model reduction. Note however that this disadvantage is shared by some widely used design methods, such as μ-synthesis or l_1 optimal control theory, that will also produce controllers with no guaranteed complexity bound. Application of some well established methods in order reduction (noteworthy, weighted balanced truncation) usually succeed in producing controllers of manageable order. The example of section 4 suggests that substantial order reduction can be accomplished without performance degradation. Research is currently under way addressing this issue.

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