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Suboptimal Control of Constrained Nonlinear Systems via Receding Horizon State Dependent Riccati Equations.

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Abstract

Feedback stabilization of systems subject to constraints has been a long-standing problem in control theory. In contrast with the case of LTI plants where several techniques for optimizing performance have recently appeared, very few results are available for the case of nonlinear systems. In this paper we propose a new controller design method, based on the combination of Receding Horizon and Control Lyapunov Functions, for nonlinear systems subject to input constraints. The main result of the paper shows that this control law renders the origin an asymptotically stable equilibrium point in the entire region where stabilization with constrained controls is feasible, while, at the same time, achieving near-optimal performance.

1 Introduction

Feedback stabilization of systems subject to input constraints has been a long-standing problem in control theory (see [2] for an excellent survey of the literature and [24, 18] for some recent contributions). In the case where the plant itself is linear, time-invariant, considerable progress has been made in the past few years, leading to controllers capable of globally (or semiglobally) stabilizing the plant, while optimizing some measure of performance, usually given in terms of ℓ^p disturbance rejection [11, 12, 3, 14, 19, 23].

On the other hand, the problem of optimizing performance in nonlinear systems is considerably less developed, even in the absence of input constraints. Common design techniques for unconstrained nonlinear systems include Jacobian linearization (JL) [9], feedback linearization (FL) [9], the use of control Lyapunov functions (CLF)[1, 16, 5], recursive backstepping [9], and recursive interlacing [13]. While these methods provide powerful tools for designing globally (or semiglobally) stabilizing controllers, performance of the resulting closed loop systems can vary widely, as illustrated in [22] using a simplified model of a thrust vectored aircraft.

In the case of input constrained systems, if a CLF is known then an admissible control action can be found using Artstein–Sontag's formula [1, 16, 10]. More general control restrictions, including rate bounds have been addressed in [15]. A difficulty with these techniques is that most of the methods available in the literature for finding the required CLF (such as feedback linearization and backstepping) do not allow for taking control constraints into consideration. Moreover, as indicated above, performance of the resulting system is strongly dependent on the choice of CLF.

Motivated by the approach pursued in [20, 21, 22] in this paper we propose a suboptimal controller for nonlinear systems subject to input constraints. The main result of the paper shows that this controller, obtained by combining Receding Horizon (RH) and Control Lyapunov Function (CLF) techniques, stabilizes the plant in the entire region where stabilization with constrained controls is feasible. Moreover, it provides near optimal performance. Additional results include a discussion on obtaining suitable CLFs for nonlinear systems subject to control constraints and on extending these techniques to handle state or output constraints.

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2 Preliminaries

2.1 Notation and Definitions

In the sequel we consider the following class of control-affine nonlinear systems:

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x \in R^n$ and $u \in R^m$ represent the state and control variables, and the vector fields $f(.,.)$ and $g(.,.)$ are known C^1 functions.

Definition 1 *A C^1 function $V : R^n \to R_+$ is a Constrained Control Lyapunov function (CCLF) for the system (1) with respect to a given set Ω_u if it is radially unbounded in x and*

$$\inf_{u\in\Omega_u} [L_f V(x) + L_g V(x)u] \leq -\sigma(x) < 0, \qquad \forall x \neq 0 \tag{2}$$

where $\sigma(.)$ is a positive definite function, and where $L_h V(x) = \frac{\partial V}{\partial x} h(x)$ denotes the Lie derivative of V along h.

Definition 2 *Given a compact, convex set $\Omega_u \subset R^m$, a control law $u(t)$ is admissible with respect to Ω_u if $u(.) \in \mathcal{L}^\infty$ and $u(t) \in \Omega_u$ for all $t \geq 0$.*

2.2 The Constrained Quadratic Regulator Problem

Consider the nonlinear system (1). In this paper we address the following problem:

Problem 1 *Given an initial condition x_o and a compact, convex control constraint set Ω_u, find an admissible state-feedback control law $u[x(t)]$ that minimizes the following performance index:*

$$J(x_o, u) = \frac{1}{2}\int_0^\infty [x'Q(x)x + u'R(x)u]\, dt,\ x(0) = x_o \tag{3}$$

where $Q(.)$ and $R(.)$ are C^1, positive definite matrices[1].

It is well known ([17], section 8.5) that this problem is equivalent to solving the following Hamilton-Jacobi-Bellman partial differential equation:

$$\begin{array}{l} 0 = \min_{u\in\Omega_u} \left\{ \frac{\partial V}{\partial x} [f(x) + g(x)u] + \frac{1}{2}u'Ru + \frac{1}{2}x'Qx \right\} \\ \text{subject to: } V(0) = 0 \end{array} \tag{4}$$

[1] This condition can be relaxed to $Q(x) \geq 0$

If this equation admits a C^1 nonnegative solution V, then the optimal control is given by:

$$u(x) = \min_{u\in\Omega_u} \left\{ \frac{\partial V}{\partial x} g(x)u + \frac{1}{2}u'Ru \right\}$$

and $V(x)$ is the corresponding optimal cost (or storage function).

3 A Finite Horizon Approximation

Unfortunately, the complexity of equation (4) prevents its solution, except in some very simple, low dimensional cases. To solve this difficulty, motivated by the idea first introduced in [20] for linear systems and extended in [22] to the nonlinear case, in this section we introduce a finite horizon approximation to Problem 1. Assume that a CCLF $\Psi(x)$, in the sense of Definition 1, is known in a region $0 \in S \subseteq R^n$. Let $c = \inf_{x\in\partial S} \Psi(x)$, where ∂S denotes the boundary of S and define the set

$$S_c = \{x : \Psi(x) < c\} \tag{5}$$

Consider the following finite horizon performance index:

$$\begin{array}{ll} J_\Psi(x_o, u) = & \frac{1}{2}\int_t^{t+T} (x'Qx + u'Ru)\, dt + \Psi[x(t+T)] \\ & x(t) = x_o \end{array} \tag{6}$$

Then we propose the following Receding Horizon type law:

$$u_\Psi(t) = v(t) \text{ where:} v(s) = \underset{v\in\mathcal{U}}{\operatorname{argmin}} J_\Psi[x(s), u] \tag{7}$$

subject to:

$$\begin{array}{l} \frac{1}{2}(v'Rv + x'Qx) + \dot{\Psi})|_{T+s} \leq 0 \\ x(T+s) \in S_c \end{array} \tag{8}$$

where $\mathcal{U}$ denotes the set of control laws admissible with respect to Ω_u.

Theorem 1 *The control law u_Ψ has the following properties:*

1. *It is admissible.*
2. *It renders the origin an asymptotically stable equilibrium point of (1) in the entire region where the system is stabilizable with a bounded control action.*
3. *Coincides with the globally optimal control law when $\Psi(x) = V(x)$, where $V(x) \in C^1$ satisfies the Hamilton-Jacobi-Bellman PDE (4).*

Proof: By construction, the control law is admissible. Denote by $J_o(x)$ the value of the finite horizon index (6) corresponding to the control law (7). To show that this control law renders the origin an asymptotically stable equilibrium point, we will show that $J_o(x)$ is a Lyapunov function for the closed loop system. To this effect, consider an initial condition x_o and denote by u^*, x^* the optimal control and associated trajectory respectively. Then

$$J_o[x(t+dt)] \leq \frac{1}{2}\int_{t+dt}^{T+t}(x^{*\prime}Qx^* + v^{*\prime}Rv^*)\,dt + \Psi[x^*(T+t)]$$

$$+\min_{v\in\Omega_u}\left\{\frac{1}{2}x^{*\prime}(T+t)Qx^*(T+t) + \frac{1}{2}v'Rv + \dot{\Psi}[x^*(T+t)]\right\}dt$$

$$= J_o[x(t)] - \frac{1}{2}\left[x^*(t)'Qx^*(t) + u^{*\prime}(t)Ru^*(t)\right]dt$$

$$+\min_{v\in\Omega_u}\left\{\frac{1}{2}x^{*\prime}(T+t)Qx^*(T+t) + \frac{1}{2}v'Rv + \dot{\Psi}[x^*(T+t)]\right\}dt$$

Therefore, if (8) holds then we have that

$$\begin{array}{rcl} \dot{J}_o & = & \lim_{dt\to 0}\frac{J_o[x(t+dt)] - J_o[x(t)]}{dt} \\ & \leq & -\frac{1}{2}\left[x^{*\prime}(t)Qx^*(t) + u^{*\prime}(t)Ru^*(t)\right] < 0 \end{array} \quad (9)$$

To complete the proof of item 2, let $\mathcal{R}$ denote the region where Problem 1 is feasible. Given any initial condition $x_o \in \mathcal{R}$, there exists an admissible control law $u_a(x)$ such that the corresponding trajectory $\lim_{t\to\infty} x(t) = 0$. Hence, there exists some finite $T(x_o)$ such that $x[T(x_o)] \in S_c$. By construction the set S_c is controlled invariant with respect to some admissible control law u_S. It follows that the control:

$$u = \begin{cases} u_a(x) & x \notin S_c \\ u_S[x(s)] & x \in S_c \end{cases} \quad (10)$$

is admissible and renders $J_\Psi(x,u)$ finite. Thus, the optimization problem (7) subject to (8) is feasible at $t=0$. Denote by u^*, x^* the solution to this problem in $[0,dt]$ and the associated trajectory, respectively. Consider now the problem (7)-(8) starting from the initial condition $x^*(dt)$. It follows that the control law:

$$u = \begin{cases} u^*(s) & s \in [dt, T] \\ u_S[x(s)] & s \in [T, T+dt] \end{cases} \quad (11)$$

is admissible, since it steers the system along the feasible trajectory x^* in $[dt,T]$ and keeps it inside the set S_c in $[T, T+dt]$. Thus $x^*(t+dt) \in \mathcal{R}$, i.e., the control action u_Ψ renders $\mathcal{R}$ invariant. This establishes the fact that the control action u_Ψ is admissible for all t. Asymptotic stability follows from (9).

To prove item 3.- note that when $\Psi(x) = V(x)$ then from the Hamilton Jacobi equation (4) we have that

$$x(t+T)'Qx(t+T) + L_f\Psi\Big|_{x(t+T)} + \min_{v\in\Omega_u}\left\{v'Rv + L_g\Psi\Big|_{x(t+T)}v\right\} = 0 \quad (12)$$

Thus in this case the constraints (8) are redundant. From a dynamic programming argument it follows that

$$\begin{array}{rcl} J_o(x_o) & = & \inf_{u\in\mathcal{U}}\left\{\int_0^T [x'Qx + u'Ru]\,dt + V[x(T)]\right\} \\ & = & \inf_{u\in\mathcal{U}}\int_0^\infty [x'Qx + u'Ru]\,dt \\ & = & V(x_o) \end{array} \quad (13)$$

Next, we show that the suboptimality level of the proposed algorithm decreases monotonically along the trajectories of the system.

Theorem 2 *Let $J_o(x)$ denote the optimal value of the performance index (6) and define the approximation error $e(x) = J_o(x) - V(x)$, where $V(x) \in C^1$ is a solution to the HJB equation (4). Then $\frac{d}{dt}e(x) \leq 0$ along the trajectories generated by the control law u_Ψ.*

Proof: From (4) it follows that for all $u \in \Omega_u$

$$\frac{\partial V}{\partial x}[f(x) + g(x)u] + \frac{1}{2}u'Ru + \frac{1}{2}x'Qx \geq 0 \quad (14)$$

In particular, this implies that along the trajectories of the system (1) generated by the control law u_Ψ we have:

$$\begin{array}{rcl} 0 & \leq & \frac{\partial V}{\partial x}[f(x) + g(x)u_\Psi] + \frac{1}{2}u_\Psi'Ru_\Psi + \frac{1}{2}x'Qx \\ & = & \frac{d}{dt}V + \frac{1}{2}u_\Psi'Ru_\Psi + \frac{1}{2}x'Qx \end{array} \quad (15)$$

From (9) it follows that the evolution of $e(t)$ along the trajectory satisfies:

$$\frac{d}{dt}e = \frac{d}{dt}J - \frac{d}{dt}V \leq -\frac{1}{2}\left[x'Qx + u'_\Psi Ru_\Psi\right] - \frac{d}{dt}V \leq 0 \quad (16)$$

where the last inequality follows from (15).

Remark 1 *The result above formalizes the intuitive fact that minimizing the upper bound (6) moves the system in the "right" direction, since the suboptimality level decreases monotonically along the trajectories.*

4 Incorporating State or Output Constraints

Assume that in addition to the control constraint $u \in \Omega_u$, the system is subject to state constraints of the form

$x \in \Omega_x$, where Ω_x is a convex set containing the origin in its interior[2]. As we briefly show next, the proposed controller can be readily modified to handle these constraints. The only changes that are required is to select the invariant set S_c in (8) so that $S_c \subseteq \Omega_x$ [3], and to modify the optimization (7) so that the state constraints are taken into accout. Note that Theorem 1 still holds under these conditions. In particular, the controller is guaranteed to stabilize the system in the entire region where the problem is feasible, and yields optimal performance when $\Psi(x) = V(x)$.

5 Selecting suitable CLFs

In principle, a simple way of finding a CCLF is to find first a CLF using any of the methods available in the literature, such as feedback linearization and backstepping (see for instance [6, 5]), and then considering an invariant set S_c where the associated control action does not exceed the bounds[4]. However this approach may require the use of a large horizon T in the optimization to guarantee that the set S_c is reached (i.e. the constraints (8) are feasible). In addition, in order to minimize the suboptimality level incurred by the proposed algorithm, the CLF should be "close" to the value function V. In this section we propose a method for finding a suitable local CCLF. This approach is motivated by the empirically observed success of the SDRE method, briefly covered in the Appendix.

In the sequel we consider for simplicity the case where the control constraints are of the form $\|u\|_\infty \leq u_{max}$ but the method can be easily generalized to more general constraints. Begin by rewriting the nonlinear system (1) into the following linear-like form:

$$\dot{x} = A(x)x + B(x)u \tag{17}$$

and assume that, for every x the pair $[A(x), B(x)]$ is stabilizable (in the linear sense). Consider now the following Riccati equation, parametric in x and τ:

$$0 = A'(x)P(x,\tau) + P(x,\tau)A(x) + \frac{1}{\tau}Q(x) \tag{18}$$

$$-P(x,\tau)B(x)R^{-1}(x)B'(x)P(x,\tau) \tag{19}$$

and the associated control law:

$$u = -R^{-1}(x)B'P(x,\tau)x \tag{20}$$

[2]This set can represent either state constraints or originate from output constraints of the form $h(x) \leq h_{max}$.

[3]This is always possible since $0 \in \text{int}\{\Omega_x\}$.

[4]This idea was originally proposed for the case of LTI systems in [20].

where $P(x,\tau)$ is the positive definite solution of (18). Finally, consider the the mapping $\tau : R^n \to R^+$ defined implicitly by the solution to the following equation:

$$x'P(x,\tau)x - c(\tau) = 0 \tag{21}$$

where

$$c(\tau) = \min_i \frac{u_{max}^2}{b_i(x)'P(x,\tau)b_i(x)} \tag{22}$$

Note that $\hat{P}(x) \doteq P(x,1)$ is precisely the solution to the SDRE associated with the system (17) and hence is a CLF in a neighborhood $\mathcal{N}$ of the origin. Let $S_{\mathcal{N}} \subseteq \mathcal{N}$ denote the largest set inside $\mathcal{N}$ that is rendered invariant by the control law

$$\hat{u} = -R^{-1}B(x)'\hat{P}(x)x \tag{23}$$

and define the sets:

$$\begin{aligned} \mathcal{E} &= \{x : x'\hat{P}(x)x \leq c(1)\} \\ S_1 &= \mathcal{E} \cap S_{\mathcal{N}} \end{aligned} \tag{24}$$

By construction, the set S_1 is invariant with respect to the control law (23). Moreover, it can be easily shown that inside this set $\hat{u}$ is admissible. Motivated by these observations we propose the following CCLF:

$$\Psi(x) = \begin{cases} \frac{1}{2}x'P[x,\tau(x)]x & x \in S_{\mathcal{N}}/S_1 \\ \frac{1}{2}x'P[x,1]x & x \in S_1 \end{cases} \tag{25}$$

with associated control action

$$u(x) = \begin{cases} -R^{-1}B'(x)P[x,\tau(x)]x & x \in S_{\mathcal{N}}/S_1 \\ -R^{-1}B'(x)P[x,1]x & x \in S_1 \end{cases} \tag{26}$$

In the sequel we show that, if $x'\hat{P}(x)x$ is a CLF for the system (1) in the region S_1, then $\Psi(x)$ is a CCLF in some region $S_c \supseteq S_1$.

Theorem 3 *Given some $\tau_{max} \geq 1$, assume that, for every fixed $\tau \in [1, \tau_{max}]$, there exists an invariant region S_τ where $\phi(x,\tau) \doteq x'P[x,\tau(x)]x$ is a CLF for the system (1). Define the region $S_c \doteq (\cap_\tau S_\tau) \cup S_1$. Then $\Psi[x,\tau(x)]$ is a CCLF for (1) in the region S_c.*

<u>Proof:</u> Begin by noting that, by construction, the control action u is admissible. To establish that Ψ is a CLF, differentiate (18) with respect to τ to obtain:

$$0 < \frac{1}{\tau^2}Q(x) + [A(x) - B(x)R^{-1}B'(x)P(x)]'\frac{\partial P}{\partial \tau} \tag{27}$$

$$+\frac{\partial P}{\partial \tau}[A(x) - B(x)R^{-1}B'(x)P(x)] \tag{28}$$

Since $P(x,\tau)$ is stabilizing by construction, the matrix $A_c(x) = [A(x) - B(x)R^{-1}B'(x)P(x)]$ is Hurwitz. Since

$Q > 0$, this fact, combined with (27) establishes that $\frac{\partial}{\partial\tau}P(x,\tau) < 0$. From (22) it follows that $\frac{\partial c(\tau)}{\partial\tau} > 0$. Consider now $\frac{d}{dt}\Psi(x)$ along the trajectories generated by the control action

$$u = -B'(x)P[x,\tau(x)]x \qquad (29)$$

Differentiating (21) we have that:

$$\phi_t + \left[x'\frac{\partial P}{\partial\tau}x - \frac{\partial c}{\partial\tau}\right]\dot{\tau} = 0 \qquad (30)$$

where $\phi_t \doteq \dot{x}'Px + x'P\dot{x} + \sum_{i=1}^{n} x'\frac{\partial P(x,\tau)}{\partial x_i}x\dot{x}_i$. Solving for $\dot{\tau}$ yields:

$$\dot{\tau} = -\frac{\phi_t}{x'\frac{\partial P}{\partial\tau}x - \frac{\partial c}{\partial\tau}} < 0 \qquad (31)$$

where the inequality follows from the facts that $\frac{\partial c(\tau)}{\partial\tau} > 0$, $\frac{\partial P(x,\tau)}{\partial\tau} < 0$, and by assumption, $\phi_t < 0$. Finally, note that from (31) we have that:

$$\frac{d}{dt}x'P[x,\tau(x)]x = \phi_t + x'\frac{\partial P}{\partial\tau}x\dot{\tau} = \frac{\partial c}{\partial\tau}\dot{\tau} < 0 \qquad (32)$$

6 Conclusions

A large number of realistic control problems involve designing a controllers for systems subject to time-domain constraints on the control action. This problem has been the subject of considerably attention in the past two decades, and in the case of LTI dynamics, several techniques are available for finding stabilizing controllers that, at the same time, optimize some measure of performance (for instance in the ℓ^2 or ℓ^∞ induced norm sense). On the other hand, very few results are available for nonlinear systems.

In this paper we propose a procedure for synthesizing state-feedback controllers for nonlinear, control affine systems subject to control constraints and performance requirements expressed in terms of a quadratic performance index. The proposed controller is obtained by combining Receding Horizon and Control Lyapunov ideas, following in the spirit of [22], and it only necessitates knowing a (not necessarily optimal) local Constrained Control Lyapunov Function. As we show in the paper, such a function can be readily obtained using as a starting point the State Dependent Riccati Equation (SDRE) approach.

The main result of the paper shows that the proposed control law is guaranteed to stabilize the system (in the entire region where stabilization with bounded controls is feasible) and outperform the control law obtained when using the CCLF alone. Thus, the proposed approach provides a simple way to improve the performance that can be achieved from a given CLF.

An additional advantage of the proposed framework is that it can be easily extended to incorporate additional constraints on outputs or states.

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A The SDRE approach to nonlinear regulation

In this section we briefly cover the details of the SDRE approach [4]. The main idea of the method is to recast the the nonlinear system (1) into a State Dependent Coefficient (SDC) linear-like form:

$$\dot{x} = A(x)x + B(x)u \tag{33}$$

and to solve pointwise along the trajectory the corresponding algebraic Riccati equation:

$$A'(x)P(x) + P(x)A(x) - P(x)B(x)R^{-1}(x)B'(x)P(x) + Q(x) = 0 \tag{34}$$

The suboptimal control law is given by $u_{sdre} = -R^{-1}(x)B'P(x)x$. In the sequel we briefly review the properties of this control law. The corresponding proofs can be found in the appropriate references.

Lemma 1 ([4]) *Assume that $Q(x) = C'(x)C(x)$ and that there exists a neighborhood Ω of the origin where the pairs $\{A(x), B(x)\}$ and $\{A(x), C(x)\}$ are pointwise stabilizable and detectable respectively and all the matrix functions involved are C^1. Then the control law u_{sdre} renders the origin a locally asymptotically stable equilibrium point of the closed-loop system.*

Lemma 2 ([4]) *The SDRE control law and its associated state and co-state trajectories satisfy the necessary optimality condition: $\frac{\partial H}{\partial u} = 0$, where $H = x'Q(x)x + u'R(x)u + \lambda'[f(x) + B(x)u]$ and where λ denotes the co-states.*

Lemma 3 ([4]) *Assume that the parametrization (33) is stabilizable and all the matrices involved along with their gradients are bounded in a neighborhood Ω of the origin. Then the SDRE control law and its associated state and co-state trajectories asymptotically satisfy at a quadratic rate[5] following necessary condition for optimality:*

$$\dot{\lambda} = -\frac{\partial H}{\partial x}$$

in the sense that

$$\|\dot{\lambda} + \frac{\partial H}{\partial x}\| \leq x'Ux$$

for some constant matrix $U > 0$ and all $x \in \Omega$.

Lemma 4 ([7]) *Let $P(x)$ denote a solution to the SDRE (34). If there exists a positive definite function $V(x)$ such that $\frac{\partial V(x)}{\partial x} = P(x)x$ then u_{sdre} is the globally optimal control law.*

[5] i.e. $\|\dot{\lambda} + \frac{\partial H}{\partial x}\| \to 0$ as $O(\|x\|^2)$ as $x \to 0$.