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A Set Induced Norm Approach to the Robust Control of Constrained Systems

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Abstract

Most realistic control problems involve both some type of time–domain constraints and model uncertainty. However, there presently exist few design methods capable of simultaneously addressing both issues. We recently proposed to address this class of problems by using a "constrained robustness measure", generated by a constraint-set induced operator norm, to assess the stability properties of a family of systems. In this paper we explore the properties of this constrained–robustness measure and we extend the theoretical framework to include control as well as state constraints. These results are applied to the problem of designing fixed–order stabilizing feedback controllers for systems subject to structured parametric model uncertainty and time–domain constraints.

1. Introduction

A substantial number of control problems can be summarized as the problem of designing a controller capable of achieving acceptable performance under system uncertainty and design constraints. However, this problem is far from solved, even in the simpler case where the system under consideration is linear. Several methods have been proposed recently to deal with constrained control problems under the assumption of exact knowledge of the model (see [1] and references therein). However, such an assumption can be too restrictive, preventing their application in realistic problems.

On the other hand, during the last decade a considerable amount of time has been spent analyzing the question of whether some relevant properties of a system (most notably asymptotic stability) are preserved under the presence of unknown perturbations. This research effort has led to procedures for designing "robust" controllers, capable of achieving desirable properties under various classes of plant perturbations while, at the same time, satisfying frequency–domain constraints. However, most of these design procedures cannot accommodate directly time domain constraints (which precludes their use in cases such as when there exist physically motivated "hard" bounds on the states or control effort), although some progress has been recently made in this direction [2–5].

In [6–7] we proposed to approach time–domain constrained systems using an operator norm–theoretic approach. We introduced a simple robustness measure that indicated how well the family of systems under consideration satisfied a given set of time–domain constraints and we proposed a design method yielding controllers that maximized this robustness measure. In this paper we extend our formalism to include control as well as a more general description of state constraints and we explore the properties of the resulting constrained robustness measure. These theoretical results are applied to the problem of designing stabilizing controllers for systems subject to structured parametric model uncertainty and time–domain constraints. We show that in cases of practical interest the synthesis problem can be reduced to a convex, albeit in general non-differentiable, optimization problem.

The paper is organized as follows: In section II we introduce the concepts of *constrained stability* and *robust constrained stability* and we use these concepts to give a formal definition of the *robust constrained stability analysis* and *robust constrained stability design* problems. The *analysis* problem is studied in section III where we give necessary and sufficient conditions for constrained stability. We use these results to define a constrained robustness measure and we show that, under mild dynamics of the system. In section IV we apply the results of section III to the *design* problem and we show that in cases of practical interest our approach yields a well behaved optimization problem. Finally, in section V, we summarize our results and we indicate directions for future research.

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2. Definitions and Background Results

2.1 Preliminary Definitions

• **Def. 1:** Consider the linear, time invariant, discrete time, autonomous system modeled by the difference equation:

$$\underline{x}_{k+1} = A\underline{x}_k,\ k = 0,1\ldots \qquad (S^a) \text{ subject to the constraint:}$$

$$\underline{x} \in \mathcal{G} \subset R^n \tag{1}$$

where $A \in R^{n*n}$ and where $\underline{x}$ indicates x is a vector quantity. The system (S^a) is *Constrained Stable* if for *any* point $\underline{\bar{x}} \in \mathcal{G}$, the trajectory $\underline{x}_k(\underline{\bar{x}})$ originating in $\underline{\bar{x}}$ remains in $\mathcal{G}$ for all k.

Remark 1: A nonempty subset $\mathcal{S} \subset R^n$ is a *positively invariant set* of the system (S^a) if for any initial state $\underline{x}_o \in \mathcal{S}$, the trajectory $\underline{x}_k(\underline{x}_o) \in \mathcal{S}\ \forall\ k$, or equivalently [8] if and only if $\underline{x} \in \mathcal{S}$ implies $A\underline{x} \in \mathcal{S}$. Therefore, it follows that the system (S^a) is constrained stable *iff* it has the set $\mathcal{G}$ as a positively invariant set.

• **Def. 2:** Consider the family of linear discrete–time systems modeled by the difference equation:

$$\underline{x}_{k+1} = (A + \Delta)\underline{x}_k \qquad (S^a_\Delta)$$

where Δ belongs to some perturbation set $\mathcal{D} \subseteq R^{n*n}$. The system (S^a) is *Robustly Constrained Stable* with respect to the set $\mathcal{D}$ if (S^a_Δ) is constrained stable for all perturbation matrices $\Delta \in \mathcal{D}$.

We proceed now to restrict the class of constraints allowed in our problem. The introduction of this restriction, while not affecting significantly the number of real-world problems that can be handled by our formalism, introduces more structure into the problem. This additional structure plays a key role in section III where we derive necessary and sufficient conditions for constrained stability.

2.2 Constraint Qualification Hypothesis

In this paper, we will limit ourselves to constraints of the form:

$$\underline{x} \in \mathcal{G} \subset R^n \tag{2}$$

where $\mathcal{G}$ is a convex, compact, balanced set (i.e a convex compact set such that $\underline{x} \in \mathcal{G} \Rightarrow \lambda\underline{x} \in \mathcal{G}$ for $|\lambda| \le 1$ [9]) containing the origin in its interior.

• **Def. 3:** [9] The *Minkowsky Functional* (or gauge) p of a balanced convex set $\mathcal{G}$ containing the origin in its interior is defined by

$$p(\underline{x}) = \inf_{r>0}\left\{r : \frac{\underline{x}}{r} \in \mathcal{G}\right\} \tag{3}$$

A well known result in functional analysis (see for instance [9]) establishes that p defines a seminorm in R^n. Furthermore, when $\mathcal{G}$ is compact, this seminorm becomes a norm. In the sequel, we will denote this norm as $\|\underline{x}\|_{\mathcal{G}} \stackrel{\Delta}{=} p(\underline{x})$

Remark 2: The set $\mathcal{G}$ can be characterized as the unity ball in $\|.\|_{\mathcal{G}}$ i.e. $\mathcal{G} = \{\underline{x}: \|\underline{x}\|_{\mathcal{G}} \leq 1\}$.

2.3 Statement of the Problem

Consider the LTI system represented by the following state-space realization:

$$\underline{x}_{k+1} = A\underline{x}_k + B\underline{u}_k \qquad (S)$$

subject to the constraint:

$$\underline{x}_k \in \mathcal{G} \subset R^n$$

where $\underline{x} \in R^n$ represents the state and $\underline{u} \in R^m$ represents the control input. Then, the basic problems that we address in this paper are the following:

• **Robust Constrained Stability Analysis Problem:** Given the nominal system (S) and a linear feedback control law $\underline{u}_k = F\underline{x}_k$, determine if the resulting closed-loop system is constrained-stable. If the nominal closed-loop system is constrained-stable, determine the maximum allowable level of model uncertainty (in the sense of some previously defined norm) such that the constraints are satisfied for any initial condition $\underline{x} \in \mathcal{G}$.

• **Linear Robust Constrained Control Synthesis Problem:** Given the system (S) find a *linear* controller such that the resulting closed-loop system is constrained stable and satisfies some additional specifications such as:

i) maximum robustness against structured model uncertainty of the form $A = A_o + \Delta$, $\Delta \in \mathcal{D}$

ii) bounds on the control effort of the form $\underline{u}_k \in \Omega \subset R^m$, where Ω is a compact, convex balanced set containing the origin in its interior.

3. Constrained Stability Analysis

Consider the system (S^a) and let $\|.\|_G$ denote the operator norm induced in R^{n*n} by $\mathcal{G}$ (i.e. $\|A\|_{\mathcal{G}} \stackrel{\Delta}{=} \sup_{\|\underline{x}\|_{\mathcal{G}}=1} \|A\underline{x}\|_{\mathcal{G}}$). From definition 1 it follows that (S^a) is constrained stable *iff* $\|A\|_{\mathcal{G}} \leq 1$. Moreover, (S^a) is robustly constrained stable with respect to a given set $\mathcal{D}$ *iff* $\|A + \Delta\|_{\mathcal{G}} \leq 1$ for all $\Delta \in \mathcal{D}$. This observation can be used to define a robustness measure as follows:

• **Def. 4:** Consider the system (S^a). The *constrained stability measure* $\varrho_{\mathcal{G}}^{\mathcal{N}}$ is defined as:

$$\varrho_{\mathcal{G}}^{\mathcal{N}} \stackrel{\Delta}{=} \begin{cases} 0 & \text{if } \|A\|_{\mathcal{G}} > 1; \\ \max_{\Delta \in \mathcal{D}} \|\Delta\|_{\mathcal{N}} & \text{if } \|A + \Delta\|_{\mathcal{G}} < 1 \ \forall \ \Delta \in \mathcal{D}; \\ \min_{\Delta \in \mathcal{D}} \{\|\Delta\|_{\mathcal{N}}: \|A + \Delta\|_{\mathcal{G}} = 1\} & \text{otherwise.} \end{cases}$$

where $\|.\|_{\mathcal{N}}$ denotes a suitable operator norm defined in $\mathcal{D}$. In the special case where the induced operator norm $\|.\|_{\mathcal{G}}$ is used in the set $\mathcal{D}$, we will denote the constrained stability measure as $\varrho_{\mathcal{G}}$.

Remark 3: Let the set $\mathcal{B}\Delta^{\mathcal{N}}$ be the intersection of $\mathcal{D}$ with the origin centered ball of radius $\varrho_{\mathcal{G}}^{\mathcal{N}}$, i.e:

$$\mathcal{B}\Delta^{\mathcal{N}} = \left\{\Delta \in \mathcal{D}: \|\Delta\|_{\mathcal{N}} \leq \varrho_{\mathcal{G}}^{\mathcal{N}}\right\}$$

Then, from definition 4 it follows that the family (S_{Δ}^a) is constrained stable for all perturbations $\Delta \in \mathcal{B}\Delta^{\mathcal{N}}$.

Remark 4: In principle $\varrho_{\mathcal{G}}^{\mathcal{N}}$ can be a *non-continuous* function of A. In the sequel we will show that under some assumptions that are commonly verified in practice, $\varrho_{\mathcal{G}}^{\mathcal{N}}$ is a *continuous, concave* function of the dynamics matrix A.

• **Theorem 1:** Assume that the perturbation set $\mathcal{D}$ is is a *closed cone* with vertex at the origin [10], (i.e. $\Delta^o \in \mathcal{D} \iff \alpha\Delta^o \in \mathcal{D} \ \forall \ 0 \leq \alpha$). Then $\varrho_{\mathcal{G}}^{\mathcal{N}}$ is a *continuous, concave* function of A.

Proof: The proof of the theorem is given in Appendix A.

Remark 5: Note that the class of sets considered in this theorem includes as a particular case sets of the form:

$$\mathcal{D} = \left\{\Delta: \Delta = \sum_1^m \mu_i E_i; \ \mu_i \geq 0, \ E_i \text{ given}\right\} \qquad (4)$$

which has been the object of much interest lately ([11-13] and references therein).

In the next lemma we introduce a *lower bound* of the constrained stability measure and we show that for *unstructured* perturbations (i.e. the case where $\mathcal{D} \equiv R^{n*n}$) this lower bound is saturated.

• **Lemma 1:**

$$\varrho_{\mathcal{G}} \geq 1 - \|A\|_{\mathcal{G}} \qquad (5)$$

Furthermore, for the unstructured perturbation case, i.e. the case where $\mathcal{D} \equiv R^{n*n}$, condition (5) is saturated.

Proof: The first part of the lemma can be easily proved from definition 4 and the triangle inequality. The second part follows by noting that for $\Delta^o \stackrel{\Delta}{=} \frac{(1-\|A\|_{\mathcal{G}})A}{\|A\|_{\mathcal{G}}}$ (5) is saturated $\diamond$.

Remark 6: Note that the results of Lemma 1 can be used to find a lower bound for the constrained robustness measure in the general case when an operator norm different from $\|.\|_{\mathcal{G}}$ is used in the set $\mathcal{D}$. Since all finite dimensional matrix norms are equivalent [14], it follows that, given any norm $\mathcal{N}$ in the set $\mathcal{D}$, there exist a constant c such that $\|.\|_{\mathcal{G}} \leq c\|.\|_{\mathcal{N}}$. Hence $\varrho_{\mathcal{G}}^{\mathcal{N}} \leq \frac{\varrho_{\mathcal{G}}}{c}$.

3.1 Quadratic Constraints Case:

In this section we particularize our theoretical results for the special case where the constraint region is an hyperellipsoid. In this case, without loss of generality, we have:

$$\mathcal{G} = \{\underline{x}: \underline{x}'P\underline{x} \leq 1, \ P \in R^{n*n} \text{ positive definite}\}$$

Hence $\|\underline{x}\|_{\mathcal{G}}^2 = \underline{x}'P\underline{x}$ and:

$$\begin{aligned} \|A\|_{\mathcal{G}}^2 &= \max_{\underline{x}} \left\{\frac{\underline{x}'A'PA\underline{x}}{\underline{x}'P\underline{x}}\right\} \\ &= \max_{\underline{x}} \left\{\frac{\underline{x}'L'L'^{-1}A'L'LAL^{-1}L\underline{x}}{\underline{x}'L'L\underline{x}}\right\} \\ &= \max_{\|\underline{y}\|_2=1} \|LAL^{-1}\underline{y}\|_2^2 = \|LAL^{-1}\|_2^2 = \|\tilde{A}\|_2^2 \end{aligned} \qquad (6)$$

where $L'L = P$ and $\tilde{A} = LAL^{-1}$. In this case our approach yields a generalization of the well known technique of estimating the robustness measure by using quadratic based Lyapunov functions, (see [15] and references therein).

• *Example 1:* (multilinearly correlated perturbations) In the case of quadratic constraints and multilinearly correlated uncertainty, the lower bound on ϱ given by (5) can be tightened as follows. Assume that the set $\mathcal{D}$ is given by:

$$\begin{aligned} \mathcal{D} = \left\{\Delta \in R^{n*n}: L\Delta L^{-1} = U\begin{pmatrix}\tilde{\Delta} \\ 0\end{pmatrix}\right\} \\ \tilde{\Delta} \in R^{m*n}, \ U'U = I_n, \ L'L = P \end{aligned} \qquad (7)$$

Since the euclidian norm is invariant under multiplications by a unitary matrix we have:

$$\begin{aligned} \|A+\Delta\|_{\mathcal{G}} &= \|L(A+\Delta)L^{-1}\|_2 \\ &= \|\tilde{A} + U\begin{pmatrix}\tilde{\Delta} \\ 0\end{pmatrix}\|_2 = \|\begin{pmatrix}A_1 \\ A_2\end{pmatrix} + \begin{pmatrix}\tilde{\Delta} \\ 0\end{pmatrix}\|_2 \\ &= \|\begin{pmatrix}A_1+\tilde{\Delta} \\ A_2\end{pmatrix}\|_2 \end{aligned} \tag{8}$$

A well known result on matrix dilations establishes [16] that:

$$\|\begin{pmatrix}X \\ A_2\end{pmatrix}\|_2 \leq 1 \iff \|A_2\|_2 \leq 1$$
$$\text{and } X = Y(I - A_2'A_2)^{\frac{1}{2}},\ \|Y\|_2 \leq 1$$

hence it follows that:

$$\|A+\Delta\|_{\mathcal{G}} = 1 \iff \|(A_1+\tilde{\Delta})N\|_2 = 1 \tag{9}$$

where $N \triangleq (I - A_2'A_2)^{\frac{-1}{2}}$. Finally, by defining $\|\Delta\|_{\mathcal{N}} \triangleq \|\tilde{\Delta}N\|_2$ and using the results of Lemma 1, we get:

$$\varrho_{\mathcal{G}}^{\mathcal{N}} = 1 - \|A_1 N\|_2 \tag{10}$$

Remark 7: Note that when $A_2 = 0$ we recover the results of Lemma 1, since in this case $\varrho = 1 - \|A_1\|_2 = 1 - \|A\|_{L'L}$

• *Example 2:* (unstructured perturbation)

In this case, Theorem 2 yields $\varrho_{\mathcal{G}} = 1 - \|A\|_{\mathcal{G}}$ where:

$$\|A\|_{\mathcal{G}}^2 = \|A\|_P^2 = \max_{\underline{x}} \left(\frac{\underline{x}'A'PA\underline{x}}{\underline{x}'P\underline{x}}\right) \tag{11}$$

Consider now the case where $\varrho_{\mathcal{G}} > 0$. Then, there exists Q positive definite such that:

$$A'PA - P = -Q \tag{12}$$

and:

$$\|A\|_{\mathcal{G}}^2 = \max_{\underline{x}} \left(1 - \frac{\underline{x}'Q\underline{x}}{\underline{x}'P\underline{x}}\right) \leq 1 - \frac{\sigma_{min}(Q)}{\sigma_{Max}(P)} \tag{13}$$

Hence:

$$\varrho_{\mathcal{G}} = 1 - \|A\|_{\mathcal{G}} \geq 1 - \left(1 - \frac{\sigma_{min}(Q)}{\sigma_{Max}(P)}\right)^{\frac{1}{2}} \tag{14}$$

A common technique in state space robust analysis is to obtain robustness bounds from equation (12) ([17-18]). This case can be accommodated by our formalism by recognizing the fact that once P is selected, the system becomes effectively constrained to remain within an hyperellipsoidal region. It has been suggested ([17-18]) that good robustness bounds can be obtained from (12) when P is selected such that $Q = I$. In this case our approach yields:

$$\varrho_{\mathcal{G}} = 1 - \|A\|_{\mathcal{G}} = 1 - \left(1 - \frac{1}{\sigma_{Max}(P)}\right)^{\frac{1}{2}} \tag{15}$$

which coincides with the robustness bound found in [18].

• *Example 3:* (Unstructured perturbation, A semisimple) Consider the case where A is semisimple, i.e.

$$\begin{aligned} A &= L^{-1}\Lambda L \\ \Lambda &= diag\left\{\begin{pmatrix}\sigma_1 & \omega_1 \\ -\omega_1 & \sigma_1\end{pmatrix}, \ldots, \begin{pmatrix}\sigma_p & \omega_p \\ -\omega_p & \sigma_p\end{pmatrix}, \sigma_{p+1}, \ldots, \sigma_n\right\} \end{aligned} \tag{16}$$

Then, the maximum of the stability measure, $\varrho_{\mathcal{G}}$, over all possible positive definite matrices P, is achieved for $P = L'L$.

Proof: The proof follows by noting that $\|A\|_{L'L} = \rho(A)$ where $\rho(.)$ denotes the spectral radius, which is a lower bound for any matrix norm [14] ◇.

3.2 Polyhedral Constraints

Consider now the case where the region $\mathcal{G}$ is polyhedral, i.e. the case where:

$$\mathcal{G} = \{\underline{x}: |G\underline{x}| \leq \underline{\omega}\} \tag{17}$$

where $G \in R^{p*n}$, rank$(G) = n$, $\underline{\omega} \in R^p$, $\omega_i > 0$ and the $|.|$ should be interpreted on a component by component sense. Although this case is of practical importance, up to date a technique to estimate the robustness of such systems was unavailable, except perhaps to fit an hyperellipsoidal region within the admissible region and then use some of the bounds available for the quadratic case. Such a technique is clearly inappropriate since it guarantees robust stability *only* in a certain subregion of the region of interest. In this section we show that polyhedral regions fit naturally within our formalism and that in this case $\varrho_{\mathcal{G}}^{\mathcal{N}}$ can be efficiently computed as the minimum of the solution of p Linear Programming problems.

• **Theorem 2:** Let $\varrho_i^{\mathcal{N}}$ be the solution of the following optimization problem:

$$\varrho_i^{\mathcal{N}} = \min_{\Delta \in \mathcal{D}} \{\|\Delta\|_{\mathcal{N}}: \|H + \Delta H\|_1^{(i)} \geq 1\} \tag{18}$$

where:

$$W = diag\{w_i\},\ H \triangleq W^{-1}GA(G'G)^{-1}G'W$$
$$\Delta H \triangleq W^{-1}G\Delta(G'G)^{-1}G'W$$

and where $\|M\|_1^{(i)}$ indicates the l_1 norm of the i^{th} row of the matrix M. Then:

$$\varrho_{\mathcal{G}}^{\mathcal{N}} = \min_{1 \leq i \leq p} \{\varrho_i^{\mathcal{N}}\} \tag{19}$$

Proof: It is easily shown that:

$$\|\underline{x}\|_{\mathcal{G}} = \max_{1 \leq i \leq p} \left\{\frac{|G\underline{x}|_i}{\omega_i}\right\} = \|W^{-1}G\underline{x}\|_\infty \tag{20}$$

From the definition of H we have that $W^{-1}GA = HW^{-1}G$. Hence:

$$\|A\underline{x}\|_{\mathcal{G}} = \|W^{-1}GA\underline{x}\|_\infty = \|HW^{-1}G\underline{x}\|_\infty$$

and $\|A\|_{\mathcal{G}} = \|H\|_\infty$ Assume that the lemma is false and that there exist $\tilde{\varrho}$ and $\tilde{\Delta}$ such that:

$$\|A + \tilde{\Delta}\|_{\mathcal{G}} = 1;\ \|\tilde{\Delta}\|_{\mathcal{N}} = \tilde{\varrho} < \varrho_{\mathcal{G}}^{\mathcal{N}} \tag{21}$$

Since $\|A + \tilde{\Delta}\|_{\mathcal{G}} = 1$ there exists i^o such that $\|H + \tilde{\Delta}H\|_1^{(i^o)} = 1$, $\|H + \tilde{\Delta}H)\|_1^{(j)} \leq 1$, $j \neq i^o$, but this implies (eq. (18)) that $\varrho_{i^o}^{\mathcal{N}} \leq \tilde{\varrho}$ which contradicts (21) ◇.

• *Example 4:* (unstructured perturbation) Consider the following case:

$$A = \begin{pmatrix}0.8 & 0.5 \\ -0.0208 & 0.5083\end{pmatrix}\ G = \begin{pmatrix}1.0 & 2.0 \\ -1.5 & 2.0\end{pmatrix}\ \underline{\omega} = \begin{pmatrix}5.0 \\ 10.0\end{pmatrix} \tag{22}$$

Then, from the definition of H, we have that:

$$H = \begin{pmatrix}0.7583 & 0.0 \\ -0.2083 & 0.55\end{pmatrix},\ \|A\|_{\mathcal{G}} = 0.7583 \tag{23}$$

and, from Theorem 2,

$$\varrho_i = \min_{\|\Delta\|_{\mathcal{G}}} \left\{\|\Delta\|_{\mathcal{G}}: \sum_{j=1}^{2} |H + \Delta|_{ij} = 1\right\}\ i = 1,2 \tag{24}$$

Casting the problems (24) into a linear programming form and solving we have that:

$$\varrho_1 = 0.2417,\ \varrho_2 = 0.2417 \text{ and } \varrho_{\mathcal{G}} = \min_{1 \leq i \leq 2} \varrho_i = 0.2417$$

Note that $\varrho_{\mathcal{G}} = 1 - \|A\|_{\mathcal{G}} = 0.2417$ as shown in Lemma 1.

4. Application to Robust Controllers Design

Consider the *Linear Robust Constrained Control Synthesis Problem* introduced in section 2.3. Let $p_\Omega(u)$ be the Minkowsky gauge for the set Ω and denote by $\|.\|_\Omega$ the corresponding norm induced in R^m. It follows that, given a feedback control law of the form $\underline{u}_k = F\underline{x}_k$, the control bounds are satisfied if and only if:

$$\|F\|_{\mathcal{G},\Omega} \triangleq \sup_{\|\underline{x}\|_{\mathcal{G}} \leq 1} \|F\underline{x}\|_\Omega \leq 1$$

Hence a full state feedback matrix F that solves the synthesis problem can be found solving the following optimization problem:

$$\max_F \{\varrho_{\mathcal{G}}^{\mathcal{N}}(F)\} \tag{25}$$

subject to:

$$\begin{aligned} \varrho_{\mathcal{G}}^{\mathcal{N}}(F) &\triangleq \min_{\Delta \in \mathcal{D}} \{\|\Delta\|_{\mathcal{N}} : \|A + BF + \Delta\|_{\mathcal{G}} = 1\} \\ \|F\|_{\mathcal{G},\Omega} &\leq 1 \end{aligned} \tag{26}$$

Since from Theorem 1, $\varrho_{\mathcal{G}}^{\mathcal{N}}(F)$ is a concave function, and since $\|F\|_{\mathcal{G},\Omega} \leq 1$ is a convex constraint, it follows that (25) has a global optimum. Hence, the problem of finding the *maximally* robust controller leads to convex, albeit non-differentiable, optimization problems, which can be solved using a number of techniques ([19]). In the remainder of this section, we give several design examples using the proposed technique.

• *Example 5:* Consider the following system:

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 \\ 0.505 & -0.51 \end{pmatrix} \; B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \mathcal{G} &= \{\underline{x} : \|\underline{x}\|_2 \leq 1\} \end{aligned} \tag{27}$$

The open-loop system has poles at $s_1 = 0.5$ and $s_2 = -1.01$. Assume that the perturbation set is such that changes the position of the poles while maintaining constant their sum, i. e:

$$\mathcal{D} = \left\{\Delta : \Delta = \mu E,\; E \triangleq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\; \mu \in \Re \right\} \tag{28}$$

Note that $\|E\|_2 = 1$ hence $\|\Delta\|_2 = |\mu|$.

In this case, the solution to the unconstrained maximally robust control problem can be computed by solving a matrix dilation problem [16]. Rewrite the dynamics matrix as:

$$A = \begin{pmatrix} x_1 & x_2 \\ a_1 & a_2 \end{pmatrix}$$

where x_i denote elements that can be modified using state-feedback. Since matrix dilations are norm-increasing we have that:

$$\begin{aligned} \|A + \mu E\|_2 &\geq \max\{\|(\,a_1 \quad a_2 + \mu\,)\|_2\} \\ &= \sqrt{a_1^2 + (a_2 + \mu)^2} \end{aligned} \tag{29}$$

Define now:

$$\begin{aligned} \mu^0 &= \operatorname{argmin}\{|\mu|, \mu \in \Re : a_1^2 + (a_2 + \mu)^2 = 1\} \\ &= \sqrt{(1 - a_1^2)} - |a_2| \end{aligned} \tag{30}$$

From (29) and (30) it follows that $\|A + \mu^o E\|_2 \geq 1$ which implies that $\varrho_2(F) \leq \mu^o$ for all F. Furthermore, from the definition of μ^o it follows that if F is selected such that $x_1 = x_2 = 0$, then $\varrho_2(F) = \mu^o$. Hence, this choice of F yields the solution to the unconstrained problem. In this particular example we have:

$$F^o = (\,0 \quad 1\,),\; \varrho_2 = 0.3531 \tag{31}$$

Consider now a feedback matrix F and let A_{cl} be the corresponding *closed-loop* matrix, i.e:

$$A_{cl} = A + BF = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} + \mu \end{pmatrix} \tag{32}$$

The corresponding value of the robustness measure can be computed using standard results on matrix dilations [16] as follows: The set Υ of numbers μ such that $\|A_{cl}\|_2 \leq 1$ can be parametrized as:

$$\Upsilon = \left\{\mu : \mu = -a_{22} - y a_{11} z + (1 - y^2)^{\frac{1}{2}} w (1 - z^2)^{\frac{1}{2}}\right\} \tag{33}$$

where:

$$\begin{aligned} y &= \frac{a_{21}}{(1 - a_{11}^2)^{\frac{1}{2}}} \\ z &= \frac{a_{12}}{(1 - a_{11}^2)^{\frac{1}{2}}} \\ w &\in \Re,\; |w| \leq 1 \end{aligned} \tag{34}$$

From (33) it follow that the constrained stability margin of A_{cl} is given by:

$$\varrho_2(F) = |a_{22} + y a_{11} z - (1 - y^2)^{\frac{1}{2}} (1 - z^2)^{\frac{1}{2}} \operatorname{sign}(a_{22} + y a_{11} z)|$$

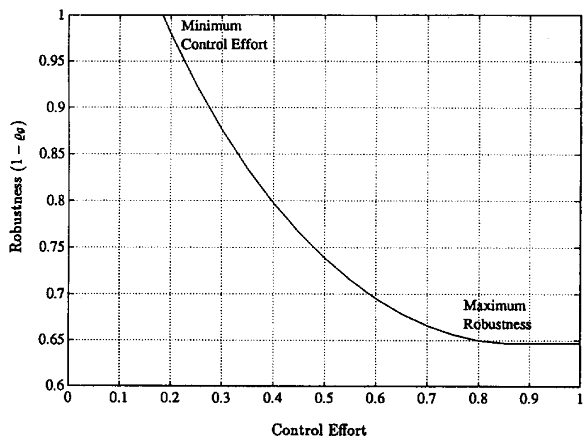

Fig 1. Robustness vs. Control Effort for Example 5

Figure 1 shows $\varrho_2(F)$ versus $\|F\|_2$, the norm of the solution to (25). For $\|F\|_2 = 1$, we recover the unconstrained solution, for $\|F\|_2 = 0.1850$, we get the minimum control effort capable of stabilizing (in the constrained sense) the nominal system. Note the trade-off between control effort and robustness. In particular, there exist a region where the curve is flat, i.e. the control effort can be reduced while essentially maintaining the same robustness obtained with a "maximum robustness" type design.

• *Example 6:* Polyhedral constraints, unstructured perturbation

Consider the following system:

$$\begin{aligned} A &= \begin{pmatrix} 0.8 & 0.5 \\ -0.4 & 1.2 \end{pmatrix} \; B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ G &= \begin{pmatrix} 1.0 & 2.0 \\ -1.5 & 2.0 \end{pmatrix} \; \underline{\omega} = \begin{pmatrix} 5.0 \\ 10.0 \end{pmatrix} \; \Omega = \{u : |u| \leq \gamma\} \end{aligned} \tag{35}$$

Since the constraint sets $\mathcal{G}$ and Ω are polyhedral, the synthesis problem can be cast in the following format:

$$\min_F \epsilon$$

subject to:

$$\begin{aligned} \|A + BF\|_{\mathcal{G}} &\leq \epsilon \\ \|F\|_{\mathcal{G},\Omega} &\leq 1 \end{aligned}$$

which can be transformed into an LP problem and solved using the simplex method. Note that a similar design algorithm was proposed by Vassilaki et. al. [20], although in their case the goal was to find admissible linear controllers for systems under polyhedral constraints, without taking into account robustness considerations. Figure 2 shows the constrained robustness measure versus γ, the bound on the control effort. Note that the minimum control effort required to stabilize the system is $\gamma = 2.6$.

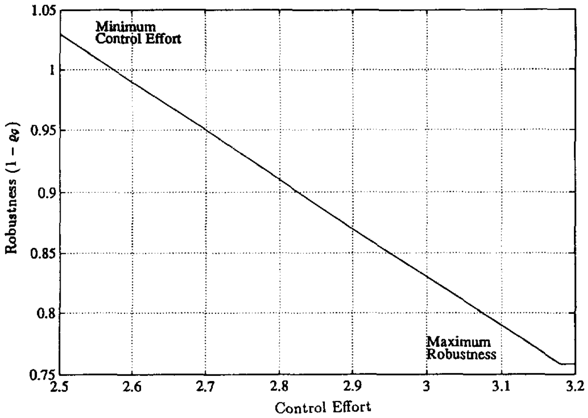

Fig 2. Robustness vs. Control Effort for Example 6

5. Conclusions

Most realistic control problems involve both some type of time-domain constraints and certain degree of model uncertainty. However, few of the control design methods currently available focus only on one aspect of the problem. Following the spirit of [6–7], in this paper we propose to approach time–domain constraints using an operator norm induced by the constraints to assess the stability properties of a family of systems. Specifically, in section II we introduced a robustness measure that indicates how well the family of systems under consideration satisfies a given set of constraints. In section III we explored the properties of this robustness measure for the case of additive parametric model uncertainty and we showed that our formalism provides a generalization of the well known technique of estimating robustness bounds from the solution of a Lyapunov equation. We then proposed, in section IV, a synthesis procedure for fixed order controllers, based upon maximization of the robustness measure subject to additional performance constraints such as bounds on the control effort. There we showed that the proposed design procedure leads to convex optimization problem. We believe that the results presented here will provide a valuable new approach to the problems of robust controllers analysis and design for linear systems. Further, since our approach is based purely upon time–domain analysis, we have reasons to believe the theory could be extended to encompass non–linear systems in a much more direct fashion than other currently used techniques.

Perhaps the more severe limitation of the theory in its present form arises from the fact that the incorporation of additional performance constraints of the form of a bound on the norm of a relevant transfer function results in non–convex optimization problems. We are currently looking into a solution to this problem by using an observer–based parametrization of all stabilizing controllers. It is expected that this formulation will be able to handle more general performance constraints as well as dynamic uncertainty, at the price of resulting in higher order controllers.

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Appendix A

Proof of Theorem 1 We begin by introducing two preliminary results:

• **Lemma 2:** Consider the system (S^a). Assume that the perturbation set D is *a closed cone with vertex at the origin* [10], i.e. $\Delta^o \in \mathcal{D} \iff \alpha\Delta^o \in \mathcal{D}\ \forall\ 0 \leq \alpha$ and that (S^a) is constraint stable (i.e. $\|A\|_{\mathcal{G}} < 1$). Let:

$$\Delta^o = \underset{\Delta\in\mathcal{D}}{\operatorname{argmin}}\,\{\|\Delta\|_{\mathcal{N}}: \|A+\Delta\|_{\mathcal{G}} = 1\} \tag{A1}$$

and consider a sequence $A^i \rightarrow A$ such that $\|A^i\|_{\mathcal{G}} < 1$. Finally, define the sequence λ^i as:

$$\lambda^i = \min_{\lambda\in\Re^+}\left\{\lambda: \|A^i + \lambda\Delta^o\|_{\mathcal{G}} = 1\right\} \tag{A2}$$

Then the sequence λ^i has an accumulation point at 1.

Proof: Since $\|A^i\|_{\mathcal{G}} < 1$ and since $\mathcal{D}$ is a closed cone it follows that λ^i is well defined. Furthermore, from (A2) it follows that:

$$\lambda^i \leq \frac{1+\|A^i\|_{\mathcal{G}}}{\|\Delta^o\|_{\mathcal{G}}} \leq \frac{2}{\|\Delta^o\|_{\mathcal{G}}} \tag{A3}$$

Hence from Bolzanno–Weierstrass' theorem [21] it follows that λ^i has an accumulation point $\tilde{\lambda}$ and that there exist a subsequence $\tilde{\lambda}^i \rightarrow \tilde{\lambda}$. Hence:

$$\|A^i + \tilde{\lambda}^i\Delta^o\|_{\mathcal{G}} = 1$$

and since $A^i \rightarrow A$ then:

$$\|A + \tilde{\lambda}\Delta^o\|_{\mathcal{G}} = 1 \tag{A4}$$

Assume that $\tilde{\lambda} < 1$ and let $\hat{\Delta} \stackrel{\Delta}{=} \tilde{\lambda}\Delta^o$ Then $\|\hat{\Delta}\|_{\mathcal{N}} < \|\Delta^o\|_{\mathcal{N}}$, $\|A+\hat{\Delta}\|_{\mathcal{G}} = 1$ and $\hat{\Delta} \in \mathcal{D}$ (since $\mathcal{D}$ is a cone) which contradicts (A1). Assume now that $\tilde{\lambda} > 1$. Then, for i large enough, $\tilde{\lambda}^i > 1$, which together with (A2) implies that:

$$\|A^i + \Delta^o\|_{\mathcal{G}} < 1 \tag{A5}$$

and hence:

$$\|A + \Delta^o\|_{\mathcal{G}} < 1 \tag{A6}$$

which contradicts (A1). Therefore $\tilde{\lambda} = 1$ ⋄.

• **Lemma 3:** Let $\rho_1 > 0, \rho_2 > 0$ and $0 \leq \lambda \leq 1$ be given numbers and assume that $\mathcal{D}$ is a cone with vertex at the origin. Consider the following sets:

$$\begin{aligned}
\rho_1 B\Delta &= \{\Delta \in \mathcal{D}: \|\Delta\|_{\mathcal{N}} \leq \rho_1\} \\
\rho_2 B\Delta &= \{\Delta \in \mathcal{D}: \|\Delta\|_{\mathcal{N}} \leq \rho_2\} \\
\rho B\Delta &= \{\Delta \in \mathcal{D}: \|\Delta\|_{\mathcal{N}} \leq \rho \stackrel{\Delta}{=} \lambda\rho_1 + (1-\lambda)\rho_2\}
\end{aligned} \tag{A7}$$

Then $\rho B\Delta \subseteq \lambda\rho_1 B\Delta + (1-\lambda)\rho_2 B\Delta$

Proof: Consider any $\Delta^o \in \rho B\Delta$. Then:

$$\begin{aligned}
\Delta^o &= \frac{\|\Delta^o\|_{\mathcal{N}}}{\rho}\left[\frac{\rho\Delta^o}{\|\Delta^o\|_{\mathcal{N}}}\right] \\
&= \frac{\|\Delta^o\|_{\mathcal{N}}}{\rho}\left[\lambda\rho_1\frac{\Delta^o}{\|\Delta^o\|_{\mathcal{N}}} + (1-\lambda)\rho_2\frac{\Delta^o}{\|\Delta^o\|_{\mathcal{N}}}\right] \\
&= [\lambda\Delta_1 + (1-\lambda)\Delta_2]
\end{aligned} \tag{A8}$$

where:

$$\begin{aligned}
\Delta_1 &= \alpha\rho_1\frac{\Delta^o}{\|\Delta^o\|_{\mathcal{N}}} \\
\Delta_2 &= \alpha\rho_2\frac{\Delta^o}{\|\Delta^o\|_{\mathcal{N}}} \\
\alpha &= \frac{\|\Delta^o\|_{\mathcal{N}}}{\rho} \leq 1
\end{aligned} \tag{A9}$$

The proof is completed by noting that from (A9) and the hypothesis it follows that $\Delta_1 \in \rho_1 B\Delta$ and $\Delta_2 \in \rho_2 B\Delta$ ⋄.

Proof of Theorem 1

Assume that $\varrho_{\mathcal{G}}^{\mathcal{N}}$ is *not* continuous. Then, given $\epsilon > 0$, for every $\delta > 0$ there exist A_δ such that $\|A_\delta - A\|_{\mathcal{G}} \leq \delta$ and $|\varrho_{\mathcal{G}}^{\mathcal{N}}(A_\delta) - \varrho_{\mathcal{G}}^{\mathcal{N}}| > \epsilon$. Hence there exist a sequence $A^i \rightarrow A$ such that $\varrho_{\mathcal{G}}^{\mathcal{N}^i} \not\rightarrow \varrho_{\mathcal{G}}^{\mathcal{N}}$. Furthermore, it is easily seen that the sequence $\varrho_{\mathcal{G}}^{\mathcal{N}^i}$ is bounded and therefore is contains a convergent subsequence. It follows that there exist a sequence $A^i \rightarrow A$ such that $\varrho_{\mathcal{G}}^{\mathcal{N}^i} \rightarrow \tilde{\varrho} \neq \varrho_{\mathcal{G}}^{\mathcal{N}}$. Let:

$$\Delta^i = \underset{\Delta\in\mathcal{D}}{\operatorname{argmin}}\left\{\|\Delta\|_{\mathcal{N}}: \|A^i + \Delta\|_{\mathcal{G}} = 1\right\} \tag{A10}$$

From (A10) it follows that $\|\Delta^i\|_{\mathcal{G}} \leq 1 + \|A^i\|_{\mathcal{G}}$. It follows then that the sequence Δ^i is bounded and therefore, since $R^{n\times n}$ with a finite dimensional matrix norm is complete and since $\mathcal{D}$ is a closed set, it has an accumulation point $\tilde{\Delta}$ (Bolzano Weierstrass) and a convergent subsequence $\tilde{\Delta}^i \rightarrow \tilde{\Delta}$ such that $\|A + \tilde{\Delta}\|_{\mathcal{G}} = 1$. Furthermore, from the definition of Δ^o it follows that

$$\tilde{\varrho} = \|\tilde{\Delta}\|_{\mathcal{N}} > \|\Delta^o\|_{\mathcal{N}} = \varrho_{\mathcal{G}}^{\mathcal{N}} \tag{A11}$$

Hence, for i large enough,

$$\|\tilde{\Delta}^i\|_{\mathcal{N}} > \|\Delta^o\|_{\mathcal{N}} \tag{A12}$$

Applying Lemma 3, we have that there exist a sequence $\lambda^i \rightarrow 1$ such that:

$$\lambda^i = \min_{\lambda\in\Re^+}\left\{\lambda: \|A^i + \lambda\Delta^o\|_{\mathcal{G}} = 1\right\} \tag{A13}$$

From (A12) and since $\lambda^i \rightarrow 1$ it follows that for i large enough

$$\begin{aligned}
\|\lambda^i\Delta^o\|_{\mathcal{N}} &< \|\tilde{\Delta}^i\|_{\mathcal{N}} \\
\|A^i + \lambda^i\Delta^o\|_{\mathcal{G}} &= 1
\end{aligned} \tag{A14}$$

and, since $\mathcal{D}$ is a cone, $\lambda^i\Delta^o \in \mathcal{D}$, which contradicts (A10). The proof is completed by noting that since *all* finite dimensional matrix norms are equivalent [14] then continuity in the $\|.\|_{\mathcal{G}}$ norm implies continuity in any other norm defined over $R^{n\times n}$ ⋄.

To prove concavity, start by considering a convex linear combination $A = \lambda A_1 + (1-\lambda)A_2$, $\lambda \leq 1$ of given matrices A_1 and A_2. Then, from Lemma 4 it follows that:

$$\begin{aligned}
\max_{\Delta\in\rho B\Delta}\|A+\Delta\|_{\mathcal{G}} &\leq \max_{\substack{\Delta_1\in\rho_1 B\Delta \\ \Delta_2\in\rho_2 B\Delta}}\|\lambda(A_1+\Delta_1) + (1-\lambda)(A_2+\Delta_2\|_{\mathcal{G}} \\
&\leq \lambda\max_{\Delta_1\in\rho_1 B\Delta}\|A_1+\Delta_1\|_{\mathcal{G}} + (1-\lambda)\max_{\Delta_2\in\rho_2 B\Delta}\|A_2+\Delta_2\|_{\mathcal{G}}
\end{aligned} \tag{A15}$$

Consider now the case where $\rho_1 = \varrho_{\mathcal{G}}^{\mathcal{N}}(A_1)$ and $\rho_2 = \varrho_{\mathcal{G}}^{\mathcal{N}}(A_2)$. Then it follows from the definition of $\varrho_{\mathcal{G}}^{\mathcal{N}}$ that both maximizations in the right hand side of (A15) yield 1 and therefore:

$$\max_{\Delta\in\rho B\Delta}\|A+\Delta\|_{\mathcal{G}} \leq 1 \tag{A16}$$

Hence, from the definition of $\varrho_{\mathcal{G}}^{\mathcal{N}}$:

$$\varrho_{\mathcal{G}}^{\mathcal{N}}[\lambda A_1 + (1-\lambda)A_2] \geq \varrho = \lambda\varrho_{\mathcal{G}}^{\mathcal{N}}(A_1) + (1-\lambda)\varrho_{\mathcal{G}}^{\mathcal{N}}(A_2)\ \diamond$$