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Norm Based Robust Control of Constrained Discrete Time Linear Systems

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Abstract

Most realistic control problems involve some types of constraints. However, up to date all the algorithms that deal with constrained problems assume that the system is perfectly known. On the other hand, during the last decade a considerable amount of time has been spent in the robust control problem. However, in its present form, the robust control theory can address only the idealized situation of completely unconstrained problems. In this paper we present a theoretical framework to analyze the stability properties of constrained discrete time systems under the presence of uncertainty and we show that this formalism provides a unifying approach, including as a particular case the well known technique of estimating robustness bounds from the solution of a Lyapunov equation. These results are applied to the problem of designing feedback controllers capable of stabilizing a family of systems while at the same time satisfying state–space constraints.

I. Introduction

A large class of problems frequently encountered in practice involves the control of linear systems with states restricted to closed convex regions of the space. Several methods have been proposed recently to deal with this class of problems (see [1] for a thorough discussion and several examples), but as a rule, all of these schemas assume exact knowledge of the dynamics involved (i.e. exact knowledge of the model). Such an assumption can be too restrictive, ruling out cases where good qualitative models of the plant are available but the numerical values of various parameters are unknow or even change during operation. On the other hand, during the last decade a considerable amount of time has been spent analyzing the question of whether some relevant quantitative properties of a system (most notably asymptotic stability) are preserved under the presence of unknown perturbations. This research effort has led to procedures for designing controllers, termed "robust controllers", capables of achieving desirable properties under various classes of perturbations. However, these design procedures cannot accommodate directly time domain constraints, although some progress has been made recently in this direction [2–4].

In this paper we present a theoretical framework to analyze the stability properties of constrained discrete time systems under the presence of uncertainties and we apply this framework to the problem of designing feedback controllers capable of stabilizing a family of systems while at the same time satisfying state space constraints. We believe that the results presented here will provide a valuable new approach for achieving what has been stated as the ultimate objective in control design: *"Achieve acceptable performance under perhaps substantial system uncertainty and under design constraints imposed either by the technology or by the nature of the physical system."*

The paper is organized as follows: In section II we introduce the concepts of *constrained stability* and *robust constrained stability* and we use these concepts to give a formal definition of the *robust constrained stability analysis* and *robust constrained stability design* problems. The *analysis* problem is studied in section III where we give necessary and sufficient conditions for constrained robustness and where we show that our approach includes as a special case the well known technique of estimating robustness bounds from the solution of a Lyapunov equation. In section IV we apply the results of section III to the *design* problem and we show that in cases of practical interest our approach yields a well behaved optimization problem. Finally, in section V, we summarize our results and we indicate directions for future research.

II. Definitions and Statement of the Problem

In this section we introduce a formal definition of the robust constrained control problem. We begin by introducing the concept of *constrained stability:*

- **Def. 1:** Consider the linear, time invariant, discrete time, unforced system modeled by the difference equation:

$$\underline{x}_{k+1} = A\underline{x}_k,\ k = 0, 1 \dots \qquad (S)$$

subject to the constraint

$$\underline{x} \in \mathcal{G} \subseteq R^n \qquad (1)$$

where $A \in R^{n*n}$ and where $\underline{x}$ indicates x is a vector quantity. The system (S) is *Constraint Stable* (C-stable) if for *any* point $\underline{\tilde{x}} \in \mathcal{G}$, the trajectory $\underline{x}_k(\underline{\tilde{x}})$ originating in $\underline{\tilde{x}}$ remains in $\mathcal{G}$ for all k.

We proceed to introduce now a restriction on the class of constraints allowed in our problem. As it will become apparent latter, the introduction of this restriction, termed the *constraint qualification hypothesis*, while not affecting significantly the number of real-world problems that can be handled by our formalism [5], introduces more structure into the problem. This additional structure is used in Lemma 1 to show that the constraints induce a norm in $\mathcal{G}$. In turn, this norm will play a key role in section III where we derive necessary and sufficient conditions for constrained stability.

Constraint Qualification Hypothesis

In this paper, we will limit ourselves to constraints of the form:

$$\underline{x} \in \mathcal{G} = \{\underline{x} \in R^n : (G(\underline{x}))_i \leq \omega_i,\ i = 1 \dots p\} \tag{2}$$

where $\underline{\omega} \in R^p$, $\omega_i > 0$ and where $G: R^n \to R^p$ has the following properties:

$$\begin{aligned} G(\underline{x})_i &\geq 0,\ i = 1 \dots p\ \forall\, \underline{x} \\ G(\underline{x}) = 0 &\iff \underline{x} = 0 \\ G(\underline{x} + \underline{y})_i &\leq G(\underline{x})_i + G(\underline{y})_i,\ i = 1 \dots p\ \forall\, \underline{x}, \underline{y} \\ G(\lambda \underline{x}) &= \lambda G(\underline{x}),\ 0 \leq \lambda \leq 1 \end{aligned} \tag{3}$$

In the next Lemma we show that $G(.)$ induces a norm, and we characterize $\mathcal{G}$ in terms of this norm.

- **Lemma 1:** Let:

$$v(x) = \max_{1 \leq i \leq p} \left\{ \frac{G(\underline{x})_i}{\omega_i} \right\} = \|W^{-1} G(\underline{x})\|_\infty \triangleq \|\underline{x}\|_{\mathcal{G}} \tag{4}$$

where $W = diag(\omega_1, \dots, \omega_p)$. Then $v(.)$ defines a norm in R^n and the set $\mathcal{G}$ can be characterized as:

$$\mathcal{G} = \{\underline{x} : \|\underline{x}\|_{\mathcal{G}} \leq 1\} \tag{5}$$

Proof: The proof of the lemma follows by noting that the constraint qualification hypothesis (3) implies that:

$$\|\underline{x}\|_{\mathcal{G}} = \|W^{-1} G(x)\|_\infty \tag{6}$$

satisfies the conditions for a norm in R^n.

Next, we take into account uncertainty in the dynamics by extending the concept of constrained stability to a family of systems and we define a quantitative way of measuring the "size" of the smallest destabilizing perturbation.

- **Def. 2:** Consider the system (S). Let the perturbed system (S_Δ) be defined as:

$$\underline{x}_{k+1} = (A + \Delta)\underline{x}_k \tag{S_Δ}$$

where Δ belongs to some perturbation set $\mathcal{D} \subseteq R^{n*n}$. The system (S) is *Robust Constraint Stable* (RC-stable) with respect to the set $\mathcal{D}$ if (S_Δ) is C-stable for all perturbation matrices $\Delta \in \mathcal{D}$.

- **Def. 3:** Let $\|.\|_{\mathcal{N}}$ be an operator norm defined in the set $\mathcal{D}$, and define the set $B\Delta^{\mathcal{N}}$ as the intersection of $\mathcal{D}$ and the origin centered $\mathcal{N}$-norm unity ball in parameter space, i.e.:

$$B\Delta^{\mathcal{N}} = \{\Delta \in \mathcal{D} : \|\Delta\|_{\mathcal{N}} \leq 1\}$$

The *Constrained Stability Measure* with respect to the norms $\|.\|_{\mathcal{N}}$ and $\|.\|_{\mathcal{G}}$, $\varrho_{\mathcal{G}}^{\mathcal{N}}$, is defined as:

$$\varrho_{\mathcal{G}}^{\mathcal{N}} = \max\{\mu : (S_\Delta) \text{ is C-stable with respect to } \mu B\Delta^{\mathcal{N}}\}$$

In the particular case that the induced operator norm $\|.\|_{\mathcal{G}}$ is used in the set $\mathcal{D}$, we will denote the Constrained Stability Measure as $\varrho_{\mathcal{G}}$ and the set $B\Delta^{\mathcal{N}}$ as $B\Delta$.

With the concepts introduced in this section, we are ready now to give a formal definition to our problem:

Robust Constrained Stability Analysis Problem:

Given the family of linear time invariant discrete time systems represented by (S_Δ) compute the constrained stability measure $\varrho_{\mathcal{G}}^{\mathcal{N}}$.

Linear Robust Constrained Stability Design Problem:

Given the family of linear time invariant discrete time systems represented by:

$$\underline{x}_{k+1} = (A + \Delta)\underline{x}_k + B\underline{u}_k$$

find a constant feedback matrix F such that for the closed-loop system:

$$\underline{x}_{k+1} = (A + BF + \Delta)\underline{x}_k \tag{S_{cl_Δ}}$$

the constrained stability measure is maximized.

III. Theoretical Results

In this section we present the basic results that are required to solve the *analysis* problem. These results will be used in section IV to solve the *design* problem. We begin by presenting a necessary and sufficient condition for Robust Constrained Stability of a family of systems. This result is then used to compute the actual value and lower bounds on the constrained stability measure introduced in the last section.

- **Theorem 1:** The system (S) is RC-stable with respect to the set $\mathcal{D}$ iff:

$$\|A + \Delta\|_{\mathcal{G}} \leq 1 \forall\, \Delta \in \mathcal{D} \tag{7}$$

where $\|.\|_{\mathcal{G}}$ denotes the induced operator norm, i.e.:

$$\|A + \Delta\|_{\mathcal{G}} = \max_{\|\underline{x}\|_{\mathcal{G}} = 1} \{\|(A + \Delta)\underline{x}\|_{\mathcal{G}}\} \tag{8}$$

Proof: Assume that (S_Δ) is constrained stable. Then, for any $\underline{x} \in \mathcal{G}$, $(A + \Delta)\underline{x} \in \mathcal{G}$. Hence, from Lemma 1 we have:

$$\begin{aligned} \|A + \Delta\|_{\mathcal{G}} &= \max_{\|\underline{x}\|_{\mathcal{G}} = 1} \{\|(A + \Delta)\underline{x}\|_{\mathcal{G}}\} \\ &= \max_{\underline{x} \in \mathcal{G}} \{\|(A + \Delta)\underline{x}\|_{\mathcal{G}}\} \leq 1 \end{aligned} \tag{9}$$

Conversely, assume that $\|A + \Delta\|_{\mathcal{G}} \leq 1$ and let $\underline{x}$ be an arbitrary point in $\mathcal{G}$ such that $\|\underline{x}\|_{\mathcal{G}} \neq 0$. Then we have:

$$\frac{\|(A + \Delta)\underline{x}\|_{\mathcal{G}}}{\|\underline{x}\|_{\mathcal{G}}} \leq \max_{\underline{x} \in \mathcal{G}} \left\{ \frac{\|(A + \Delta)\underline{x}\|_{\mathcal{G}}}{\|\underline{x}\|_{\mathcal{G}}} \right\} = \|A + \Delta\|_{\mathcal{G}} \leq 1 \tag{10}$$

and therefore

$$\|(A + \Delta)\underline{x}\|_{\mathcal{G}} \leq \|\underline{x}\|_{\mathcal{G}} \leq 1 \tag{11}$$

which implies that $(A + \Delta)\underline{x} \in \mathcal{G}$. The proof is completed by noting that if $\|\underline{x}\|_{\mathcal{G}} = 0$ then $(A + \Delta)\underline{x} = 0 \in \mathcal{G}$ ◇.

Remark: Note that if $\|A + \Delta\|_{\mathcal{G}} < 1$ for all $\Delta \in \mathcal{D}$ then $(A + \Delta)$ is a contraction mapping and the system (S_Δ) is asymptotically stable [6].

• **Corollary 1:**

$$\varrho_G^{\mathcal{N}} = \min_{\Delta \in \mathcal{D}} \{ \|\Delta\|_{\mathcal{N}} : \|A + \Delta\|_G = 1 \} \tag{12}$$

In the next lemma we introduce a *lower bound* of the constrained stability measure. In Theorem 2 we show that for *unstructured* perturbations (i.e. the case where $\mathcal{D} \equiv R^{n*n}$) this lower bound is saturated.

• **Lemma 2:**

$$\varrho_G \geq 1 - \|A\|_G \tag{13}$$

Proof: Let Δ_1 be such that $\|A + \Delta_1\|_G = 1$. Then:

$$1 = \|A + \Delta_1\|_G \leq \|A\|_G + \|\Delta_1\|_G \tag{14}$$

or

$$\|\Delta_1\|_G \geq 1 - \|A\|_G \tag{15}$$

Hence:

$$\varrho_G = \min_{\mathcal{D}} \|\Delta_1\|_G \geq 1 - \|A\|_G \diamond \tag{16}$$

• **Theorem 2:** For the unstructured perturbation case, i.e. the case where $\mathcal{D} \equiv R^{n*n}$, condition (13) is saturated.

Proof: Let:

$$\lambda = 1 - \|A\|_G \tag{17}$$

and define:

$$\Delta^o \triangleq \frac{\lambda A}{\|A\|_G} \tag{18}$$

Then:

$$\|\Delta^o\|_G = \lambda \tag{19}$$

and

$$\begin{aligned} \|A + \Delta^o\|_G &= \|A + \frac{\lambda A}{\|A\|_G}\|_G \\ &= \|A\|_G \left(1 + \frac{\lambda}{\|A\|_G}\right) \\ &= \|A\|_G + \lambda = 1 \end{aligned} \tag{20}$$

Hence, from the definition of ϱ_G we have that:

$$1 - \|A\|_G = \lambda = \|\Delta^o\|_G \geq \varrho_G \tag{21}$$

but, from Lemma 2 we have that:

$$\varrho_G \geq 1 - \|A\|_G \tag{22}$$

Hence it follows that $\varrho_G = 1 - \|A\|_G \diamond$

3.1 Quadratic Constraints Case:

In this subsection we particularize our theoretical results for the special case where the constraint region is an hyperellipsoid, i.e. the case where:

$$G(\underline{x}) = (\underline{x}^T P \underline{x})^{\frac{1}{2}}, \; P \in R^{n*n} \text{ positive definite} \tag{23}$$

We will show that in this case our approach yields a generalization of the well known technique of estimating the robustness measure by using quadratic based Lyapunov functions, (see [7] and references therein) by obtaining robustness bounds previously derived in this context. Moreover, using our approach we will show that in some cases these bounds give the *actual* value of the constrained stability measure.

Example 1: (unstructured perturbation)

In this case, Theorem 2 yields $\varrho_G = 1 - \|A\|_G$ where:

$$\|A\|_G^2 = \|A\|_P^2 = \max_{\underline{x}} \left(\frac{\underline{x}^T A^T P A \underline{x}}{\underline{x}^T P \underline{x}} \right) \tag{24}$$

Consider now the case where $\varrho_G > 0$. Then, there exists Q positive definite such that:

$$A^T P A - P = -Q \tag{25}$$

and:

$$\begin{aligned} \|A\|_G^2 &= \max_{\underline{x}} \left(1 - \frac{\underline{x}^T Q \underline{x}}{\underline{x}^T P \underline{x}}\right) \\ &\leq 1 - \frac{\sigma_{min}(G)}{\sigma_{Max}(P)} \end{aligned} \tag{26}$$

Hence:

$$\varrho_G = 1 - \|A\|_G \geq 1 - \left(1 - \frac{\sigma_{min}(Q)}{\sigma_{Max}(P)}\right)^{\frac{1}{2}} \tag{27}$$

A common technique in state space robust analysis is to obtain robustness bounds from equation (25) ([8, 9]). This case can be accommodated by our formalism by recognizing the fact that once P is selected, the system becomes effectively constrained to remain within an hyperellipsoidal region. It has been suggested ([8, 9]) that good robustness bounds can be obtained from (25) when P is selected such that $Q = I$. In this case our approach yields:

$$\varrho_G = 1 - \|A\|_G = 1 - \left(1 - \frac{1}{\sigma_{Max}(P)}\right)^{\frac{1}{2}} \tag{28}$$

which coincides with the robustness bound found by Sezer and Siljak [9]. Note however that our derivation shows this bound to be exact.

Example 2: (Unstructured perturbation, A semisimple)

Consider the case where A is semisimple, i.e.

$$\begin{aligned} &A = L^{-1} \Lambda L \\ &\Lambda = diag\left\{ \begin{pmatrix} \sigma_1 & \omega_1 \\ -\omega_1 & \sigma_1 \end{pmatrix}, \ldots, \begin{pmatrix} \sigma_p & \omega_p \\ -\omega_p & \sigma_p \end{pmatrix}, \sigma_{p+1}, \ldots, \sigma_n \right\} \end{aligned} \tag{29}$$

Then, the maximum of the stability measure, ϱ_G, over all possible positive definite matrices P, is achieved for $P = L^T L$.

Proof: From (24) and (29) we have:

$$\begin{aligned} \|A\|_P^2 &= \max_{\underline{x}} \left\{ \frac{\underline{x}^T A^T P A \underline{x}}{\underline{x}^T P \underline{x}} \right\} \\ &= \max_{\underline{x}} \left\{ \frac{\underline{x}^T L^T L^{-T} A^T L^T L A L^{-1} \underline{x}}{\underline{x}^T L^T L \underline{x}} \right\} \\ &= \max_{\|\underline{y}\|_2 = 1} \|L A L^{-1} \underline{y}\|_2^2 \\ &= \|L A L^{-1}\|_2^2 = \sigma_{Max}^2(\Lambda) \end{aligned} \tag{30}$$

From (29) it follows that:

$$\sigma_{Max}(\Lambda) = \max_i |\lambda_i^A| = \rho(A) \tag{31}$$

where λ_i^A denotes the eigenvalues of A and $\rho(.)$ denotes the spectral radius. Since the spectral radius is always *smaller* than any other matrix norm [10] we have that:

$$\|A\|_M \geq \rho(A) = \|A\|_{L^T L} \tag{32}$$

and therefore:

$$\begin{aligned} &\varrho_{L^T L} = 1 - \|A\|_{L^T L} \\ &\quad \geq \varrho_M = 1 - \|A\|_M \; \forall \, M \in R^{n*n}, \text{ positive definite} \diamond \end{aligned} \tag{33}$$

3.2 Polyhedral Constraints

In this subsection we consider the case where the region $\mathcal{G}$ is polyhedral, i.e. the case where:

$$G(x) = |G\underline{x}| \tag{34}$$

where $G \in R^{n*n}$ and $\text{rank}(G) = n$, i.e. $(G^T G)$ is non-singular. We begin by showing that in this case the induced norm of an operator M, $\|M\|_{\mathcal{G}}$, can be expressed in terms of the *infinity norm* of an operator H linearly related to M. This result is used to obtain a particular expression of condition (12) which in turn allows for the computation of the constrained stability measure as the solution of a Linear Programming problem.

- **Lemma 3:** Let $M \in R^{n*n}$ and define $H \triangleq GM(G^T G)^{-1}G^T$. Then,

$$\|M\|_{\mathcal{G}} = \|W^{-1}HW\|_\infty \tag{35}$$

Proof: From the definition of H we have that:

$$GM = HG \tag{36}$$

Using (36) and Lemma 1 we have:

$$\begin{aligned} \|M\|_{\mathcal{G}} &= \max_{\|\underline{x}\|_{\mathcal{G}}=1} \|M\underline{x}\|_{\mathcal{G}} = \max_{\|\underline{x}\|_{\mathcal{G}}=1} \|W^{-1}GM\underline{x}\|_\infty \\ &= \max_{\|W^{-1}G\underline{x}\|_\infty=1} \|W^{-1}HG\underline{x}\|_\infty \\ &= \max_{\|\underline{y}\|_\infty=1} \|W^{-1}HW\underline{y}\|_\infty = \|W^{-1}HW\|_\infty \diamond \end{aligned} \tag{37}$$

The results of Lemma 3 can be used to efficiently compute $\varrho_{\mathcal{G}}^{\mathcal{N}}$ as the minimum of the solution of p Linear Programming problems as follows:

- **Lemma 4:** Let $\varrho_i^{\mathcal{N}}$ be the solution of the following optimization problem:

$$\varrho_i^{\mathcal{N}} = \min_{\Delta \in \mathcal{D}} \{\|\Delta\|_{\mathcal{N}} : \|W^{-1}(H + \Delta H)W\|_1^{(i)} \geq 1\} \tag{38}$$

where $\|M\|_1^{(i)}$ indicates the L_1 norm of the i^{th} row of the matrix M and where H and ΔH are defined as in Lemma 3. Then:

$$\varrho_{\mathcal{G}}^{\mathcal{N}} = \min_{1 \leq i \leq p} \{\varrho_i^{\mathcal{N}}\} \tag{39}$$

Proof: The proof (ommited for space reasons) is based upon assuming that (39) is false and showing that this leads to a contradiction.

Example 3: (unstructured perturbation)

Consider the following case:

$$A = \begin{pmatrix} 0.8 & 0.5 \\ -0.0208 & 0.5083 \end{pmatrix} \; G = \begin{pmatrix} 1.0 & 2.0 \\ -1.5 & 2.0 \end{pmatrix} \; \underline{\omega} = \begin{pmatrix} 5.0 \\ 10.0 \end{pmatrix} \tag{40}$$

Then, from the definition of H, we have that:

$$H = \begin{pmatrix} 0.7583 & 0.0 \\ -0.4167 & 0.55 \end{pmatrix}, \; \|A\|_{\mathcal{G}} = 0.7583 \tag{41}$$

and, from Lemma 4,

$$\varrho_i = \min_{\|\Delta\|_{\mathcal{G}}} \left\{ \|\Delta\|_{\mathcal{G}} : \sum_{j=1}^{2} \frac{|H + \Delta|_{ij}\omega_j}{\omega_i} = 1 \right\} \; i = 1, 2 \tag{42}$$

Casting the problems (42) into a linear programming form and solving we have that:

$$\varrho_1 = 0.2417, \; \varrho_2 = 0.2417 \;\text{ and }\; \varrho_{\mathcal{G}} = \min_{1 \leq i \leq 2} \varrho_i = 0.2417$$

Note that in this case $\varrho_{\mathcal{G}} = 1 - \|A\|_{\mathcal{G}} = 0.2417$ as shown in Theorem 2.

IV. Application to Robust Controllers Design

In this section we apply our formalism to solve the *linear robust constrained stability design problem* introduced in section II. From Theorem 1 it follows that a full state feedback matrix F such that the constrained stability measure, $\varrho_{\mathcal{G}}^{\mathcal{N}}$, of the closed loop system is maximized can be selected by solving the following max-min problem:

$$\max_F \left\{ \min_{\Delta \in \mathcal{D}} \|\Delta\|_{\mathcal{N}} \right\} \tag{43}$$

subject to:

$$\|A + BF + \Delta\|_{\mathcal{G}} = 1$$

Define:

$$\varrho_{\mathcal{G}}^{\mathcal{N}}(F) \triangleq \min_{\Delta \in \mathcal{D}} \{\|\Delta\|_{\mathcal{N}} : \|A + BF + \Delta\|_{\mathcal{G}} = 1\} \tag{44}$$

then (43) is equivalent to the following optimization problem:

$$\max_F \{\varrho_{\mathcal{G}}^{\mathcal{N}}(F)\} \tag{45}$$

Note that since the function defined is (44) is in general *non-differentiable*, non-smooth optimization techniques must be used to solve (45). Moreover, in general nothing can be stated about the existence of local minima of (44). Hence a general non-smooth optimization algorithm could conceivably get trapped at local extrema. However, in the next theorem we show that for a case of practical interest, (45) reduces to the well behaved problem of finding the maximum of a concave function.

- **Theorem 3:** Consider the particular case where $\mathcal{D}$ is a cone with vertex at the origin, (i.e. $\Delta \in \mathcal{D} \iff \lambda\Delta \in \mathcal{D}$; $\lambda \geq 0$). Then $\varrho_{\mathcal{G}}^{\mathcal{N}}(F)$ is a **concave** function.

The proof of the theorem is given in Appendix A. Note that the class of sets considered in this theorem includes as a particular case sets of the form:

$$\mathcal{D} = \left\{ \Delta : \Delta = \sum_1^m \mu_i E_i ; \; \mu_i \geq 0, \; E_i \text{ given} \right\} \tag{46}$$

which has been the object of much interest lately ([11-13] and references therein).

At the present time, we are investigating several methods of solving (45), and a future article is planned to report the results. In this paper, we will limit ourselves to the restricted case of *unstructured* perturbations. In this case, from Theorem 2 we have that $\varrho_G = 1 - \|A + BF\|_G$. Hence, (45) reduces to solving the following *convex* minimization problem:

$$F = \underset{F}{\operatorname{argmax}}\, \varrho_G = \underset{F}{\operatorname{argmin}}\, \|A + BF\|_G \qquad (47)$$

In the remainder of this section, we will indicate how problem (47) can be solved for the particular cases of quadratic and polyhedral constraints. We begin by considering quadratic constraints:

4.1 Quadratic Constraints Case

In this case (47) can be solved using standard results on matrix dilations [14]. Let $P = L^T L$ and assume that $rank(B) = m$. Then, since the 2-norm is invariant under orthonormal transformations we have that:

$$\|A + BF\|_{L^TL} = \|L(A + BF)L^{-1}\|_2 = \|\tilde{A} + \tilde{B}\tilde{F}\|_2 \qquad (48)$$

where $\tilde{A} = QLA(QL)^{-1}$, $\tilde{B} = QLB$, $\tilde{F} = F(QL)^{-1}$ and where Q is an orthonormal matrix such that:

$$\tilde{B} = \begin{pmatrix} B_1 \\ O \end{pmatrix},\ B_1 \text{ invertible} \qquad (49)$$

Then:

$$\tilde{A} + \tilde{B}\tilde{F} = \begin{pmatrix} A_1 + B_1\tilde{F} \\ A_2 \end{pmatrix} \qquad (50)$$

and it follows that the optimal F is such that $A_1 + B_1\tilde{F} = 0$, i.e. $\tilde{F}^o = -B_1^{-1}\tilde{A}_1$ and that $\min_F \|A + BF\|_{L^TL} = \|A_2\|_2 = \sigma_{Max}(A_2)$.

Example 4:

Consider the system:

$$A = \begin{pmatrix} 0.8 & 0.5 \\ -0.4 & 1.2 \end{pmatrix}\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}\ \underline{\omega} = 1$$

A direct application of (50) yields:

$$F^o = (\,0.4 \quad -1.2\,),\ \|A + BF^o\|_I = 0.9434,\ \varrho_I = 0.0566$$

4.2 Polyhedral Constraints

When the constraints are polyhedral, (47) can be cast in the following format:

$$\min_F \epsilon \qquad (51)$$

subject to:

$$\|A + BF\|_G \leq \epsilon \qquad (52)$$

By using (35), the inequalities (52) can be transformed into:

$$|G(A + BF)(G^TG)^{-1}G^T|\underline{\omega} \leq \epsilon\underline{\omega} \qquad (53)$$

The optimization problem defined by (51) and (53) can be cast into a Linear Programming problem and solved using the simplex method.

Example 5:

Consider the following system:

$$A = \begin{pmatrix} 0.8 & 0.5 \\ -0.4 & 1.2 \end{pmatrix}\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$G = \begin{pmatrix} 1.0 & 2.0 \\ -1.5 & 2.0 \end{pmatrix}\ \underline{\omega} = \begin{pmatrix} 5.0 \\ 10.0 \end{pmatrix}$$

Using Linear Programming we get:

$$F = (\,0.3792 \quad -0.6917\,)$$

$$A_{cl} = \begin{pmatrix} 0.8 & 0.5 \\ -0.0208 & 0.5083 \end{pmatrix},\ eig(A_{cl}) = \begin{pmatrix} 0.55 \\ 0.7583 \end{pmatrix}$$
$$\|A_{cl}\|_G = 0.7583 = \rho(A)$$

where A_{cl} denotes the closed-loop matrix, $eig(A_{cl})$ its eigenvalues and $\rho(A)$ its spectral radius. Hence we have:

$$\varrho_G = 1 - \|A_{cl}\|_G = 0.2417$$

V. Conclusions

As we mention in the introduction, the ultimate objective in control design can perhaps be summarized as [2]: "*Achieve acceptable performance under perhaps substantial system uncertainty and under design constraints*". This statement looks deceptively simple, but up to date design techniques focus either only on the uncertainty issue or only on the constraint satisfaction issue. In this paper we presented a theoretical framework capable of simultaneously addressing both issues. Since most physically generated constraints have a natural expression in time domain, our analysis focuses in state-space robustness analysis.

In section II, we introduced the concept of *robust constrained stability* and we introduced a quantity, the *constrained stability measure*, that measures the "size" of the smallest destabilizing perturbation. In section III we presented necessary and sufficient conditions guaranteeing constrained robust stability and we showed that our formalism provides a unifying approach, including as a particular case the well-known technique of estimating robustness bounds from the solution of a Lyapunov equation. Finally, in section IV, we considered the *design* problem. There, we showed that a full state feedback matrix that maximizes the stability measure of the closed loop system can be found as the solution of a game-like problem. Although the properties of this problem are still unknown for the general case, we proved that in a specific case that has been the object of much attention lately, it leads to the well behaved problem of finding the maximum of a concave function. Finally, we considered the particular case of *unstructured* perturbations and we showed that in this case the problem reduces to the simpler case of finding the minimum of a convex (albeit perhaps non-differentiable) function.

We believe that the results presented here will provide a valuable new approach to the problems of robust controllers analysis and design for linear systems. Further, since our approach is based purely upon time-domain analysis, we have reasons to believe the theory could be extended to encompass non-linear systems in a much more direct fashion that some of the currently used techniques.

Perhaps the most severe limitation to the theory in its present form, arises from the fact that the design procedure is limited to constant linear feedback. However, it is clear that only a fraction of the feasible constrained problems admits a constant linear feedback solution. It is our goal to extend the theory to include the non-linear, optimization-based controllers that where the subject of [5].

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Appendix A: Proof of Theorem 3

The following preliminary lemma is introduced without proof:

- **Lemma 5:** Let $\rho_1 > 0, \rho_2 > 0$ and $0 \leq \lambda \leq 1$ be given numbers. Consider the following sets:

$$\begin{aligned} \rho_1 B\Delta &= \{\Delta \in \mathcal{D}: \|\Delta\|_{\mathcal{N}} \leq \rho_1\} \\ \rho_2 B\Delta &= \{\Delta \in \mathcal{D}: \|\Delta\|_{\mathcal{N}} \leq \rho_2\} \\ \rho B\Delta &= \{\Delta \in \mathcal{D}: \|\Delta\|_{\mathcal{N}} \leq \rho \triangleq \lambda\rho_1 + (1-\lambda)\rho_2\} \end{aligned} \qquad (A1)$$

Then:

$$\rho B\Delta \subseteq \lambda\rho_1 B\Delta + (1-\lambda)\rho_2 B\Delta$$

Proof of Theorem 3:

Given two matrices F_1 and F_2, consider a convex linear combination $F = \lambda F_1 + (1-\lambda)F_2$. Then:

$$\begin{aligned} \max_{\Delta \in \rho B\Delta} \|A + BF + \Delta\|_{\mathcal{G}} &\leq \max_{\substack{\Delta_1 \in \rho_1 B\Delta \\ \Delta_2 \in \rho_2 B\Delta}} \|\lambda(A + BF_1 + \Delta_1) \\ &\quad + (1-\lambda)(A + BF_2 + \Delta_2\|_{\mathcal{G}} \\ &\leq \lambda \max_{\Delta_1 \in \rho_1 B\Delta} \|A + BF_1 + \Delta_1\|_{\mathcal{G}} + \\ &\quad (1-\lambda) \max_{\Delta_2 \in \rho_2 B\Delta} \|A + BF_2 + \Delta_2\|_{\mathcal{G}} \end{aligned} \qquad (A2)$$

Consider now the case where $\rho_1 = \varrho_{\mathcal{G}}^{\mathcal{N}}(F_1)$ and $\rho_2 = \varrho_{\mathcal{G}}^{\mathcal{N}}(F_2)$. Then it follows from the definition of $\varrho_{\mathcal{G}}^{\mathcal{N}}$ that both maximizations in the right hand side of (A2) yield 1 and therefore:

$$\max_{\Delta \in \rho B\Delta} \|A + BF + \Delta\|_{\mathcal{G}} \leq 1 \qquad (A3)$$

Hence, from the definition of $\varrho_{\mathcal{G}}^{\mathcal{N}}$:

$$\varrho_{\mathcal{G}}^{\mathcal{N}}[\lambda F_1 + (1-\lambda)F_2] \geq \varrho = \lambda\varrho_{\mathcal{G}}^{\mathcal{N}}(F_1) + (1-\lambda)\varrho_{\mathcal{G}}^{\mathcal{N}}(F_2) \diamond$$

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