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SUBOPTIMAL CONTROL OF LINEAR SYSTEMS WITH STATE AND CONTROL INEQUALITY CONSTRAINTS.

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Abstract.

A suboptimal controller based upon on-line quadratic programming is described. Theoretical results are presented to show that such a controller is optimal under the assumption that there are no constraints on the computation time. Finally, an implementation of a suboptimal controller that takes such constraints into account is described.

Introduction.

A large class of problems frequently encountered in practice involves the control of linear time invariant systems with control and state inequality constraints. Noldus [1] analyzed the controllability of such systems using a Lyapunov approach. Gutman [2, 3] gives an algorithm to find maximal initial condition sets (defined below) and a time varying feedback control law to transfer such systems to the origin.

Once the problem of transfering such a system from an initial condition to the origin is determined to be feasible, a question of interest is how to effect this transfer in a way that is optimal in some sense. Classically this problem has been solved using Mathematical Programming techniques [5] - [7]. These approaches have the disadvantage of yielding open loop control laws that have to be recalculated entirely if the system is perturbed. A different approach is used by Gutman in [4] where a regulator based upon the on-line use of Linear Programming for the minimum time control of a reservoir is described.

In this paper we propose an optimal feedback controller for discrete linear time invariant systems with quadratic cost function and linear control and state inequality constraints based upon real time Quadratic Programming. The first part of the paper states the basic theorical results. In the second part, we describe a controller, based upon these results and heuristics, that takes into account the constraints imposed by the limited amount of time available between samples to solve the QP program. By casting the problem as an on-line optimization problem, we address the fundamental issue that open loop control cannot respond to present conditions. By including constraints on the computation time, we make the problem realistic.

Statement of the problem.

Given the linear, time invariant, controllable system:

$$x(k+1) = Ax(k) + Bu(k) \qquad (1)$$

with $x(0) = x_o$, $u(k) \in R^m$, $x(k) \in R^n$ and the additional constraints:

$$u(k) \in \Omega,\ x(k) \in G \qquad (2)$$

where Ω and G are compact convex polyhedrons containing the origin in their interior,

find $u(k)$ to minimize $J(x,u) = 0.5(\sum_0^\infty x^T(k)Qx(k) + u^T(k)Ru(k))$

subject to (1) and (2), with Q positive semidefinite and R positive definite.

Definitions.

The initial condition set $X \subseteq G$ is the set such that:

$X = \{\ x$: if $x(0) = x$ in (1), then there exist a sequence $u(k) \in \Omega$ such that $x(k) \in X$ for every k and $\lim_{k\to\infty} x(k) = 0\ \}$

The set X is called Ω invariant [2].

Given a constant feedback matrix K, the set X_K is defined as:

$X_K = \{$ x: if $x(0) = x$ in (1) then:
1)the feedback law $u(k) = -Kx(k)$ generates a control $u(k) \in \Omega$ for every k
2)the states $x(k)$ of the closed loop system never leave the region G
3)$\lim_{k\to\infty} x(k) = 0\ \}$

Lemma. Let K_o be the optimal feedback gain obtained using the standard Linear Quadratic procedure. Then, if the initial condition $x(0) \in X_{Ko}$, the closed loop system states $x(k) \in X_{Ko}$ for every k.

The proof follows from the fact that the system is time invariant and any point of an optimal trajectory, if taken as the initial condition, will yield the same trajectory.

Remarks: 1)If the initial condition $x(0) \in X_{Ko}$ then the solution to the constrained optimization coincides with the Linear Quadratic solution.
2) The lemma establishes that once the system reaches the region X_{Ko}, it remains there .

Theorems.

The following theorems provide the required theoretical background. Their proofs follow from the behavior of linear systems, convexity and continuity arguments and Hamilton-Jacobi-Bellman theory.

Theorem I. There exist an open ball $B(0,r) \subseteq X_{Ko}$.

Theorem II. Let $Y \subseteq G$ be a convex polyhedron given by its vertices y_i, $i = 1,n$. Then $Y \subseteq X_{Ko}$ iff $y_i \in X_{Ko}$.

Theorem III. Consider the following optimization problems:

$$\min_u \{\ J(x,u) = 0.5(\sum_0^\infty x^TQx + u^TRu)\} \qquad (3)$$

$$\min_u \{\ J_n(x,u) = 0.5(\sum_0^{n-1} x^TQx + u^TRu) + 0.5x^T(n)Sx(n)\} \qquad (4)$$

subject to (1) and (2), with Q positive semi definite and R positive definite, where S is the solution to the Riccati equation derived from the unconstrained Linear Quadratic problem*.

Claim III-a: An optimal trajectory, $x_o(k)$, k=1,2.. which solves (4) is also a solution of (3) provided that $x_o(n) \in X_{Ko}$.
Claim III-b: Consider now the optimization problems:
$\min J_n(x,u)$ (5)
$\min J_m(x,u)$ with $m > n$ (6)
Then a solution $x_o(k)$ to (5) is also a solution to (6) provided that $x_o(n) \in X_{Ko}$.

* We will assume that the initial condition of the system $x(0) \in X$, so there is at least one feasible solution to the problem.

Control algorithm.

From theorem III it follows that the solution to the optimization problem (3) can be obtained by solving the sequence of quadratic programs of type (4):

$$\min J_n(x,u) \qquad n=1,2...,u(k) \in \Omega,\ x(k) \in G$$

until a solution such that $x(n) \in X_{Ko}$ is obtained. In addition, claim III-b shows that once such a solution is obtained, no further improvement in the cost will be achieved by increasing n.

Based upon these results we propose the following real time control algorithm using on-line quadratic programming:
Consider the discrete linear time invariant system given by (1). Let y be the current state of the system, k the current time instant and ΔT the sampling interval. Then:

If $y \in X_{Ko}$ the solution coincides with that of the unconstrained LQ problem: $u(m) = -K_o x(m)$ for $m \geq k$.
If $y \notin X_{Ko}$ solve the sequence of quadratic programs given by:

$$\min_u \{ J_n(x,u) = 0.5(\sum_0^{n-1} x^TQx + u^TRu) + 0.5{*}x^T(n)Sx(n)\} \quad n=1,2.,$$

subject to (1) and (2) with $x(0) = y$ until a solution $x(k)$, $u(k)$ and a number n such that $x(n) \in X_{Ko}$ are obtained. If such a solution has not been reached during the interval ΔT available for computations use the solution of the last QP program solved as the control law.

Note that theorem II provides an easy way of constructing a region contained in X_{Ko} that is required by the algorithm.

A simple example:

Consider the system given by [5]: $x(k+1) = Ax(k) + Bu(k)$ with:

$$A = \begin{bmatrix} 1.0 & 0.2212 \\ 0.0 & 0.7788 \end{bmatrix} \qquad B = \begin{bmatrix} 0.0288 \\ 0.2212 \end{bmatrix} \qquad (7)$$

The objective is to drive the system to the origin with unspecified final time and with minimum energy, so the matrices Q and R are selected to be the identity of appropiate dimensions.
Assume that the admissible states and control are restricted to:

$$|x_1| \leq 1.5, \quad |x_2| \leq 0.3, \quad |u| \leq 0.5 \qquad (8)$$

The unconstrained Linear Quadratic solution is given by:

$$u = -K_o x, \quad K_o = [\, 0.8831 \quad 0.8811 \,] \qquad (9)$$

It is easily verified that the points (0.5,0); (0,0.3); (-0.5,0); and (0,-0.3) belong to the region X_{Ko}. Hence by theorem II the square that has these points as vertices is entirely contained in X_{Ko}. This square was employed as a criteria for stopping the formulation of succesive quadratic programs.

Figure 1 shows the response of the controller to the initial conditions (1.0,0.3). For the on-line quadratic programming we employed Wolfe's algorithm [8] and the controller was limited to computing only 3 terms of the sequence ($n \leq 3$). Note that up to t = 1 sec. the system is limited by the control constraint. From t = 1.5 s to 2.5 s the control is unsaturated and the system is limited by the constrains on the velocity.

Conclusions.

In this paper we present a feedback controller based upon a sequence of quadratic programs that are solved at each sampling interval. From the results of the theorems presented it is clear that the performance of this controller approaches the performance of the true optimal controller when the sampling interval ΔT is large enough to allow for the computation of several terms of the sequence.

We believe that this controller will be valuable for the control of systems where the classical approaches of computing and storing a family of extremal curves or solving a Hamilton-Jacobi type equation in real time are not applicable. An example of such a situation could be a microprocessor controlled robotic system.

There are many open questions which remain to be resolved. Perhaps most important, the stability properties of the controller are unclear when there is not enough time to compute the true minimum and the last computed control law is used. A related question is that of finding maximal regions in the state space where the proposed controller can be proved to be stabilizing. Another need is to find algorithms to construct maximal regions contained in X_{Ko}. Finaly, a very practical concern is to find time-optimal and memory-optimal quadratic programming algorithms that will allow for the computation of enough terms of the sequence with the limited resources that one expects to have on a microprocessor based system.

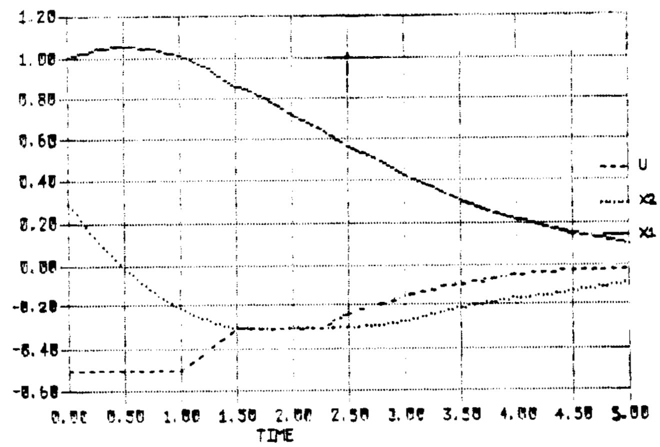

Figure 1. $x_1(t)$, $x_2(t)$ and $u(t)$ for the system (7).

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