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Set–Membership Identification of Parametric Systems

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Abstract— **This paper considers the problem of robust, set membership identification of parametric LTI plants, using frequency domain data. We consider the case of noisy data, and provide tractable, LMI–based conditions for computing inner and outer approximations to the set of parameters so that the resulting plant is consistent with a given *a priori* information and interpolate the experimental data. The results presented here are useful for model (in)validation, robust control synthesis and fault diagnosis.**

I. INTRODUCTION

Many problems of practical interest lead to uncertainty structures involving unknown but bounded parameters. Examples include model (in)validation and control synthesis for plants where the structure of the model is known, for instance from first principles, up to the value of some parameters. In these cases, covering the uncertainty structure with dynamic (non–parametric) uncertainty may lead to (potentially very) conservative results, since it allows for not physically realizable situations. For example, in the case of electric circuits, components always have some tolerance that can be represented by a given interval. However, if such uncertainty is modeled as non–parametric, then additional uncertainty with no-physical meaning is introduced, i.e. a real interval is covered by a complex ball.

Analysis/Synthesis with parametric uncertainty has been a long standing problem in the control community, see for instance ([1], [2]). Unfortunately, while in many cases of practical importance the analysis problem is tractable, it is well known that robust control synthesis for parametric uncertainty leads to generically NP–hard problems. Nevertheless, in the past few years, there has been a renewed interest in the subject, motivated by the emergence of a *risk–adjusted* approach to robust control [3], [4], [5], [6]. Here, in return for an arbitrarily small probability of performance violation, one obtains tractable problems whose complexity grows only polynomialy with the size of the uncertainty. Clearly, pursuing this approach requires availability of identification methods capable of producing the required plant/uncertainty description with minimal conservatism.

Control oriented identification has been extensively studied since the early 1990's and is by now a relatively well understood subject (see for instance the textbook [7] or Chap. 10 in [8]). Further, the initial results dealing with purely time or frequency domain measurements and non–parametric plants have been extended in a number of ways including combinations of time and frequency domain measurements ([9]) as well as parametric and non–parametric models ([10]). Nevertheless, in spite of its success, this framework can provide only a *dynamic* uncertainty description[1], and thus leads to excessive conservatism in cases where the uncertainty is known to be purely parametric.

Set-membership identification of purely parametric plants has been addressed in a number of papers. Milanese and co–workers [11], have pursued an approach that yields a set of parameters whose response belongs to the set of experimental evidence (usually time-domain). The exact set is approximated using the best adjusted inner and outer box or ellipsoid, by formulating a optimization problem. However, in the case of models that are non-linear in the parameters (as in parametric LTI models using frequency or time domain data), the associated optimization problem is generally non-convex. Alternatively, and also for this case, the approach in [12] can be used. It computes a tight approximation of the consistent set of parameters, by using set inversion via interval analysis (branch & bound). However, in the non-linear parameter case either, bounding the set of consistent parameters using optimization or set-inversion leads to computational complexity which is exponential in the number of parameters.

A different approach has been pursued in [2] (pp. 582-592) and [13], [14], [15], [16], proposing interval model identification algorithms, that, starting from a set of measured frequency responses, generate interval plants whose frequency response covers the experimental data. This approach has been further extended in [17] considering noisy measurements, and exploiting Kharitonov's theorem to obtain hard bounds on the values of the parameters. Both

[1]This is accomplished by exploiting information based complexity results to overbound the diameter of the information.

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of these approaches, obtain a parameter set which covers the experimental data, but is not related to the consistency set. The latter is understood as the set of models included in the candidate model set (*a priori* information) which may reproduce the experimental measurements (*a posteriori* information) within the error bounds (*a priori* information), sometimes called *feasible* set. Thus, these approaches can neither exploit additional information available for the plant (such as bounds on the $\mathcal{H}_\infty$ or $\mathcal{H}_2$ norms) to bound the behavior of the identified model at frequencies other than those used in the experiments nor provide worst case bounds on the identification error.

In this paper we address the problem of robust identification from a deterministic, set membership standpoint, applied to LTI models with a LFT dependency on its uncertain parameters. As discussed before, for these class of models, existing methods as those in [12] and [11] lead to complex computational problems. Our main result is an efficient algorithm that, starting from some *a priori* information about the plant and experimental data (frequency domain), generates a nominal model that preserves the parametric structure of the plant, along with hard bounds on the values of the unknown parameters. By exploiting integral quadratic constraints (IQCs) results, the problem of finding these bounds is recast as a Linear Matrix Inequality convex optimization one that can be efficiently solved with existing software (see [18], [19]). Moreover, since the proposed algorithm is interpolatory, it is optimal within a factor of two [7], [8], and worst case bounds on the identification error can be computed by exploiting information based complexity concepts.

The paper is organized as follows. Section II introduces the notation and some background on robust interpolation with frequency domain data. Section III presents the main results: given measurements corrupted by bounded additive noise, it provides tractable, LMI–based conditions for computing inner and outer approximations to the set of parameters so that the resulting plant is consistent with some given *a priori* information and interpolates the experimental data. An algorithm to solve this problem is also presented. Section IV presents a simple example to which this procedure is applied. Finally, Section V presents some conclusions and possible directions for future research.

II. Preliminaries

A. Notation

By $\mathcal{L}_\infty$ we will denote the Lebesgue space of complex valued matrix functions essentially bounded on the unit circle, equipped with the norm:

$$\|G(z)\|_\infty \doteq ess \sup_{|z|=1} \bar{\sigma}\left(G(z)\right)$$

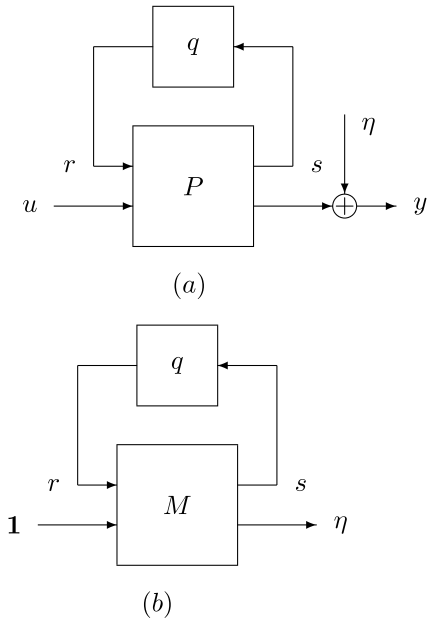

Fig. 1. (a) Robust Parametric Identification Set–up. (b) Equivalent Formulation.

where $\bar{\sigma}$ denotes the largest singular value. By $\mathcal{H}_\infty$ we denote the subspace of functions in $\mathcal{L}_\infty$ with a bounded analytic continuation inside the unit disk, equipped with the norm $\|G(z)\|_\infty \doteq ess\sup_{|z|<1} \bar{\sigma}\left(G(z)\right)$. Similarly, $\mathcal{H}_2$ denotes the Hilbert space of of matrix valued complex functions $F(s)$ with analytic continuation inside the unit disk, and square integrable there, equipped with the usual $\mathcal{H}_2$ norm $\|F\|^2_{\mathcal{H}_2} \doteq \frac{1}{2\pi}\int_0^{2\pi} Trace[F^*(j\omega)F(j\omega)]d\omega < \infty$. ℓ_2 denotes the space of bounded vector functions $x(e^{j\omega})$ equipped with the norm $\|x\|_2^2 \doteq \int_0^{2\pi} \|x^f(e^{j\omega})\|^2 \frac{d\omega}{2\pi}$, and $\mathcal{B}\ell_2(\epsilon)$, the origin centered ϵ radius closed ball in this space.

This paper considers discrete time, single input-single output, causal, linear time invariant (LTI) systems represented by the convolution kernel $\{h\}$ in $y_k = (h*u)_k \doteq \sum_{j=0}^{k} h_{k-j}u_j$, or, alternatively by the complex–valued transfer function $H(z) = \sum_{k=0}^{\infty} h_k z^k$.

Given a matrix M, M^* denotes its Hermitian transpose. As usual, $M > 0$ ($M \geq 0$) indicates that M is positive definite (positive semi-definite). Finally $\mathcal{F}_u(M,q)$ denotes the upper linear fractional interconnection of the matrices M, q:

$$\mathcal{F}_u(M,q) \doteq M_{21}q(I - M_{11}q)^{-1}M_{12} + M_{22}.$$

B. Statement of the Problem

In this paper we will consider parametric systems of the form shown in Figure 1(a), consisting of the Linear Fractional Transformation (LFT) interconnection of a known, stable LTI plant P and a real uncertainty structure $q = \operatorname{diag}\{q_1 I_{n_1}, \ldots, q_r I_{n_r}\}$. Our goal is to establish hard bounds on the values of the parameters q_i such that the $\mathcal{F}_u(P,q)$: (i) is consistent with some given *a priori* information, and (ii) interpolates, within some experimental error, a set of experimental measurements. In particular, in the sequel, we consider *a priori* information of the form:

$$\begin{aligned} \mathcal{S} &= \{\mathcal{F}_u(P,q),\ q \in \mathbf{\Delta}_r\} \\ \mathbf{\Delta}_r &= \{q = \operatorname{diag}(q_1 I_{n_1}, \ldots, q_r I_{n_r}) \\ &\quad : \|Wq\|_\infty \leq 1,\ W \text{ given}\} \\ \mathcal{N}_f &= \mathcal{B}\ell_2(\epsilon_f) \end{aligned}$$

for the set of plants and measurement noise, respectively. Here $\mathbf{\Delta}_r \subset \mathbb{R}^{n\times n}$. Moreover, in the sequel we will assume, by absorbing W into P if necessary, that $W = I$. The experimental information consists of N_f measurements of the frequency response, $\mathbf{y}^f$, both corrupted by additive, ℓ^2 bounded noise:

$$y_i^f = H(e^{\jmath\omega_i}) + \eta^f,\ \eta^f \in \mathcal{N}_f \tag{1}$$

where H denotes the transfer function of the (unknown) plant and vector $\mathbf{H} = \begin{bmatrix} H(e^{\jmath\omega_1}) & \cdots & H(e^{\jmath\omega_{N_f}}) \end{bmatrix}^T$ its frequency response.

Definition 2.1: Given: (i) *a priori* information of the form (1), and (ii) a vector $\mathbf{y}^f$ of frequency domain experimental measurements, the consistency set $\mathcal{T}(\mathbf{y}^f)$ is defined as the set of all plants consistent with the *a priori* information that could have generated the available *a posteriori* experimental data, that is:

$$\mathcal{T}(\mathbf{y}^f) \doteq \left\{H \in \mathcal{S},\ \left\|\mathbf{y}^f - \mathbf{H}\right\|_2 \leq \epsilon_f\right\} \tag{2}$$

Definition 2.2: The *a priori* information and *a priori* experimental data are said to be consistent if and only if $\mathcal{T}(\mathbf{y}^f) \neq \emptyset$

Note that $\mathcal{T}(\mathbf{y}^f)$ is the *smallest* set of models that are indistinguishable using the available information. Thus its radius and diameter give lower and upper bounds on the worst–case identification error (see chapter 10 in [8]).

Using these concepts, we can now formally state the problem under consideration as follows:

Problem 2.1: Given an unknown parametric plant g, the *a priori* sets of candidate models and noise (1), and a finite set of frequency domain experimental data $(\mathbf{y}^f)$:

- Determine whether the consistency set $\mathcal{T}(\mathbf{y}^f)$ is non-empty.
- If $\mathcal{T}(\mathbf{y}^f) \neq \emptyset$, find hyperboxes $\underline{\mathcal{H}}$ and $\overline{\mathcal{H}}$ such that (i) $\mathcal{F}_u(P,q) \in \mathcal{T} \Rightarrow q \in \overline{\mathcal{H}}$ and (ii) $q \in \underline{\mathcal{H}} \Rightarrow \mathcal{F}_u(P,q) \in \mathcal{T}$

III. Main Results

In this section we propose a convex, LMI based optimization algorithm to compute tight approximations to the hyperboxes $\underline{\mathcal{H}}$ and $\overline{\mathcal{H}}$. We begin by recasting the problem into the equivalent form shown in Figure 1 (b), that involves finding the minimum value, over q, of $\|\mathcal{F}_u(M,q)\|_2$, where $M(e^{\jmath\omega})$ is a transfer matrix built from the *a priori* and *a posteriori* information.

A. Problem Transformation

In principle, finding $\underline{\mathcal{H}}$ and $\overline{\mathcal{H}}$ in Problem 2.1 can be recast respectively, into the following (non-convex) constrained optimization forms (s.t. stands for *subject to*):

$$\begin{aligned} &\max_q \|Wq\|_\infty \text{ s.t. } \|y^f - H(q)\|_2 < \epsilon_f \\ &\min_q \|Wq\|_\infty \text{ s.t. } \|y^f - H(q)\|_2 \geq \epsilon_f \end{aligned} \tag{3}$$

Rather than attempting to solve these problems directly, in the sequel we will exploit a combination of interpolation theory and integral quadratic constraint techniques to first reduce them to a finite–dimensional, albeit possibly non–convex, optimization and then obtain tractable convex relaxations. The main idea of the method relies on the following observation: since only a finite set of input/output measurements is available, we can assume, without loss of generality, that both $u(e^{\jmath\omega})$ and $y(e^{\jmath\omega})$ are the impulse responses of some known systems $S_u, S_y \in \mathcal{RH}_\infty$[2]. It follows then that the condition

$$y = \mathcal{F}_u(P,q)u + \eta;\ \eta \in \mathcal{N}_f \tag{4}$$

is equivalent to

$$\begin{aligned} &\begin{bmatrix} s \\ \eta_n \end{bmatrix} = \underbrace{\begin{bmatrix} P_{11} & P_{12}S_u \\ -\frac{1}{\epsilon_f}P_{21} & \frac{1}{\epsilon_f}\left(S_y - P_{22}S_u\right) \end{bmatrix}}_{M(e^{\jmath\omega})} \begin{bmatrix} r \\ \mathbf{1} \end{bmatrix} \\ &r(e^{\jmath\omega}) = qs(e^{\jmath\omega}), \\ &\|\eta_n\|_2 \leq 1 \end{aligned} \tag{5}$$

where $\mathbf{1}$ denotes the unit impulse, η_n has been scaled to belong to $\mathcal{B}\ell_2$, and $(e^{\jmath\omega})$ has been eliminated in the first equation due to space limitations.

Thus, finding $\underline{\mathcal{H}}$ and $\overline{\mathcal{H}}$ in Problem 2.1 is equivalent to solving the following optimization problems:

Problem 3.1: Given the experimental information $\{u(e^{\jmath\omega}), y(e^{\jmath\omega})\}$, the nominal plant P and noise level ϵ_f, find

$$\max_q \|q\|_\infty \text{ subject to } \|\mathcal{F}_u(M,q)\mathbf{1}\|_2 \leq 1 \tag{6}$$

and

$$\min_q \|q\|_\infty \text{ subject to } \|\mathcal{F}_u(M,q)\mathbf{1}\|_2 > 1 \tag{7}$$

B. A Convex Relaxation for $\underline{\mathcal{H}}$

In this section we introduce an LMI based algorithm for computing an inner approximation to the hyperbox $\underline{\mathcal{H}}$. The main idea is to use the formulation introduced above to reduce the problem to a robust $\mathcal{H}_2$ problem, which can then be solved proceeding as in [20], with suitable modifications

[2] S_u and S_y can be found for instance by solving a boundary Nevanlinna–Pick interpolation problem.

to account for the fact that here the uncertainty under consideration is real.

Theorem 3.1: Consider a given stable discrete–time LTI system $M(z) \in \mathcal{RH}_\infty$ and a real uncertainty structure $q \in \mathbf{\Delta}_r$. If there exist hermitian matrices $X(e^{\jmath\omega}) \doteq \text{block}-\text{diag}\{X_j\}, G(e^{\jmath\omega}) \doteq \text{block}-\text{diag}\{G_i\},\ X > 0$, and a real transfer function $v(e^{\jmath\omega}) \geq 0$, such that the following inequalities hold:

$$M^* \begin{bmatrix}\alpha_\ell^2 X & 0\\ 0 & I\end{bmatrix} M - \begin{bmatrix} X & 0\\ 0 & v\end{bmatrix} + \jmath\left(\begin{bmatrix} G & 0\\ 0 & 0\end{bmatrix} M - M^* \begin{bmatrix} G & 0\\ 0 & 0\end{bmatrix}\right) \leq 0$$

$$\text{and} \tag{8}$$

$$\int_0^{2\pi} v(e^{\jmath\omega}) \frac{d\omega}{2\pi} \leq 1$$

then $\|\mathcal{F}_u(M,q)\|_2 \leq 1$ for all q, $|q_i| \leq \alpha_\ell$.

Proof: Given in the Appendix.

C. A Convex Relaxation for $\overline{\mathcal{H}}$

In this section we show that an upper bound to the solution of problem (3.1) can be found by solving a set of convex LMI optimization problems. To this effect, we begin by presenting a theorem guaranteeing that $\|\mathcal{F}_u(M,q)\|_2 > 1$[3].

Theorem 3.2: Consider a given stable discrete–time LTI system $M(z) \in \mathcal{RH}_\infty$ and a real uncertainty structure $q \in \mathbf{\Delta}_r$. If there exist hermitian matrices $X_1(e^{\jmath\omega}) \doteq \text{block}-\text{diag}\{X_{1,j}\}\}, X_2(e^{\jmath\omega}) = \text{block}-\text{diag}\{X_{2,j}\}, G(e^{\jmath\omega}) \doteq \text{block}-\text{diag}\{G_i\},\ X_1 > 0,\ X_2 > 0$, and a real transfer function $v(e^{\jmath\omega}) \geq 0$, such that the following inequalities hold:

$$M^* \begin{bmatrix} -X_1(e^{\jmath\omega}) & 0 & 0 & 0\\ 0 & \alpha_i^2 X_i & 0 & 0\\ 0 & 0 & -X_2(e^{\jmath\omega}) & 0\\ 0 & 0 & 0 & 1\end{bmatrix} M - \begin{bmatrix} -X_1 & 0 & 0 & 0\\ 0 & X_i & 0 & 0\\ 0 & 0 & -X_2(e^{\jmath\omega}) & 0\\ 0 & 0 & 0 & v(e^{\jmath\omega})\end{bmatrix} + \jmath\left(\begin{bmatrix} G & 0\\ 0 & 0\end{bmatrix} M - M^* \begin{bmatrix} G & 0\\ 0 & 0\end{bmatrix}\right) \geq 0$$

$$\text{and} \tag{9}$$

$$\int_0^{2\pi} v(e^{\jmath\omega}) \frac{d\omega}{2\pi} > 1$$

[3]Recall that the $\mathcal{H}_2$ norm of a transfer function is equal to the ℓ_2 norm of its impulse response.

then

$$\|\mathcal{F}_u(M,q)\|_2 > 1, \quad \forall \begin{cases} |q_j| \leq 1,\ j = 1, \cdots, n\\ |q_i| \geq \alpha_i,\ i \neq j\end{cases}$$

Proof: Given in the Appendix.

Direct application of this result leads to the following algorithm for finding an approximation to $\overline{\mathcal{H}}$.

Algorithm 1: Given the *a priori* information P, ϵ_f and experimental data $\{u(e^{\jmath\omega}), y(e^{\jmath\omega})\}$:

0.- Form the system M defined in (5). Set i =1.
1.- Find the minimum value of α_i such that conditions (9) hold.
2.- Scale M by $\begin{bmatrix} I & 0 & 0\\ 0 & \alpha_i & 0\\ 0 & 0 & I\end{bmatrix}$. Set i=i+1. If $i \leq n$ go to step 1.
3.- The desired hyperbox is given by $\overline{\mathcal{H}} \doteq \{q \colon |q_i| \leq \alpha_i\}$

IV. Example

Consider the problem of computing the inner and outer bounds for the following system:

$$P_{11} = \begin{bmatrix} 0 & 0\\ 0 & 0\end{bmatrix} \quad P_{12} = \begin{bmatrix} 1\\ 1\end{bmatrix}$$

$$P_{21} = \begin{bmatrix} 1 & 1\end{bmatrix} \quad P_{22} = \frac{0.2(z+1)^2}{3z^2 - 8z + 6}$$

The uncertainty structure is $\Delta = \text{diag}(\delta_1,\ \delta_2)$, measurement noise bound $\epsilon_f = 0.18$, and the candidate model is $P_{22}(z)$. Without loss of generality, the actual system is coincident with the candidate one. The idea is to provide inner and outer bounds on the consistency set, by setting the set of models $f(z) = F_u(P, \Delta) = P_{22}(z) + \delta_1 + \delta_2$. The inner and outer bounds on the parameters should box the measurement noise.

Using theorem 3.1, we have computed the set of parameters that define an inner approximation to the consistency set, and therefore consistent with *a priori* and *a posteriori* information. The inner consistency parameter set $|\delta_i| \leq \alpha_\ell = 0.06$, $i = 1, 2$ is contained within the measurement noise bounds, as illustrated in figure 3. In this case, the worst combination of uncertainties is $\delta_1 + \delta_2 \leq 2\alpha_\ell \leq \epsilon_f$, which verifies the theory with little conservatism.

For the outer bound, in this example we note that the parameter combination $\delta_1 = -\delta_2$ cannot be invalidated, i.e. the resulting model coincides with the actual one $f(z) = P_{22}(z)$. As a consequence $\alpha_i > 1$ $(i = 1, 2)$, otherwise the resulting set would include points of the set $\delta_1 = -\delta_2$. The outer parameter bound is therefore $|\delta_i| < \alpha_i = 1.21$ $(i = 1, 2)$ which contains, the consistency set, as well as the measurement bounds as seen in figure 3.

V. Conclusions and Future Research

In this paper we address the problem of robust identification of parametric plants from a deterministic, set membership perspective. The goal is to provide hard inner and outer

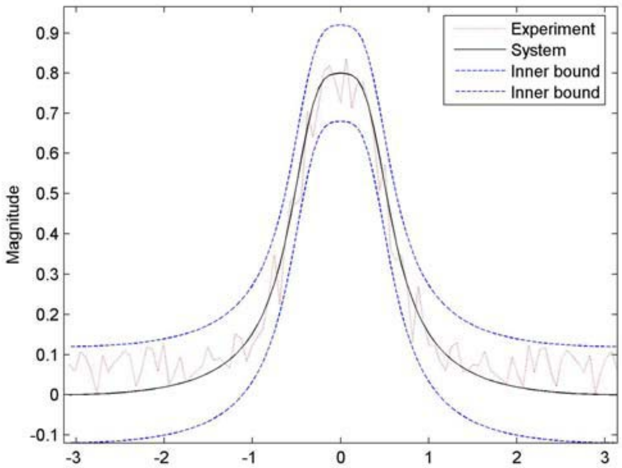

Fig. 2. Inner bounds for the consistency data set (dashed) vs. experimental outcome (dotted) and real system (full).

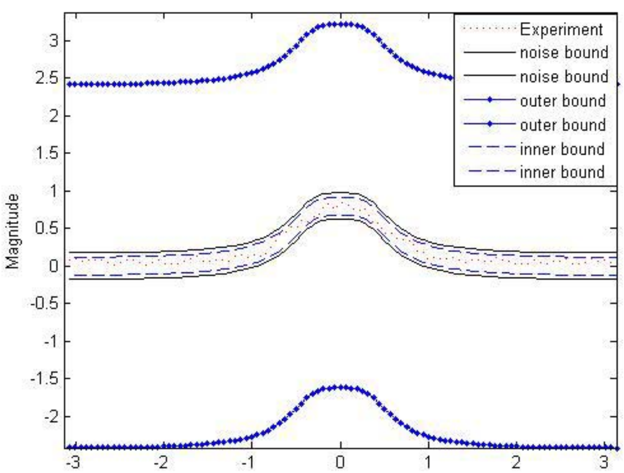

Fig. 3. Inner (dashed) and outer (dash-dot) bounds for the consistency data set vs. experimental (dotted) and noise bounds (full).

bounds on a set of parameters such that the resulting plant satisfies some given a priori information and interpolates, within the measurement error level, some experimental data. By exploiting IQC techniques, these problems are reduced to convex LMI optimizations. This technique can serve as a fast first step for a branch and bound optimization program which may compute the exact consistency set. Research is currently under way to extend these results to time–domain data and to time–varying parameters.

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APPENDIX

Theorem 3.1

Proof: For simplicity we will assume non–repeated parameters, which implies that the scaling matrices X and G are diagonal. The case of repeated parameters follows along similar lines. Pre/post–multiplying the first inequality in (8) by $\begin{bmatrix} r^* & I \end{bmatrix}$ and its hermitian transpose yields, frequency by frequency (see Figure 1):

$$\sum_{i=1}^{n_q} x_i(\alpha_\ell^2 |s_i|^2 - |r_i|^2) + |\eta|^2 - v + \jmath \sum_{i=1}^{n_q} (r_i^* g_i s_i - s_i^* g_i r_i) \leq 0 \quad (10)$$

Since $r_i = q_i s_i$ and q_i is real, the last term on the right hand side vanishes, leading to:

$$|\eta|^2 \leq v - \sum_{i=1}^{n_q} x_i(\alpha_\ell^2 - |q_i|^2)|s_i|^2 \leq v \quad (11)$$

The proof follows now by integrating this last equation, using the second inequality in (8), and the fact that the $\mathcal{H}_2$ norm of a SISO LTI transfer function coincides with the energy of its impulse response. ■

Theorem 3.2

Proof: Proceeding as in the proof above, start by pre/post–multiplying the first inequality in (9) by $\begin{bmatrix} r_1^* & \dots & r_i^* & \dots r_{n_q}^* & 1 \end{bmatrix}$, leading to:

$$\sum_{\substack{j=1 \\ j \neq i}}^{n_q} x_j \quad (|r_j|^2 - |s_j|^2) + x_i(\alpha_i^2 |s_i|^2 - |r_i|^2) + |\eta|^2 - v \geq 0$$

$$\iff$$

$$|\eta|^2 \geq v + \sum_{\substack{j=1 \\ j \neq i}}^{n_q} x_j \quad (1 - |q_j|^2)|s_j|^2 + x_i |s_i|^2 (|q_i|^2 - \alpha_i^2) \geq v$$

where, as before, we used the fact that since the uncertainty is real the term involving the G scales vanishes. The proof follows now by integrating this last equation, using the second inequality in (9). ■

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