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Robust Control of Systems under Mixed Time/Frequency Domain Constraints via Convex Optimization

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Abstract

A successful controller design paradigm must take into account both model uncertainty and design specifications. Model uncertainty can be addressed using the $\mathcal{H}_\infty$ or l_1 robust control theory. However, these frameworks cannot accommodate the realistic case where the design specifications include both time and frequency domain constraints. In this paper we propose an approach that takes explicitly into account both mixed time/frequency domain constraints and model uncertainty. This is achieved by minimizing a set-induced operator norm, subject to additional frequency domain performance specifications such as bounds on the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm of relevant transfer functions. We show that this formulation results in a convex optimization problem that can be exactly solved. Thus, the conservatism inherent in some previous approaches is eliminated.

1. Introduction

A large number of control problems require designing a controller capable of achieving acceptable performance under system uncertainty and design specifications, usually including both time and frequency domain constraints. However, despite its practical importance, this problem remains to a large extent still open, even in the simpler case where the system under consideration is linear.

The problem of controlling linear systems under time domain constraints has been solved only in the rather idealized case where the dynamics are completely known (see for instance [1-2] and references therein). Clearly such an assumption can be too restrictive, resulting in controllers that are seldom suitable for real-world applications.

During the last decade a large research effort has led to procedures for designing robust controllers capable of achieving desirable properties under various classes of model uncertainty. In particular, a powerful framework has been developed, addressing the issues of robust stability and robust performance in the presence of norm-bound perturbations by minimizing an $\mathcal{H}_\infty$ bound [3]. The $\mathcal{H}_\infty$ framework, combined with μ-analysis [4] (in order to exploit the structure of the uncertainty) has been successfully applied to a number of hard practical control problems (see for instance [5]). However, in spite of this success, it is clear that plain $\mathcal{H}_\infty$ control can only address a subset of the common performance requirements since, being a frequency domain method, it can not address time domain specifications. Recently some progress has been made in this direction [6-7]. However, most of the proposed methods rely on a number of approximations, and this may preclude finding a solution if the design specifications are tight. In [8-9] time-domain constraints over a finite horizon are incorporated into an $\mathcal{H}_\infty$ optimal control problem which is then exactly solved. However, at this stage constraints over an infinite horizon can be handled only indirectly.

A different approach to robust control has been pursued in [10-11], where robustness and disturbance rejection are approached using the l_1 optimal control theory introduced by Vidyasagar [10] and developed by Pearson and coworkers [11]. These methods are attractive since they allow for an explicit solution to the robust performance problem. However, they cannot accommodate some common classes of frequency domain specifications (such as $\mathcal{H}_2$ or $\mathcal{H}_\infty$ bounds).

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We recently proposed [12-13] to approach time-domain constrained systems using an operator norm-theoretic approach. In this framework, robustness against model uncertainty and satisfaction of time-domain constraints are achieved by minimizing a set induced operator norm, subject to additional frequency domain constraints. In this paper we generalize the framework of [12-13] by eliminating some of the approximations used there and by considering the more general case of output feedback controllers. The main result of the paper shows that for the case of $\mathcal{H}_2$ or $\mathcal{H}_\infty$ constraints, the resulting optimization problem can be cast into a finite-dimensional convex optimization form. This approach eliminates most of the conservatism inherent in previously proposed methods.

2. Preliminaries

2.1 Notation

By l_1 we denote the space of real sequences $\{q_i\}$, equipped with the norm $\|q\|_1 = \sum_{k=0}^{\infty} |q_k| < \infty$. Given a sequence $q \in l_1$ we will denote its Z-transform by $Q(z)$. $\mathcal{L}_\infty$ denotes the Lebesgue space of complex valued transfer matrices which are essentially bounded on the unit circle with norm $\|T(z)\|_\infty \stackrel{\Delta}{=} \sup_{|z|=1} \sigma_{max}(T(z))$. $\mathcal{H}_\infty$ ($\mathcal{H}_\infty^-$) denotes the set of stable (antistable) complex matrices $G(z) \in \mathcal{L}_\infty$, i.e. analytic in $|z| \geq 1$ ($|z| \leq 1$). $\mathcal{H}_2$ denotes the space of complex matrices square integrable in the unit circle and analytic in $|z| > 1$, equipped with the norm:

$$\|G\|_2^2 = \frac{1}{2\pi} \oint_{|z|=1} Trace\{G(z)'G(z)\} z dz$$

where $'$ indicates transpose conjugate. The prefix $\mathcal{R}$ denotes real rational transfer matrices. Throughout the paper we will use packed notation to represent state-space realizations, i.e.

$$G(z) = C(zI - A)^{-1}B + D \stackrel{\Delta}{=} \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

Given two transfer matrices $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ and Q with appropriate dimensions, the lower *linear fractional transformation* is defined as:

$$\mathcal{F}_l(T, Q) \stackrel{\Delta}{=} T_{11} + T_{12}Q(I - T_{22}Q)^{-1}T_{21}$$

For a transfer matrix $G(z)$, $G^{\sim} \stackrel{\Delta}{=} G'(\frac{1}{z})$. Finally, $\underline{x}$ indicates that x is a vector quantity.

2.2 Definitions and Preliminary Results

• **Def. 1:** Consider the linear, discrete time, autonomous system modeled by the difference equation:

$$\underline{x}_k = \phi_k \underline{x}_o,\ k = 0, 1 \ldots\ \phi_0 = I_n \qquad (S^a)$$

subject to the constraint $\underline{x} \in \mathcal{G} \subset R^n$, where $\mathcal{G}$ is a compact, convex, balanced set containing the origin. The system (S^a) is *Constrained Stable* if for *any* point $\underline{\bar{x}} \in \mathcal{G}$, the trajectory $\underline{x}_k(\underline{\bar{x}})$ originating at $\underline{\bar{x}}$ remains in $\mathcal{G}$ for all k.

• **Def. 2:** Consider the family of linear discrete time systems modeled by the difference equation:

$$\underline{x}_k = \phi_{k\Delta}\underline{x}_o, \; k = 0,1 \qquad (S^a_\Delta)$$

where ϕ_{k_Δ} belongs to some family $\mathcal{P} \subset R^{n*n}$ described by the parameter Δ which takes values in a set $\mathcal{D}$. The system (S^a) is *Robustly Constrained Stable* with respect to the family $\mathcal{P}$ if every element of $\mathcal{P}$ is constrained–stable.

• **Def. 3:** The *Minkowsky Functional* p of a convex, balanced, set $\mathcal{G}$ containing the origin in its interior is defined by

$$p(\underline{x}) = \inf_{r>0}\left\{r: \frac{\underline{x}}{r} \in \mathcal{G}\right\} \tag{1}$$

A well known result in functional analysis (see for instance [14]) establishes that p defines a seminorm in R^n. Furthermore, when $\mathcal{G}$ is compact, this seminorm becomes a norm. In the sequel we will denote this norm as $\|\underline{x}\|_\mathcal{G} \stackrel{\Delta}{=} p(\underline{x})$ and we will use its properties to establish some of the key results in the paper. Of particular importance are the facts that induced norms are submultiplicative and that all finite dimensional matrix norms are equivalent [15]. The $\|.\|_\mathcal{G}$ also provides a connection with Lyapunov theory. As we show in the next lemma, (S^a) is constrained–stable iff ϕ is a contraction in this norm.

• **Lemma 1:** Consider the system (S^a). Let $\phi \stackrel{\Delta}{=} \{\phi_k\}$ and denote by $\|.\|_\mathcal{G}$ the operator norm induced in R^{n*n} by $\mathcal{G}$, i.e. $\|A\|_\mathcal{G} = \sup_{\|\underline{x}\|_\mathcal{G}=1} \frac{\|A\underline{x}\|_\mathcal{G}}{\|\underline{x}\|_\mathcal{G}}$. Finally, let $\|\phi\|_\mathcal{G} \stackrel{\Delta}{=} \sup_k \|\phi_k\|_\mathcal{G}$. Then the system (S^a) is constrained stable *iff* $\|\phi\|_\mathcal{G} \le 1$

Proof: The proof follows immediately from Def. 1 by noting that:

$$\begin{aligned}\|\phi\|_\mathcal{G} \le 1 &\iff \|\phi_k\|_\mathcal{G} \le 1 \; \forall \; k \\ &\iff \|\phi_k\bar{\underline{x}}\|_\mathcal{G} \le 1 \; \forall \; k, \; \|\bar{\underline{x}}\|_\mathcal{G} \le 1 \\ &\iff \|\underline{x}_k(\bar{\underline{x}})\|_\mathcal{G} \le 1 \forall \bar{\underline{x}} \in \mathcal{G} \iff \underline{x}_k(\bar{\underline{x}}) \in \mathcal{G} \forall \; k\end{aligned} \tag{2}$$

where $\underline{x}_k(\bar{\underline{x}})$ denotes the trajectory that originates in $\bar{\underline{x}}$ $\diamond$.

Remark 1: For the special case of systems subject to additive parametric uncertainty:

$$\underline{x}_{k+1} = (A + \Delta)\underline{x}_k \tag{3}$$

where Δ is constant, it can be easily shown that the condition $\|\phi\|_\mathcal{G} \le 1$ reduces to $\|A + \Delta\|_\mathcal{G} \le 1$, the necessary and sufficient condition derived in [16].

Remark 2: The operator norm defined in Lemma 1 can be extended to $\mathcal{RH}_\infty^{n*n}$ as follows: Let $\Phi(z) \in \mathcal{RH}_\infty^{n*n}$ and let $\{\phi_k\} = Z^{-1}\{\Phi(z)\}$. Then we can define:

$$\|\Phi(z)\|_\mathcal{G} \stackrel{\Delta}{=} \sup_k \|\phi_k\|_\mathcal{G} \tag{4}$$

Note that since $\Phi(z) \in \mathcal{RH}_\infty^{n*n}$, $\{\phi_i\} \in l_1^{n*n}$. Since $\|\phi_k\|_1$ is uniformly bounded with respect to k, it follows from the equivalence of all finite–dimensional matrix norms [15] that $\|\phi_k\|_\mathcal{G}$ is also uniformly bounded, hence $\|\Phi\|_\mathcal{G}$ is finite.

2.3 The Uncertainty Model

In this paper we will consider systems subject to *unstructured* multiplicative dynamic model uncertainty. Thus, if we denote by $\Phi^o(z)$ the z–transform of (S^a), then the family of systems under consideration will be modeled as:

$$\begin{aligned}\mathcal{P}_\delta &= \{\Phi(z): \Phi(z) = \Phi^o(z)(I_n + \Delta), \; \Delta \in \mathcal{D}_\delta\} \\ \mathcal{D}_\delta &= \{\Delta \in \mathcal{RH}_\infty^{n*n}: \|\Delta\|_{\mathcal{G},1} \le \delta\}\end{aligned} \tag{5}$$

where

$$\|\Delta\|_{\mathcal{G},1} \stackrel{\Delta}{=} \sum_{i=0}^{\infty} \|\Delta_i\|_\mathcal{G} \tag{6}$$

Note that since $\Delta \in \mathcal{RH}_\infty^{n*n}$, $\{\Delta_i\} \in l_1^{n*n}$. Hence $\|\Delta\|_{\mathcal{G},1}$ is finite.

In section 2.5 we will show that the uncertainty description (5) includes as a special case the additive parametric uncertainty (3).

2.4 The Mixed Performance Robust Control Problem:

Consider the LTI system represented by the following state–space realization:

$$\begin{aligned}\underline{x}_{k+1} &= A\underline{x}_k + B_1\underline{\omega}_k + B_2\underline{u}_k \\ \underline{\zeta}_k &= C_1\underline{x}_k + D_{12}\underline{u}_k \\ \underline{y}_k &= C_2\underline{x}_k + D_{21}\underline{\omega}_k\end{aligned} \qquad (S)$$

subject to the constraint:

$$\underline{x}_k \in \mathcal{G} \subset R^n$$

where the pairs (A, B_2) and (A, C_2) are stabilizable and detectable respectively, D_{12} has full column rank, D_{21} has full row rank, $\underline{x} \in R^n$ represents the state; $\underline{\zeta} \in R^q$ represents the variables subject to performance specifications; $\underline{y} \in R^p$ represents the outputs available to the controller, $\underline{u} \in R^m$ represents the control input; and where $\underline{\omega} \in R^r$ contains other external inputs of interest such as disturbances or commands. Then, the basic problem that we address in this paper is the following: Given the nominal system (S) subject to model uncertainty of the form (5), with additional frequency–domain performance specifications of the form:

$$\|W(z)T_{\zeta\omega}(z)\|_* \le 1 \qquad (P)$$

where $*$ indicates either $\mathcal{H}_2$ or $\mathcal{H}_\infty$, and $W(z)$ is a suitable weighting function, find a *dynamic output-feedback controller:*

$$\begin{aligned}\hat{\underline{x}}_{k+1} &= F\hat{\underline{x}}_k + G\underline{y}_k \\ \underline{u}_k &= H\hat{\underline{x}}_k + E\underline{y}_k\end{aligned} \quad \hat{\underline{x}}_o = 0 \qquad (C)$$

such that the resulting closed–loop system is robustly constrained stable (i.e. for all members of the family (5) $\underline{x}_k \in \mathcal{G}$ for any initial condition $\underline{x}_o \in \mathcal{G}$) and satisfies the performance specifications (P)

2.5 Constrained Stability Analysis

In this section we consider constrained stability in the presence of model uncertainty and we introduce a constrained robustness measure. We begin by deriving a bound on the $\|.\|_\mathcal{G}$ of the dynamics Φ for all the elements of the family $\mathcal{P}_\delta$ and showing that this bound is tight.

• **Theorem 1:** Consider the family of systems $\mathcal{P}_\delta$ (5). Then:

$$\|\Phi\|_\mathcal{G} \le \|\Phi^o\|_\mathcal{G}(1+\delta) \tag{7}$$

and there exist at least one $\hat{\Phi} \in \mathcal{P}_\delta$ such that (7) is an equality.

Proof: Let $\Psi(z) = \Phi(z)\Delta(z)$. Then:

$$\begin{aligned}\|\Psi_k\|_\mathcal{G} &= \|\sum_{i=0}^{k} \phi_i\Delta_{k-i}\|_\mathcal{G} \le \sum_{i=0}^{k}\|\phi_i\|_\mathcal{G}\|\Delta_{k-i}\|_\mathcal{G} \\ &\le (\sup_k \|\phi_k\|_\mathcal{G})\sum_{i=0}^{\infty}\|\Delta_i\|_\mathcal{G} = \|\Phi\|_\mathcal{G}\|\Delta\|_{\mathcal{G},1}\end{aligned} \tag{8}$$

From (8) it follows that:

$$\|\Phi\|_\mathcal{G} = \|\Phi^o(I + \Delta)\|_\mathcal{G} \le \|\Phi^o\|_\mathcal{G}(1 + \|\Delta\|_{\mathcal{G},1})$$

Finally, let $\hat{\Delta} \stackrel{\Delta}{=} \delta I_n$. Then $\hat{\Phi} = \Phi^o(I_n + \hat{\Delta}) \in \mathcal{P}$ and $\|\hat{\Phi}\|_\mathcal{G} = \|\Phi^o\|_\mathcal{G}(1 + \delta)$ $\diamond$.

Remark 3: The uncertainty description (5) includes as a special case systems subject to additive parametric uncertainty (3) in the sense that if there exist $\tilde{\Delta}$, $\|\tilde{\Delta}\|_\mathcal{G} \le \delta$ such that $\|A + \tilde{\Delta}\|_\mathcal{G} = 1$ then $\Delta_o = \frac{\tilde{\Delta}}{z} \in \mathcal{D}_\delta$ and it can be easily shown that $\|\Phi(z)\|_\mathcal{G} = \|\Phi^o(z)(I + \Delta_o(z))\|_\mathcal{G} = 1$.

Corollary: The family of systems described by (5) is constraint stable *iff*

$$\delta \le \frac{1 - \|\Phi^o\|_\mathcal{G}}{\|\Phi^o\|_\mathcal{G}}$$

This result can be used to define a quantitative measure of robustness in terms of the "size" of the smallest destabilizing perturbation as follows:

• **Def. 4:** Consider the system (S^a). The *constrained stability measure*, ϱ is defined as:

$$\varrho \triangleq 1 - \|\Phi\|_{\mathcal{G}}$$

From Theorem 1 it follows that the family of systems described by (5) is constraint stable iff $\delta \leq \frac{\varrho}{1-\varrho}$. Thus a larger value of ϱ indicates a system capable of accommodating larger model uncertainty.

2.6 Effect of Disturbances

In the previous section we defined a measure of stability in terms of the smallest destabilizing model perturbation and we showed that in order to maximize this measure, $\|\Phi\|_{\mathcal{G}}$ should be minimized. This analysis neglected the effect of the disturbances $\underline{\omega}$ on the states $\underline{x}$. In this section we consider this effect and we show that it can be minimized by minimizing Φ. It follows then that achieving min $\|\Phi\|_{\mathcal{G}}$ is desirable both in terms of maximizing robustness against model uncertainty *and* minimizing the effects of perturbations.

• **Lemma 2:** Let $\underline{x}(z) = T_{x\hat{\omega}}(z)\underline{\hat{\omega}}(z)$, where $\underline{\hat{\omega}} \triangleq B_1\underline{\omega}$ and $\underline{\omega} \in l_1$. Then

$$\|\underline{x}(z)\|_{\mathcal{G}} \triangleq \sup_k \|\underline{x}_k\|_{\mathcal{G}} \leq \|T_{x\hat{\omega}}\|_{\mathcal{G}} \|\underline{\hat{\omega}}\|_{1,\mathcal{G}}$$

where:

$$\|\underline{\hat{\omega}}\|_{1,\mathcal{G}} \triangleq \sum_{k=1}^{\infty} \|\underline{\hat{\omega}}_k\|_{\mathcal{G}}$$

Proof: The proof is similar to the proof of Theorem 1 ⋄.

• **Lemma 3:** Consider the system:

$$\underline{x}_{k+1} = A\underline{x}_k + \underline{\hat{\omega}}_k \qquad (S_w)$$

with initial condition $\underline{x}_o$. Then $\|T_{x\hat{\omega}}\|_{\mathcal{G}} = \|T_{xx_o}\|_{\mathcal{G}} \triangleq \|\Phi\|_{\mathcal{G}}$.

Proof: The system (S_w) is equivalent to:

$$\begin{aligned} \underline{\tilde{x}}_{k+1} &= A\underline{\tilde{x}}_k + \underline{\hat{\omega}}_k + A\delta_{k,o}\underline{x}_o \\ \underline{x}_k &= \underline{\tilde{x}}_k + \delta_{ko}\underline{x}_o \end{aligned}$$

with initial condition $\underline{\tilde{x}}_o = 0$. Taking z transforms yields:

$$\underline{x}(z) = (I - \frac{A}{z})^{-1}\underline{x}_o + (zI - A)^{-1}\underline{\hat{\omega}} \qquad (9)$$

Hence we have $T_{x\hat{\omega}}(z) = \frac{1}{z}\Phi(z)$ and, by taking inverse z transforms, $(T_{x\hat{\omega}})_k = \phi_{k-1}$. It follows that $\|T_{x\hat{\omega}}\|_{\mathcal{G}} = \sup_k \|(T_{x\hat{\omega}})_k\|_{\mathcal{G}} = \|\Phi\|_{\mathcal{G}}$ ⋄.

Remark 4: From Lemmas 2 and 3 it follows that by minimizing $\|\Phi\|_{\mathcal{G}}$ (and hence maximizing the constrained robustness measure), we are also minimizing the effects of the disturbances $\underline{\omega}$ upon the states.

3. Robust Constrained Control Synthesis

From sections 2.5 and 2.6 it follows that a robust controller guaranteeing satisfaction of the state constraints in the presence of model uncertainty can be obtained by maximizing the constrained stability measure. In [16] we showed that for the simpler case of static full-state feedback and uncertainty limited to a conic set this approach yields well-behaved optimization problems. However, in most cases maximizing the robustness measure does not necessarily guarantee a design that meets desirable specifications. Moreover, good performance and good robustness properties are usually conflicting design objectives which must be traded-off. Hence, a better design can be achieved by selecting a set of specifications and then using the extra degrees of freedom that may be available in the problem to maximize the robustness measure over the set of all controllers that satisfy the given specifications for the nominal plant. Thus, the design problem will have the general form of a non-differentiable constrained minimization problem. In this section we show that: i) with an appropriate parametrization of all the achievable closed-loop maps, this optimization problem is convex and ii) For the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ cases the structure of the problem can be used to cast it into a finite dimensional convex optimization.

3.1 Problem Transformation

The system (S) is equivalent to:

$$\begin{aligned} \underline{\tilde{x}}_{k+1} &= A\underline{\tilde{x}}_k + B_1\underline{\omega}_k + B_2\underline{u}_k + A\delta_{k,o}\underline{x}_o \\ \underline{x}_k &= \underline{\tilde{x}}_k + \delta_{k,o}\underline{x}_o \\ \tilde{z}_k &= C_1\underline{x}_k + C_1\delta_{k,o}\underline{x}_o + D_{12}\underline{u}_k \\ \underline{y}_k &= C_2\underline{x}_k + C_2\delta_{k,o}\underline{x}_o + D_{21}\underline{\omega}_k \end{aligned} \qquad (S_o)$$

with initial condition $\underline{\tilde{x}}_o = 0$. It can be easily shown (see for instance [17]) that if the pairs (A, B_2) and (A, C_2) are stabilizable and detectable respectively, then the set of all internally stabilizing controllers and all achievable closed-loop transfer functions can be parametrized in terms of a free parameter $\tilde{Q} \in \mathcal{RH}_\infty$ respectively as:

$$\begin{aligned} K &= \mathcal{F}_l(J, \tilde{Q}) \\ T_{\underline{x}\underline{x}_o}(z) &\triangleq \Phi(z) = \mathcal{F}_l(T_x, \tilde{Q}) = T_{11}^x + T_{12}^x\tilde{Q}T_{21}^x \\ T_{\zeta\underline{\omega}}(z) &= \mathcal{F}_l(T_f, \tilde{Q}) = T_{11}^f + T_{12}^f\tilde{Q}T_{21}^f \end{aligned}$$

where J, T_x and T_f have the following state-space realizations:

$$J = \left(\begin{array}{c|cc} A + B_2F + LC_2 & -L & B_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array}\right)$$

$$T_x = \left(\begin{array}{cc|cc} A_F & -B_2F & A & B_2 \\ 0 & A_L & A_L & 0 \\ \hline I & 0 & I & 0 \\ 0 & C_2 & C_2 & 0 \end{array}\right) \qquad (10)$$

$$T_f = \left(\begin{array}{cc|cc} A_F & -B_2F & B_1 & B_2 \\ 0 & A_L & B_1 + LD_{21} & 0 \\ \hline C_1 + D_{12}F & -D_{12}F & 0 & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array}\right)$$

$$\begin{aligned} A_F &= A + B_2F \\ A_L &= A + LC_2 \end{aligned}$$

where F and L are such that A_F and A_L are stable. A suitable choice for F and L is provided in the next lemma.

• **Lemma 4:** Let

$$\begin{aligned} F &\triangleq -(R + B_2'XB_2)^{-1}(D_{12}'C_1 + B_2'XA) \\ L &\triangleq -(D_{21}B_1' + C_2YA')'(S + C_2YC_2')^{-1} \end{aligned} \qquad (11)$$

where $R \triangleq D_{12}'D_{12}$, $S \triangleq D_{21}D_{21}'$ and X and Y are the unique positive solutions to the following Riccati equations:

$$\begin{aligned} &-(B_2'XA + D_{12}'C_1)'(D_{12}'D_{12} + B_2'XB_2)^{-1}(B_2'XA + D_{12}'C_1) \\ &+ A'XA - X + C_1'C_1 = 0 \end{aligned} \qquad (12)$$

$$\begin{aligned} &-(C_2YA' + D_{21}B_1')'(D_{21}D_{21}' + C_2YC_2')^{-1}(C_2YA' + D_{21}B_1') \\ &AYA' - Y + B_1B_1' = 0 \end{aligned} \qquad (13)$$

Then F stabilizes the pair (A, B_2), L' stabilizes the pair (A', C_2') and $T_{\zeta w} = G_cB_1 - NFG_f + UR_B^{\frac{1}{2}}\tilde{Q}R_L^{\frac{1}{2}}V$, where:

$$
\begin{aligned}
G_c &= \left(\begin{array}{c|c} A_F & I \\ \hline C_1 + D_{12}F & 0 \end{array}\right) \\
G_f &= \left(\begin{array}{c|c} A_L & B_1 + LD_{21} \\ \hline I & 0 \end{array}\right) \\
R_B &= D_{12}'D_{12} + B_2'XB_2 \\
R_L &= D_{21}D_{21}' + C_2YC_2' \\
U &= NR_B^{\frac{-1}{2}} \\
V &= R_L^{\frac{-1}{2}}M \\
N &= \left(\begin{array}{c|c} A_F & B_2 \\ \hline C_1 + D_{12}F & D_{12} \end{array}\right) \\
M &= \left(\begin{array}{c|c} A_L & B_1 + LD_{21} \\ \hline C_2 & D_{21} \end{array}\right)
\end{aligned}
\tag{14}
$$

Moreover, the following properties hold: i) U and V are *inner* and *co-inner* respectively, i.e. $U^{\sim}U = I, VV^{\sim} = I$; ii) G_cB_1 is orthogonal to N and G_f is orthogonal to M.

Proof: The proof follows from standard state–space manipulations and is omitted for space reasons.

3.2 Robust Constrained Stability Optimization

In this section we show that, in the absence of performance constraints, maximum constrained robustness is achieved by *constant* state feedback. Thus in this case the optimally robust controller can be found by using the simple design procedure proposed in [16]. We will use this result to show that solving the mixed performance robust control problem requires considering only a finite number of inequalities.

• **Lemma 5:** Assume that there exists F such that $\|A_F\|_{\mathcal{G}} < 1$. Then:

$$\min_{K(z)\ \text{stab}} \|\Phi(z)\|_{\mathcal{G}} = \min_F \|A + B_2F\|_{\mathcal{G}} \tag{15}$$

Proof: From (10) we have that:

$$\Phi(z) = T_{11}^x + T_{12}^x\tilde{Q}T_{21}^x \tag{16}$$

Assume, by eliminating redundant outputs if necessary, that C_2 has full row rank and define:

$$Q \triangleq \left(\tilde{Q}C_2 - F\right)\left(I - \frac{A_L}{z}\right)^{-1} \tag{17}$$

Since A_L is stable (17) defines a *bijection* over $\mathcal{RH}_\infty$. In terms of Q, $\Phi(z)$ is given by:

$$\Phi(z) = \left\{I_n + (zI_n - A_F)^{-1}(A_F + B_2Q)\right\} \tag{18}$$

Let Q_i denote the coefficients of the impulse response of the transfer matrix Q. From (18) we have that:

$$
\begin{aligned}
\phi_0 &= I_n \\
\phi_k &= A_F^k + \sum_{i=0}^{k-1} A_F^{k-1-i}B_2Q_i
\end{aligned}
\tag{19}
$$

Let $\mu = \max_k \|\phi_k\|_{\mathcal{G}} = \|\Phi(z)\|_{\mathcal{G}}$. Then from (19) we have:

$$\mu \geq \|\phi_1\|_{\mathcal{G}} = \|A + B_2(F + Q_0)\|_{\mathcal{G}} \tag{20}$$

Hence

$$\min_{Q\in\mathcal{RH}_\infty} \mu \geq \min_{\hat{F}} \|A + B_2\hat{F}\|_{\mathcal{G}} \triangleq \mu^* \tag{21}$$

The proof is completed by noting that for $F = \hat{F}$ and $Q(z) = 0$ we have $\mu = \|\phi_1\|_{\mathcal{G}} = \mu^*$ since $\|.\|_{\mathcal{G}}$ is submultiplicative, $\|A_F\|_{\mathcal{G}} \leq 1$, and $\phi_k = A_F^k$ ⋄.

• **Lemma 6:** If $Q(z)$ is such that its impulse response satisfies $\|q_i\|_2 \leq C_q\delta^i$, with $\delta < 1$, then there exists N, independent of Q, such that:

$$\|\Phi(z)\|_{\mathcal{G}} = \sup_{1\leq k\leq N} \|\phi_k\|_{\mathcal{G}}$$

Proof: Let ρ denote the spectral radius of A_F. Since A_F is stable, $|\rho| < 1$ and it can be easily shown that there exist $C_a, 1 > \lambda > \delta$ such that $\|A_F^k\|_2 \leq C_a\lambda^k$. Hence

$$
\begin{aligned}
\|\phi_k\|_2 &\leq \|A_F^k\|_2 + \sum_{i=0}^{k-1} \|A_F^{k-1-i}\|_2\|B_2\|_2\|Q_i\|_2 \\
&\leq C_a\lambda^k + \frac{C_aC_q\|B_2\|_2}{\lambda - \delta}\lambda^k \triangleq M\lambda^k
\end{aligned}
\tag{22}
$$

Let μ_u denote the minimum (over all $Q \in \mathcal{RH}_\infty$) of $\|\Phi\|_{\mathcal{G}}$, obtained by solving the optimization problem (15). Clearly $\mu_u \leq \|\Phi\|_{\mathcal{G}}$. From the equivalence of all finite dimensional matrix norms [15] it follows that there exist c such that $\|.\|_{\mathcal{G}} \leq c\|.\|_2$. Hence, by selecting N such that:

$$cM\lambda^N < \mu_u \tag{23}$$

we have that $\|\phi_k\|_{\mathcal{G}} \leq \mu_u \leq \|\Phi\|_{\mathcal{G}}\ \forall\ k \geq N$ and therefore $\max_k \|\phi_k\|_{\mathcal{G}}$ is achieved for some $k < N$ ⋄.

In the sequel we consider the following special cases: i) The perturbation $\underline{\omega}$ is a bounded power spectral signal and the performance variable $\underline{\zeta}$ is a bounded power signal (or alternatively $\underline{\omega} \in l_2$ and $\zeta \in l_\infty$), hence the appropriate induced norm is $\|T_{\zeta\omega}\|_2$; and ii) $\underline{\omega}$ and $\underline{\zeta} \in l_2$, resulting in $\|T_{\zeta\omega}\|_\infty$. The dual interpretation of the disturbances as l_1 signals, for the purpose of constrained stability, and as bounded spectral power or bounded power, for the purpose of performance, is similar to the approach used by Bernstein and Haddad [18] to address the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem.

3.3 $\mathcal{H}_\infty$ Performance Criterion

In this case the mixed performance control problem can be stated as:

$$\min_{Q\in\mathcal{RH}_\infty} \|T_{11}^x + T_{12}^x\tilde{Q}T_{21}^x\|_{\mathcal{G}} \tag{$\mathcal{H}_\infty$}$$

subject to:

$$\|T_{11}^f + T_{12}^f\tilde{Q}T_{21}^f\|_\infty \leq \gamma$$

Since U is inner and V is co-inner, we can find $U_\perp, V_\perp$ such that $(U\ U_\perp)$ and $(V\ V_\perp)'$ are unitary. Since (pre)post-multiplication by a unitary matrix preserves the ∞ norm we have that:

$$
\begin{aligned}
\|T_{\zeta\omega}\|_\infty &= \left\| \begin{pmatrix} U \\ U_\perp \end{pmatrix}^{\sim} T_{\zeta\omega}\begin{pmatrix} V & V_\perp \end{pmatrix}^{\sim} \right\|_\infty \\
&= \left\| \begin{pmatrix} U^{\sim}(G_cB_1 - NFG_f)V^{\sim} + R_B^{\frac{1}{2}}\tilde{Q}R_L^{\frac{1}{2}} & U^{\sim}(G_cB_1 - NFG_f)V_\perp^{\sim} \\ U_\perp^{\sim}(G_cB_1 - NFG_f)V^{\sim} & U_\perp^{\sim}(G_cB_1 - NFG_f)V_\perp^{\sim} \end{pmatrix} \right\|
\end{aligned}
\tag{24}
$$

Note that, in general, we have a 4-block general distance problem. In this paper, for simplicity, we will limit ourselves to the special case where the system is square and right invertible. In this case U, V are unitary and (24) reduces to:

$$\|T_{\zeta\omega}\|_\infty = \|U^{\sim}(G_cB_1 - NFG_f)V^{\sim} + R_B^{\frac{1}{2}}\tilde{Q}R_L^{\frac{1}{2}}\|_\infty \triangleq \|R + Q_B\|_\infty \tag{25}$$

where $R \triangleq U^{\sim}(G_cB_1 - NFG_f)V^{\sim}$ and $Q_B \triangleq R_B^{\frac{1}{2}}\tilde{Q}R_L^{\frac{1}{2}}$.

Problem ($\mathcal{H}_\infty$) is a convex optimization problem in $\mathcal{RH}_\infty$. However, even though this space is complete, it is not compact. Therefore a minimizing solution to ($\mathcal{H}_\infty$) may not exists. Motivated by this difficulty we introduce the additional constraint that all the poles of the closed–loop system must lay in the closed δ–disk, where $\delta < 1$ is given. Thus, the original problem is modified to:

$$\min_{\tilde{Q}\in\mathcal{RH}_\delta} \|\Phi(z)\|_{\mathcal{G}} = \|T_{11}^x + T_{12}^x\tilde{Q}T_{21}^x\|_{\mathcal{G}} \tag{$\mathcal{H}_\infty^\delta$}$$

subject to:

$$
\begin{aligned}
\Phi(z) &\in \mathcal{RH}_\delta \\
\|T_{11}^f + T_{12}^f\tilde{Q}T_{21}^f\|_\delta &\leq \gamma
\end{aligned}
$$

where $\mathcal{RH}_\delta = \{Q(z) \in \mathcal{RH}_\infty : Q(z) \text{ analytic in } |z| \geq \delta\}$ and $\|G(z)\|_\delta = \sup_{|z|=\delta} \sigma_{max}(G(z))$.

• **Theorem 2:** Let $Q_F = \sum_{i=0}^{N-1} q_i z^{-i}$ be given. Then, the condition that there exist $Q_R \in \mathcal{RH}_\delta$ such that $\|R+Q\|_\delta \leq \gamma$, where $Q = Q_F + z^{-N}Q_R$ and R has all its poles outside the disk $|z| \leq \delta$, is equivalent to a *convex* constraint of the form $\|\mathcal{Q}\|_2 \leq \gamma$ where:

$$
\begin{aligned}
\mathcal{Q} &= W^{\frac{1}{2}} \begin{pmatrix} I & 0 \\ 0 & \mathcal{H}' \end{pmatrix} L_c^{\frac{1}{2}} \\
Lc &= \begin{pmatrix} L_{11}^C & L_{12}^C \\ L_{12}^{C'} & L_{22}^C \end{pmatrix} \\
L_{11}^C &= L_o^C \\
L_{12}^C &= -\left((A_R')^{N-1} C_R' \quad (A_R')^{N-2} C_R' \ldots \quad C_R' \right) \\
L_{22}^C &= I_N \\
W'^{\frac{1}{2}} W^{\frac{1}{2}} &= \begin{pmatrix} L_o^0 & \mathcal{A} \\ \mathcal{A}' & I \end{pmatrix} \\
\mathcal{A} &= \left(A_R^{-N} B_R \quad A_R^{-(N-1)} B_R \ldots A_R^{-1} B_R \right) \\
\mathcal{H} &= \begin{pmatrix} H_N & H_{N-1} & \ldots & \ldots & H_1 \\ & H_N & H_{N-1} & \ldots & H_2 \\ & & \ddots & & \\ & & & H_N & H_{N-1} \\ & & & & H_N \end{pmatrix} \\
H_i &= q_{N-i} + B_R'(A_R')^{N-1-i} C_R' \qquad 1 \leq i \leq N-1 \\
H_N &= q_0 + D_R \\
R &= \left(\begin{array}{c|c} \delta A_R & \delta^{\frac{1}{2}} B_R \\ \hline \delta^{\frac{1}{2}} C_R & D_R \end{array} \right)
\end{aligned}
\tag{26}
$$

and where L_o^0 and L_o^C are the solutions to the following Lyapunov equations:

$$
\begin{aligned}
A_R L_o^0 A_R' - L_o^0 &= B_R B_R' \\
A_R' L_o^C A_R - L_o^C &= (A_R')^N C_R' C_R (A_R)^N
\end{aligned}
\tag{27}
$$

Proof: Consider first the case where $\delta = 1$. Let $G \triangleq R + Q_F$. The proof follows by noting that, given Q_F, there exist $Q_R \in \mathcal{RH}_\infty$ such that $\|T_{\zeta w}\|_\infty \leq \gamma$ *iff* the corresponding *unconstrained* 1 block Nehari approximation problem has a solution, i.e. if:

$$
\begin{aligned}
\min_{q_R \in \mathcal{RH}_\infty} \|G + z^{-N} q_R\|_\infty &= \min_{q_R \in \mathcal{RH}_\infty} \|z^N G + q_R\|_\infty \\
&= \min_{q_R \in RH_\infty^-} \|z^{-N} G^\sim + q_R\|_\infty \\
&= \Gamma_H(z^{-N} G^\sim) \leq \gamma
\end{aligned}
\tag{28}
$$

where Γ_H indicates the maximum Hankel singular value and where we used the facts that z^N is an inner function and that the best stable approximation to a given function coincides with the best antistable approximation to its conjugate. In order to compute Γ_H we need to compute the observability L_o and controllability L_c grammians of the stable part $\mathcal{G}$ of $z^{-N} G^\sim$. In [13] we showed, through some lengthy computations, that these grammians can be computed explicitly. Furthermore, L_c is independent of Q_F and L_o is given by:

$$
\begin{aligned}
L_o &= \begin{pmatrix} L_o^0 & \mathcal{A}\mathcal{H}' \\ \mathcal{H}\mathcal{A}' & \mathcal{H}\mathcal{H}' \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ 0 & \mathcal{H} \end{pmatrix} \begin{pmatrix} L_o^0 & \mathcal{A} \\ \mathcal{A}' & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \mathcal{H}' \end{pmatrix}
\end{aligned}
$$

Hence:

$$
\begin{aligned}
L_c^{\frac{1}{2}} L_o L_c^{\frac{1}{2}} &= \mathcal{Q}' \mathcal{Q} \\
\mathcal{Q} &\triangleq W^{\frac{1}{2}} \begin{pmatrix} I & 0 \\ 0 & \mathcal{H}' \end{pmatrix} L_c^{\frac{1}{2}}
\end{aligned}
\tag{29}
$$

From Nehari Theorem it follows that:

$$
\begin{aligned}
\|T_{\zeta w}\|_\infty \leq \gamma &\iff \rho^{\frac{1}{2}} \left(L_c^{\frac{1}{2}} L_o L_c^{\frac{1}{2}} \right) \leq \gamma \\
&\iff \|\mathcal{Q}\|_2 \leq \gamma
\end{aligned}
\tag{30}
$$

where ρ indicates the spectral radius. The proof is completed by noting that the case $\delta < 1$ follows by using the transformation $z = \delta \hat{z}$ ⋄.

By combining the results of Lemma 6 and Theorem 2, it follows that for $\mathcal{H}_\infty$ constraints, the mixed performance robust control problem can be solved by solving a finite–dimensional convex optimization problem and an unconstrained Nehari approximation problem. This result is summarized in the following theorem:

• **Theorem 3:** $Q^o(z) = Q_F^o(z) + z^{-N} Q_R^o(z)$ solves problem $(\mathcal{H}_\infty)$ **iff** $Q_F^o(z) = \sum_{i=0}^{N-1} Q_i z^{-i}$ solves the finite–dimensional convex optimization problem:

$$
\min_{Q_i} \left\{ \max_{1 \leq k \leq N_s} \|\phi_k\|_{\mathcal{G}} \right\}
$$

subject to:

$$
\|\mathcal{Q}\|_2 \leq \gamma
$$

and Q_R^o solves the unconstrained Nehari approximation problem:

$$
\min_{Q_R \in \mathcal{RH}_\delta} \|z^{-N} G^\sim + Q_R\|_\delta
$$

where $G = R + Q_F^o$ and N is selected according to Lemma 6.

Proof: Since Q satisfies the constraint $\|Q + R\|_\delta \leq \gamma$ then $\|Q\|_\delta \leq \|R\|_\delta + \gamma \triangleq C_q$. Since $Q(z)$ is analytical inside the closed disk $|z| \leq \delta$ we have that:

$$
Q_k = \frac{1}{2\pi j} \oint_\tau Q(z) z^{k-1} dz \tag{31}
$$

where τ is the circle with radius δ. From (31) it follows that:

$$
\|Q_k\|_2 \leq \|Q\|_\delta \delta^k \leq C_q \delta^k \tag{32}
$$

The proof follows now from Theorem 2 and Lemma 6⋄.

3.4 $\mathcal{H}_2$ Performance Criterion

Let q_i denote the coefficients of the impulse response of $R_B^{\frac{1}{2}} \tilde{Q} R_L^{\frac{1}{2}}$. Then, from Lemma 4 it follows that, given $\gamma \geq \|G_c B_1 - NFG_f\|_2$, all stabilizing controllers yielding $\|T_{\zeta w}\|_2 \leq \gamma$ can be parametrized in terms of $\tilde{Q}$, where q_i satisfy the following constraint:

$$
\sum_{i=0}^{i=\infty} \|R_B^{\frac{1}{2}} q_i R_L^{\frac{1}{2}}\|_F^2 \leq \gamma^2 - \|G_c B_1 - NFG_f\|_2^2 \tag{32}
$$

where $\|.\|_F$ denotes the Frobenius norm. This results follows immediately from (14) and the orthogonality of $G_c B_1, U, G_f, V$ by noting that $\|Q(z)\|_2^2 = \sum_{i=0}^{\infty} \|Q_i\|_F^2$.

As in the $\mathcal{H}_\infty$ case, to avoid the difficulties due to the non–compactness of $\mathcal{RH}_\infty$, the original problem is modified to:

$$
\min_{\tilde{Q} \in \mathcal{RH}_\delta} \|\Phi(z)\|_{\mathcal{G}} = \|T_{11}^x + T_{12}^x \tilde{Q} T_{21}^x\|_{\mathcal{G}} \tag{$\mathcal{H}_2^\delta$}
$$

subject to:

$$
\begin{aligned}
\sum_{i=0}^{i=\infty} \|R_B^{\frac{1}{2}} q_i R_L^{\frac{1}{2}}\|_F^2 &\leq \gamma^2 - \|G_c B_1 - NFG_f\|_2^2 \\
\|q_i\|_2 &\leq C_q \delta^i
\end{aligned}
\tag{33}
$$

Remark 5: The additional constraint puts an upper bound on $\|Q\|_\infty$, since it forces $\|Q\|_\infty \leq \frac{C_q}{1-\delta}$. This additional constraint improves the robustness by bounding $\|T_{\zeta w}\|_\infty$.

• **Theorem 4:** Let $Q(z) = \sum_{k=1}^{\infty} Q_k z^{-k}$. Then the solution to the problem $(\mathcal{H}_2^\delta)$ has $Q_k = 0$ for $k > N$.

Proof: The proof follows from Lemma 6 by noting that ϕ_k depends only on Q_i, $i \leq k$. Since optimization of $\|\Phi\|_{\mathcal{G}}$ requires considering only ϕ_k, $k \leq N$, terms in the impulse response of $Q(z)$ corresponding to $k > N$ can only increase the $\mathcal{H}_2$ cost while not affecting $\|\Phi\|_{\mathcal{G}}$.

4. Conclusions

Most realistic control problems involve both some type of time-domain constraints and certain degree of model uncertainty. Model uncertainty can be successfully addressed through frequency–domain constraints combined with $\mathcal{H}_\infty$ techniques. However the standard $\mathcal{H}_\infty$ formalism cannot handle time–domain constraints. Alternatively, the recently developed l_1 robust control theory can be used to deal with model uncertainty. Although this framework can incorporate time domain constraints, it cannot handle frequency domain specifications.

In this paper we propose to approach model uncertainty and time-domain constraints using an operator norm induced by the constraints to assess the stability properties of a family of systems. The proposed controller design method results in a convex optimization problem, where additional frequency domain constraints can be imposed. We showed that when these additional constraints have the form of an $\mathcal{H}_2$ or $\mathcal{H}_\infty$ bound, the resulting problem can be transformed into a finite dimensional optimization and solved exactly. Thus, the conservatism inherent in some previous approaches is eliminated. Although here we considered only the simpler case of a one–block problem, we anticipate that the results will extend naturally to the 4–block case.

Perhaps the most severe limitation of the proposed method is that may result in very large order controllers (roughly $2N$) necessitating some type of model reduction. Preliminary results suggest that substantial order reduction can be accomplished without performance degradation. Research is currently under way addressing this issue and pursuing the extension of the formalism to structured uncertainty.

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