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Brief Paper

Heuristically Enhanced Feedback Control of Constrained Systems: The Minimum Time Case*

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Key Words—Computer control; constrained systems; dynamic programming; multivariable control systems; stability; suboptimal control.

Abstract—Recent advances in computer technology have spurred new interest in the use of feedback controllers based upon the use of on-line optimization. Still, the use of computers in the feedback loop has been hampered by the limited amount of time available for computations. In this paper we propose a feedback controller based upon the use of on-line contrained optimization in the feedback loop. The optimization problem is simplified by making use of the special structure of time-optimal systems, resulting in a substantial dimensionality reduction. These results are used to show that the proposed controller yields asymptotically stable systems, provided that enough computation power is available to solve on-line a constrainted optimization problem considerably simpler than the original.

1. *Introduction*

A SUBSTANTIAL NUMBER of control problems can be summarized as the problem of designing a controller capable of achieving acceptable performance under design constraints. This statement looks deceptively simple, but even in the case where the system under consideration is linear time-invariant, the problem is far from solved.

During the last decade, substantial progress has been achieved in the design of linear controllers. By using a parametrization of all internally stabilizing linear controllers in terms of a stable transfer matrix Q, the problem of finding the "best" linear controller can be formulated as an optimization problem over the set of suitable Q (Boyd *et al.*, 1988). In this formulation, additional specifications can be imposed by further constraining the problem. However, most of these methods can address time-domain constraints only in a conservative fashion. Hence, if the constraints are tight this approach may fail to find a solution, even if the problem is feasible (in the sense of having a, perhaps non-linear, solution).

Classically, control engineers have dealt with time-domain constraints by allowing inputs to saturate, in the case of actuator constraints, and by switching to a controller that attempts to move the system away from saturating constraints, in the case of state constraints. Although these methods are relatively simple to use, they have several serious shortcomings, perhaps the most important being their inability to handle constraints in a general way. Hence, they require *ad hoc* tuning of several parameters making extensive use of simulations.

Alternatively, the problem can be stated as an optimization problem (Frankena and Sivan, 1979). Then, mathematical programming techniques can be used to find a solution (see for instance Zadeh and Whalen, 1962; Fegley *et al.*, 1971, and references therein). However, in most cases the control law generated is an open-loop control that has to be recalculated entirely, with a considerable computational effort, if the system is disturbed. Conceivably, the set of open loop control laws could be used to generate a closed loop control law by computing and storing a complicated field of extremals (Judd *et al.*, 1987). However, this alternative requires extensive amounts of off-line computation and of storage.

Because of the difficulties with the optimal control approach, other design techniques, based upon using a Lyapunov function to design a stabilizing controller, have been suggested (Gutman and Hagander, 1985). However, these techniques tend to be unnecessarily conservative. Moreover, several steps of the design procedure involve an extensive trial and error process, without guarantee of success (see example 5.3 in Gutman and Hagander, 1985).

Recently, several techniques that exploit the concept of maximally invariant sets to obtain static (Gutman and Cwikel, 1986; Vassilaki *et al.*, 1988; Benzaouia and Burgat, 1988; Bitsoris and Vassilaki, 1990; Blanchini, 1990; Sznaier, 1990; Sznaier and Sideris, 1991a) and dynamic (Sznaier and Sideris, 1991b; Sznaier, 1991) linear feedback controllers have been proposed. These controllers are particularly attractive due to their simplicity. However, it is clear that only a fraction of the feasible constrained problems admit a linear solution. Furthermore, performance considerations usually require the control vector to be on a constraint boundary and this clearly necessitates a non-linear controller capable of saturating.

Finally, in the last few years, there has been a renewed interest in the use of feedback controllers based upon the use of on-line minimization. Although this idea was initially proposed as far back as 1964 (Dreyfus, 1964), its implementation has become possible only during the last few years, when the advances in computer technology made feasible the solution of realistically sized optimization problems in the limited time available. Consequently, theoretical results concerning the properties of the resulting closed-loop systems have started to emerge only recently. In Sznaier (1989) and Sznaier and Damborg (1990) we presented a theoretical framework to analyse the effects of using on-line optimization and we proposed a controller guaranteed to yield asymptotically stable systems. However, although these theoretical results represent a substantial advance over some previously used *ad hoc* techniques, in some cases they are overly conservative, requiring the on-line solution of a large optimization problem. Since in most sampled control systems the amount of time available

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between samples is very limited, this may preclude the use of the proposed controller in many applications.

In this paper we present a suboptimal feedback controller for the minimum-time control of discrete time constrained systems. Following the approach presented in Sznaier and Damborg (1990) this controller is based upon the solution, during the sampling interval, of a sequence of optimization problems. We show that by making use of the special structure of time-optimal systems the proposed algorithm results in a significant reduction of the dimensionality of the optimization problem that must be solved on-line, hence allowing for the implementation of the controller for realistically sized problems.

The paper is organized as follows: in Section 2 we introduce several required concepts and we present a formal definition to our problem. In Section 3 we present the proposed feedback controller and the supporting theoretical results. The main result of this section shows that by relinquishing theoretical optimality we can find a stabilizing suboptimal controller that allows for a substantial reduction of the dimensionality of the optimization problem that must be solved on-line. Finally, in Section 4, we summarize our results and we indicate directions for future research.

2. *Problem formulation and preliminary results*

2.1. *Statement of the problem.* Consider the linear, time invariant, controllable discrete time systems modeled by the different equation:

$$\underset{\sim}{x}_{k+1} = A\underset{\sim}{x}_k + B\underset{\sim}{u}_k, \quad k = 0, 1, \ldots \tag{S}$$

with initial condition $\underset{\sim}{x}_0$, and the constraints

$$\begin{aligned} &\underset{\sim}{u}_k \in \Omega \subset R^m, \quad \underset{\sim}{x}_k \in \mathcal{G} \subset R^n \\ &\mathcal{G} = \{\underset{\sim}{x} : |G\underset{\sim}{x}| \le \underset{\sim}{\gamma}\}, \quad \Omega = \{\underset{\sim}{u} : |W\underset{\sim}{u}| \le \underset{\sim}{\omega}\}, \end{aligned} \tag{C}$$

where $\underset{\sim}{\gamma} \in R^p$, $\underset{\sim}{\omega} \in R^q$, γ_i, $\omega_i > 0$, $G \in R^{pn}$, $W \in R^{qm}$ with full column rank, $\underset{\sim}{x}$, indicates x is a vector quantity and where the inequalities (C) should be interpreted in a component by component sense. Furthermore, assume as usual that A^{-1} exists. Our objective is to find a sequence of admissible controls, $\underset{\sim}{u}_k[\underset{\sim}{x}_k]$, that minimizes the transit time to the origin. Throughout the paper we will refer to this optimization problem as problem (P) and we will assume that it is well posed in the sense of having a solution. In Section 3 we give a sufficient condition on $\mathcal{G}$ for (P) to be feasible.

2.2. *Definitions and preliminary results.* In order to analyse the proposed controller we need to introduce some definitions and background theoretical results. We begin by formalizing the concept of null controllable domain and by introducing a constraint-induced norm.

Definition 1. The Null Controllable domain of (S) is the set of all points $\underset{\sim}{x} \in \mathcal{G} \subset R^n$ that can be steered to the origin by applying a sequence of admissible controls $\underset{\sim}{u}_k \in \Omega \subset R^m$, such that $\underset{\sim}{x}_k \in \mathcal{G}$, $k = 0, 1, \ldots$. The Null Controllable domain of (S) will be denoted as C_∞. The Null Controllable domain in j or fewer steps will be denoted as $C_j \subseteq C_\infty$.

Definition 2. The Minkowsky Functional (or gauge) p of a convex set $\mathcal{G}$ containing the origin in its interior is defined by

$$p(\underset{\sim}{x}) = \inf_{r>0}\left\{r : \frac{x}{r} \in \mathcal{G}\right\}.$$

A well-known result in functional analysis (see for instance Conway, 1990) establishes that p defines a seminorm in R^n. Furthermore, when $\mathcal{G}$ is balanced and compact, as in our case, the seminorm becomes a norm. In the sequel, we will denote this norm as $p(\underset{\sim}{x}) = \|\Gamma^{-1}G\underset{\sim}{x}\|_\infty \triangleq \|\underset{\sim}{x}\|_{\mathcal{G}}$ where $\Gamma = \mathrm{diag}\,(\gamma_1 \cdots \gamma_p)$.

Remark 1. The set $\mathcal{G}$ can be characterized as the unity ball in $\|\cdot\|_{\mathcal{G}}$. Hence, a point $\underset{\sim}{x} \in \mathcal{G}$ iff $\|\underset{\sim}{x}\|_{\mathcal{G}} \le 1$.

Next, we formalize the concept of underestimate of the cost to go. We will use this concept to determine a sequence of approximations that converges to the solution of (P).

Definition 3. Let O be a convex open set containing the origin and such that for all the optimal trajectories starting out in O, the constraints (C) are not effective, and let $J_0(\underset{\sim}{x})$ be the optimal cost-to-go from the state $\underset{\sim}{x}$. A function $g : R^n \to R$ such that:

$$0 \le g(\underset{\sim}{x}) \le J_0(\underset{\sim}{x})\ \forall \underset{\sim}{x} \in \mathcal{G},$$

$$g(\underset{\sim}{x}) = J_0(\underset{\sim}{x})\ \forall \underset{\sim}{x} \in O,$$

will be called an underestimate of the cost-to-go relative to the set O.

The following theorem, where we show that problem (P) can be exactly solved by solving a sequence of suitable approximations, provides the theoretical motivation for the proposed controller.

Theorem 1. Let O be the set introduced in Definition 3 and let $\underset{\sim}{x}_k^u(\underset{\sim}{\xi})$ be the (unconstrained) optimal trajectory corresponding to the initial condition $\underset{\sim}{\xi} \in O$. Finally let $g(\underset{\sim}{x}) : R^n \to R$ be an underestimate relative to O. Consider the following optimization problems:

$$\min_{\mathcal{U}} \{J(\underset{\sim}{x}) = N\} = \min_{\mathcal{U}} \left\{\sum_{k=1}^{N} 1\right\}, \tag{1}$$

$$\min_{\mathcal{U}} \{J_m(\underset{\sim}{x}) = m - 1 + g(\underset{\sim}{x}_m)\} = \min_{\mathcal{U}} \left\{\sum_{k=1}^{m-1} 1 + g(\underset{\sim}{x}_m)\right\}, \quad m < N, \tag{2}$$

subject to (C) where $\mathcal{U} = \{\underset{\sim}{u}_0, \underset{\sim}{u}_1, \ldots\}$. Then an optimal trajectory, $\underset{\sim}{x}_k^0$, $k = 1, 2 \cdots m$ which solves (2), extended by defining $\underset{\sim}{x}_k^0 = \underset{\sim}{x}_k^u(\underset{\sim}{x}_m^0)$, $k = m + 1 \cdots N$, is also a solution of (1) provided that $\underset{\sim}{x}_m^0 \in O$.

Proof. The proof follows by noting that the theorem corresponds to a special case of Theorem 1 in Sznaier and Damborg (1990), with $L_k(\underset{\sim}{x}_k, \underset{\sim}{u}_k) \equiv 1$ ◇.

It follows that problem (P) can be exactly solved by using the sampling interval to solve a sequence of optimization problems of the form (2), with increasing m, until a number m_0 and a trajectory $\underset{\sim}{x}_k$ such that $\underset{\sim}{x}_{m_0} \in O$ are obtained. However this approach presents the difficulty that the asymptotic stability of the resulting closed loop system can not be guaranteed when there is not enough time to reach the region O.

In our previous work (Sznaier, 1989; Sznaier and Damborg, 1990) we solved this dfficulty by imposing an additional constraint (which does not affect feasibility) and by using an optimization procedure based upon the quantization of the control space. By quantizing the control space, the attainable domain from the initial condition can be represented as a tree with each node corresponding to one of the attainable states. Thus the original optimal control problem is recast as a tree problem that can be efficiently solved using heuristic search techniques based upon an underestimate of the cost-to-go (Winston, 1984). We successfully applied this idea to minimum time and quadratic regulator problems. However, as we noted there, in some cases the results, based upon a worst-case analysis, proved to be overly conservative. As a result, the optimization problem quickly became untractable. This phenomenon is illustrated in the following example.

2.3. *A realistic problem.* Consider the minimum time control of an F-100 jet engine. The system at intermediate power, sea level static and PLA = 83° can be represented by

Sznaier (1989):

$$A = \begin{pmatrix} 0.8907 & 0.0474 & -0.0980 & 0.2616 & 0.0689 \\ 0.0237 & 0.9022 & -0.0202 & 0.1057 & 0.0311 \\ 0.0233 & -0.0149 & 0.8167 & 0.2255 & 0.0295 \\ 0.0 & 0.0 & 0.0 & 0.7788 & 0.0 \\ -0.0979 & 0.3532 & 0.3662 & 0.6489 & 0.0295 \end{pmatrix},$$

$$\underline{\gamma} = \begin{pmatrix} 50.0 \\ 64.0 \\ 20.0 \\ 5.0 \\ 18.1 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.0213 & -0.3704 \\ 0.0731 & -0.1973 \\ -0.0357 & -0.5538 \\ 0.2212 & 0.0 \\ 0.0527 & -3.9068 \end{pmatrix},$$

$$G = I; \quad \Omega = \{\underline{u} \in R^2 : |u_1| \le 31.0; |u_2| \le 200.0\}. \tag{3}$$

The sampling time for this system is 25 msec. In this case equation (21) in Sznaier and Damborg (1990) yields $\sim 10^3$ nodes for each level of the tree, which clearly precludes the real-time implementation of the algorithm proposed there.

3. *Proposed control algorithm*

In this section we indicate how the special structure of time-optimal systems can be used to reduce the dimensionality of the optimization problem that must be solved on-line. Specifically, we use a modification of the Discrete Time Minimum Principle to show that the points that satisfy a necessary condition for optimality are the corners of a subset of Ω. Hence, only these points need to be considered by the optimization algorithm.

3.1. *The modified discrete time minimum principle.* We begin by extending the Local Discrete Minimum Principle (Butkovskii, 1963) to Frechet differentiable terminal-cost functions and constraints of the form (C). Note that in its original form, the minimum principle requires the state constraint set $\mathcal{G}$ to be open, while in our framework it is compact.

Theorem 2. Consider the problem (P′) defined as

$$\min_{u_k \in \Omega} S(\underline{x}_N), \tag{4}$$

subject to:

$$\underline{x}_{k+1} = A\underline{x}_k + B\underline{u}_k \triangleq f(\underline{x}_k, \underline{u}_k), \quad \underline{x}_0, N \text{ given} \tag{5}$$

$$\|\underline{x}_{k+1}\|_{\mathcal{G}} < \|\underline{x}_k\|_{\mathcal{G}}, \tag{6}$$

where S is Frechet differentiable. Let the co-states ψ_k be defined by the difference equation:

$$\psi'_k = \psi'_{k+1} \frac{\partial f(\underline{x}_{k+1}, \underline{u}_{k+1})}{\partial \underline{x}'} = \psi'_{k+1} A, \qquad \psi'_{N-1} = \frac{\partial S(\underline{x}_N)}{\partial \underline{x}'}. \tag{7}$$

Finally, define the Hamiltonian as:

$$H(\underline{x}_k, \underline{u}_k, \psi_k) = \psi'_k f(\underline{x}_k, \underline{u}_k). \tag{8}$$

Then, if:

$$\max_{\|x\|_{\mathcal{G}}=1} \left\{ \min_{u \in \Omega} \|A\underline{x} + B\underline{u}\|_{\mathcal{G}} \right\} < 1, \tag{9}$$

the following results hold; (i) problem (P′) is feasible; (ii) for any initial condition $\underline{x}_0 \in \mathcal{G}$ the resulting trajectory $\{\underline{x}_k\}$ is admissible; and (iii) a necessary condition for optimality is:

$$H(\underline{x}^*_k, \underline{u}^*_k, \psi^*_k) = \min_{u \in O_u \subseteq \Omega_1} H(\underline{x}^*_k, \underline{u}, \psi^*_k), \quad k = 1, \ldots, N-1, \tag{10}$$

where

$$\Omega_1(\underline{x}_k, k) = \{\underline{u} \in \Omega : \|\underline{x}_{k+1}\|_{\mathcal{G}} \le (1-\epsilon)\|\underline{x}_k\|_{\mathcal{G}}\}, \tag{11}$$

where O_u is some neighborhood of $\underline{u}$, $\epsilon > 0$ is chosen such that Ω_1 is not empty and where * denotes the optimal trajectory.

Proof. Feasibility follows from (9) and Theorem 3.1 in Gutman and Cwikel (1986) (or as a special case of Theorem 2 in Sznaier and Damborg (1990)). Since $\underline{x}_0 \in \mathcal{G}$, $\|\underline{x}_0\|_{\mathcal{G}} \le 1$. From (9) $\|\underline{x}_k\|_{\mathcal{G}} < 1$ and therefore $\underline{x}_k \in \mathcal{G}$ for all k. To prove (iii) we proceed by induction. From (6) it follows that there exists $\epsilon > 0$ such that Ω_1 is not empty. From the definition of Ω_1 it follows that for any $\underline{u}_k \in \Omega_1(\underline{x}_k, k)$ there exists a neighborhood $O_u \subseteq \Omega_1$, not necessarily open, where (6) holds. Hence, if $\underline{x}_k \in \mathcal{G}$, $\underline{x}_{k+1} = f(\underline{x}_k, \underline{u}_k) \in \mathcal{G} \, \forall \underline{u}_k \in O_u$. Let $\bar{\underline{x}}_k$ denote a non-optimal feasible trajectory obtained by employing the non-optimal control law $\bar{\underline{u}}_{N-1}$ at stage $k = N-1$. Consider a neighborhood $O_u \subseteq \Omega$ of $\underline{u}^*_{N-1}$ such that the state constraints are satisfied for all the trajectories generated employing controls in O_u. For any such trajectory $\bar{\underline{x}}$, $\bar{\underline{u}}$ we have:

$$S(\underline{x}^*_N) \le S(\bar{\underline{x}}_N). \tag{12}$$

Hence:

$$\left.\frac{\partial S}{\partial \underline{x}'}\right|_{\underline{x}^*_N} \Delta \underline{x}_N = \left.\frac{\partial S}{\partial \underline{x}'}\right|_{\underline{x}^*_N} (f(\underline{x}^*_{N-1}, \bar{\underline{u}}_{N-1}) - (f(\underline{x}^*_{N-1}, \underline{u}^*_{N-1})) \ge 0, \tag{13}$$

$$\begin{aligned} H(\underline{x}^*_{N-1}, \bar{\underline{u}}_{N-1}, \psi^*_{N-1}) &= \left.\frac{\partial S}{\partial \underline{x}'}\right|_{\underline{x}^*_N} f(\underline{x}^*_{N-1}, \bar{\underline{u}}_{N-1}) \\ &\ge \left.\frac{\partial S}{\partial \underline{x}'}\right|_{\underline{x}^*_N} f(\underline{x}^*_{N-1}, \underline{u}^*_{N-1}) \\ &= H(\underline{x}^*_{N-1}, \underline{u}^*_{N-1}, \psi^*_{N-1}). \end{aligned} \tag{14}$$

Consider now a neighborhood $O_u \subseteq \Omega$ of $\underline{u}^*_k$ such that all the trajectories obtained by replacing $\underline{u}^*_k$ by any other control in O_u satisfy the constraints and assume that (10) does not hold for some $k < N-1$. Then, there exists at least one trajectory $\bar{\underline{x}}$, $\bar{\underline{u}}$ such that:

$$H(\underline{x}^*_k, \bar{\underline{u}}_k, \psi^*_k) < H(\underline{x}^*_k, \underline{u}^*_k, \psi^*_k). \tag{15}$$

Therefore:

$$0 > \psi^{*\prime}_k (f(\underline{x}^*_k, \bar{\underline{u}}_k) - f(\underline{x}^*_k, \underline{u}^*_k)) = \psi^{*\prime}_k \Delta \underline{x}_{k+1}. \tag{16}$$

Hence:

$$\begin{aligned} H(\bar{\underline{x}}_{k+1}, \underline{u}^*_{k+1}, \psi^*_{k+1}) - H(\underline{x}^*_{k+1}, \underline{u}^*_{k+1}, \psi^*_{k++1}) &= \psi^{*\prime}_{k+1} f(\bar{\underline{x}}_{k+1}, \underline{u}^*_{k+1}) - \psi^{*\prime}_{k+1} f(\underline{x}^*_{k+1}, \underline{u}^*_{k+1}) \\ &= \psi^{*\prime}_{k+1} \left(\left.\frac{\partial f}{\partial \underline{x}'}\right|_{\underline{x}^*_{k+1}} \right) \Delta \underline{x}_{k+1} < 0, \end{aligned} \tag{17}$$

or:

$$H(\bar{\underline{x}}_{k+1}, \underline{u}^*_{k+1}, \psi^*_{k+1}) < H(\underline{x}^*_{k+1}, \underline{u}^*_{k+1}, \psi^*_{k+1}). \tag{18}$$

Using the same reasoning we have:

$$\begin{aligned} H(\bar{\underline{x}}_{k+2}, \underline{u}^*_{k+2}, \psi^*_{k+2}) &< H(\underline{x}^*_{k+2}, \underline{u}^*_{k+2}, \psi^*_{k+2}), \\ &\vdots \\ H(\bar{\underline{x}}_{N-1}, \underline{u}^*_{N-1}, \psi^*_{N-1}) &< H(\underline{x}^*_{N-1}, \underline{u}^*_{N-1}, \psi^*_{N-1}). \end{aligned} \tag{19}$$

From (19) it follows that

$$0 > \psi^{*\prime}_{N-1} \Delta \underline{x}_N = \left.\frac{\partial S}{\partial \underline{x}'}\right|_{\underline{x}^*_N} \Delta \underline{x}_N = \Delta S, \tag{20}$$

against the hypothesis that $S(\underline{x}^*_N)$ was a minimum. ◇

3.2. *Using the modified discrete minimum principle.* In this section we indicate how to use the results of Theorem 2 to generate a set of points that satisfy the necessary conditions for optimality. In principle, we could apply the discrete minimum principle to problem (P′) by taking $S(\underline{x}_N) = \|\underline{x}_N\|_2^2$ and solving a sequence of problems, with increasing N, until a trajectory $\underline{x}^*$ and a number N_0 such that $\underline{x}^*_{N_0} = 0$ are found. However note that Theorem 2 does not add any information to the problem since;

$$\psi^{*\prime}_{N-1} = \frac{\partial S(\underline{x}^*_N)}{\partial \underline{x}'}\bigg|_0 = \underline{x}|_0 = 0. \qquad (21)$$

It follows that $\psi_k = 0\,\forall k$, and hence the optimal trajectory corresponds to a "singular arc". Therefore, nothing can be inferred *a priori* about the controls. In order to be able to use the special structure of the problem, we would like the co-states, ψ, to be non-zero.

Consider now the special case of problem (P′) where $S(\underline{x}_N) = \frac{1}{2}\|\underline{x}_N\|_2^2$ (with fixed terminal time N). Let n be the dimension of the system (S) and assume that the initial condition $\underline{x}_0$ is such that the origin cannot be reached in N stages. Then, from (7):

$$\begin{aligned}\psi^{*\prime}_k &= \psi^{*\prime}_{k+1}A,\\ \psi^{*\prime}_{N-1} &= \underline{x}_N \neq \underline{0}.\end{aligned} \qquad (22)$$

It follows (since A is regular) that $\psi^*_k \neq 0\,\forall k$. Furthermore, since (S) is controllable, C_n (null controllability region in n steps) has dimension n (Sznaier, 1989). It follows that, by taking N large enough $\underline{x}_N \in C_n$. Hence an approximate solution to (P) can be found by solving (P′) for N such that $\underline{x}_N \in C_n$ and by using Linear Programming to find the optimal trajectory from $\underline{x}_N$ to the origin.

Theorem 3. The optimal control sequence $\mathcal{U} = \{\underline{u}^*_0 \cdots \underline{u}^*_{N-1}\}$ that solves problem (P′) is always in the boundary of the set Ω_1. Further, the control sequence can always be selected to be a corner point of such a set.

Proof. Since the constraints are linear and $\psi^*_k \neq \underline{0}$, it follows that the control $\underline{u}^*_k$ that solves (10) belongs to the boundary of the set $\Omega_1(\underline{x}_k, k)$. Further, except in the case of degeneracies, i.e. when ψ^*_k is parallel to one of the boundaries of Ω_1, the control $\underline{u}^*_k$ must be a corner point of the set. In the case of degeneracies, all the points of the boundary parallel to the co-state yield the same value of the Hamiltonian and therefore the optimal control $\underline{u}_k$ still can be selected to be a corner of Ω_1. ◇

3.3. *Algorithm* H_{MP}. In this section we apply the results of Theorem 3 to obtain a suboptimal stabilizing feedback control law. From Theorem 3 it follows that problem (P′) can be solved by using the following algorithm.

Algorithm H_{MP} (Heuristically Enhanced Control using the minimum principle).

(1) Determine ϵ for equation (11) (for instance using Linear Programming off-line). Let $O = C_1$, and determine an underestimate $g(\cdot)$ relative to O.
(2) Let $\underline{x}_k$ be the current state of the system:
   (a) If $\underline{x}_k \in C_n$, null controllability region in n steps, solve problem (P) exactly using Linear Programming.
   (b) If $\underline{x}_k \notin C_n$ generate a tree by considering all possible trajectories starting at $\underline{x}_k$ with controls that lie in the corners of the polytope $\Omega_1(\underline{x}_k, k)$. Search the tree for a minimum cost trajectory to the origin, using heuristic search algorithms and $g(\cdot)$ as heuristics.
   (c) If there is no more computation time available for searching and the region O has not been reached, use the minimum partial cost trajectory that has been found.
(3) Repeat step 2 until the region the origin is reached.

Remark 2. Note that by solving problem (P′) instead of (P) we are relinquishing optimality, strictly speaking, since the trajectory that brings the system closer to C_n is not necessarily the trajectory that will yield minimum transit time to the origin. However, for any "reasonable" problem, we would expect both trajectories to be close in the sense of yielding approximately equal transit times (in the next section we will provide an example where this expectation is met).

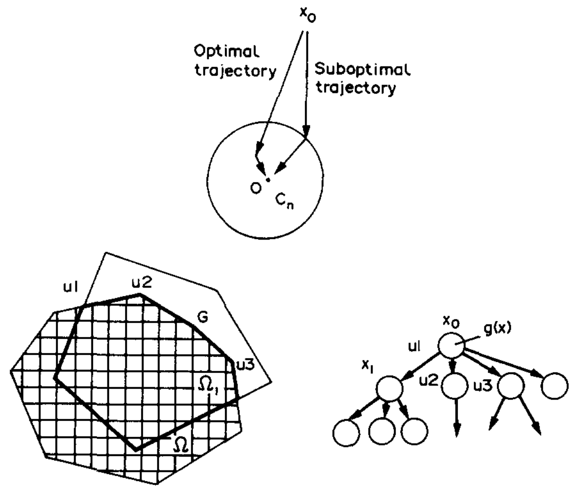

FIG. 1. Using the discrete minimum principle to limit the search.

Remark 3. Algorithm H_{MP} considers at each level of the tree only the control that lie in the corners of Ω_1, as illustrated in Fig. 1, therefore presenting a significant reduction of the dimensionality of the problem. However, the algorithm requires finding the vertices of a polytope given by a set of inequalities in R^m and this is a non-trivial computational geometry problem.

Theorem 4. The closed loop system resulting from the application of algorithm H_{MP} to problem (P′) is asymptotically stable, provided that there is enough computational power available to search one level of the tree during the sampling interval.

Proof. From Theorem 2 it follows that $\|\underline{x}_k\|_{\mathcal{G}}$ is monotonically decreasing in $\mathcal{G}$. Hence the system is guaranteed to reach the region C_n. But, since the solution to (P) is known in this region, it follows that the exact cost-to-go is a Lyapunov function for the system in C_n. Thus the resulting closed-loop system is asymptotically stable. ◇

3.4. *The heuristic for algorithm* H_{MP}. In order to complete the description of algorithm H_{MP} we need to provide a suitable underestimate $g(\underline{x})$. In principle, an estimate of the number of stages necessary to reach the origin can be found based upon the singular value decomposition of the matrices A and B, using the same technique that we used in Sznaier and Damborg (1990). However, in many cases of practical interest such as the F-100 jet engine of Section 2.3, the limitation in the problem is essentially given by the state constraints (i.e. the control authority is large). In this situation, this estimate yields an unrealistically low value for the transit time, resulting in poor performance.

The performance of the algorithm can be improved by considering an heuristic based upon experimental results. Recall that optimality depends on having, at each time interval, an underestimate $g(\underline{x})$ of the cost-to-go. Consider now the Null Controllability regions (C_k). It is clear that if they can be found and stored, the true transit time to the origin is known. If the regions are not known but a supraestimate C^s_k such that $C_k \subseteq C^s_k$ is available, a suitable underestimate $g(\underline{x})$, can be obtained by finding the largest k such that $\underline{x} \in C^s_k$ and $\underline{x} \notin C^s_{k-1}$. However, in general these supraestimates are difficult to find and characterize (Sznaier, 1989). Hence, it is desirable to use a different heuristic, which does not require the use of these regions. From the convexity of Ω and $\mathcal{G}$ it follows that the regions C_k

are convex. Therefore, a subestimate C_{sk} such that $C_{sk} \subseteq C_k$ can be found by finding points in the region C_k and taking C_{sk} as their convex hull. Once a subestimate of C_k is available, an estimate $\bar{g}(\underset{\sim}{x})$ of the cost-to-go can be found by finding the largest k such that $\underset{\sim}{x} \in C_{sk}$ and $\underset{\sim}{x} \notin C_{sk-1}$. Note that this estimate is not an underestimate in the sense of Definition 3. Since $C_{sk} \subseteq C_k$ then $\underset{\sim}{x}_k \in C_k \not\Rightarrow \underset{\sim}{x}_k \in C_{sk}$ and therefore $\bar{g}(\underset{\sim}{x}_k)$ is not necessarily $\leq k$. Thus, Theorem 1 that guarantees that once the set O has been reached the true optimal trajectory has been found is no longer valid. However, if enough points of each region are considered so that the subestimates are close to the true null controllability regions, then the control law generated by algorithm H_{MP} is also close to the true optimal control.

3.5. *Application to the realistic example.* Figure 2 shows a comparison between the trajectories for the optimal control law and algorithm H_{MP} for Example 2.3. In this particular case, the optimal control law was computed off-line by solving a sequence of linear programming problems, while algorithm H_{MP} was limited to computation time compatible with an on-line implementation. The value of ϵ was set to 0.01 (using linear programming it was determined that the maximum value of ϵ compatible with the constraints is 0.025) and each of the regions C_{sk} was found as the convex hull of 32 points, using optimal trajectories generated off-line. Note that in spite of being limited to running time roughly two orders of magnitude smaller than the computation time used off-line to find the true optimal control solution, algorithm H_{MP} generates a solution that takes only 25% more time to get to the origin (25 vs 31 stages).

Figure 3 shows the results of applying algorithm H_{MP} when the heuristic is perfect (i.e. the exact transit time to the origin is known). By comparing Figs 2 and 3 we see that most of the additional cost comes from the approximation made in Theorem 2, while the use of an estimate of the cost-to-go based upon the subestimates C_{sk} (rather than a "true" underestimate as required by Theorem 1) adds only one stage to the total transit time.

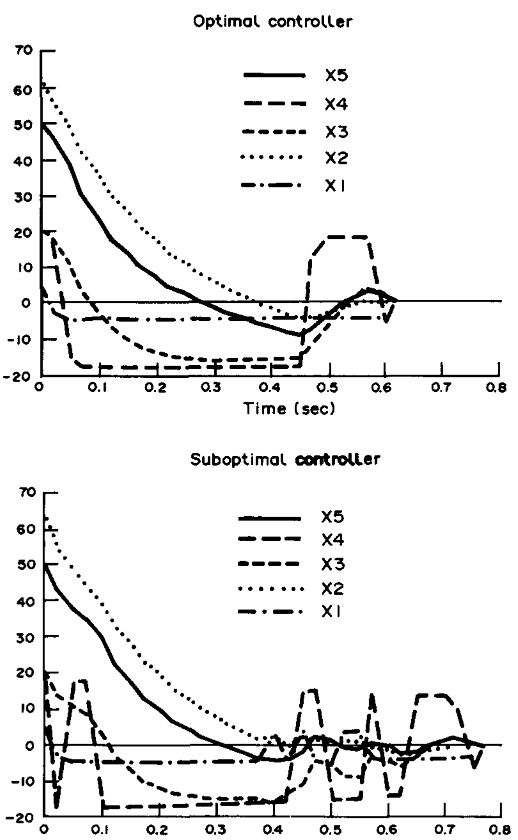

FIG. 2. Optimal control vs algorithm H_{MP} for the Example of Section 3.4.

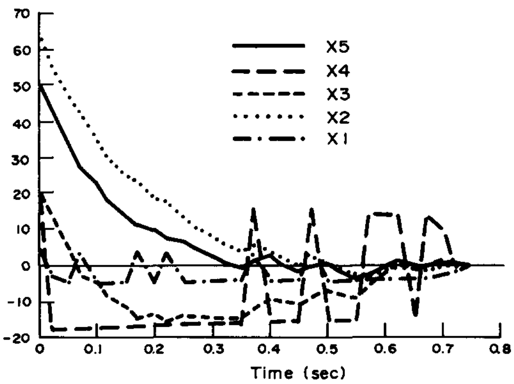

FIG. 3. Algorithm H_{MP} with perfect information.

4. *Conclusions*

Following the idea presented by Sznaier (1989) and Sznaier and Damborg (1990), in this paper we propose to address time-domain constraints by using a feedback controller based upon the on-line use of a dynamic-programming approach to solve a constrained optimization problem. Theoretical results are presented showing that this controller yields asymptotically stable systems, provided that the solution to an optimization problem, considerably simpler than the original, can be computed in real-time. Dimensionality problems common to dynamic programming approaches are circumvented by applying a suitably modified discrete time minimum principle, which allows for checking only the vertices of a polytope in control space. This polytope is obtained by considering the intersection Ω_1 of the original control region Ω with the region obtained by projecting the state constraints into the control space. The proposed approach results in a substantial reduction of the dimensionality of the problem (two orders of magnitude for the case of the example presented in Section 2.3). Hence, the proposed algorithm presents a significant advantage over previous approaches that use the same idea, especially for cases, such as Example 2.3, where the time available for computations is very limited.

We believe that the algorithm presented in this paper shows great promise, especially for cases where the dimension of the system is not small. Note however, that the algorithm requires the real time solution of two non-trivial computational geometry problems in R^n; determining the inclusion of a point in a convex hull and finding all the vertices of a polytope. Recent work on trainable non-linear classifiers such as artificial neural nets and decision trees may prove valuable in solving the first problem.

Perhaps the most serious limitation to the theory in its present form arises from the implicit assumptions that the model of the system is perfectly known. Since most realistic problems involve some degree of uncertainty, clearly this assumption limits the domain of application of the proposed controller. We are currently working on a technique, patterned along the lines of the Norm Based Robust Control framework introduced by Sznaier (1990), to guarantee robustness margins for the resulting closed-loop system. A future paper is planned to report these results.

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