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Brief Paper

Control of Constrained Discrete Time Linear Systems Using Quantized Controls†

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Key Words—Computer control; control systems; controllability; digital control; discrete time systems; feedback control; multivariable control systems; numerical control; suboptimal control.

Abstract—The theory of control of continuous-time systems with control constraints is extended to the case where the controls are of the form $u_i = n_i/s$, where n_i is an integer and s is a scaling factor. These results permit the analysis of the controllability of digital control systems with quantized controls. They also provide the theoretical framework for recently suggested real-time suboptimal controllers, based on the application of artificial intelligence techniques. Such an application is presented at the end of the paper.

1. *Introduction*

THE THEORY of control of continuous-time systems with control constraints is well known. The original results due to Lee and Marcus (1967) have been extended in a number of ways to account for different classes of constraints; see for example Jacobson *et al.* (1980). These results, however, have not been extended to cases such as digital controllers, where it is necessary to account for quantization effects.

The quantization effects may result from natural constraints, such as the presence of a computer with a finite word length in the controller. Alternatively, they may be artificially imposed as in the case of Heuristically Enhanced Optimal Control (Guez, 1986), where the control space is partitioned into a finite set to simplify the search for an optimal trajectory.

Traditionally, quantization effects have been treated by adding noise sources and non-linear quantizers to the system (Kuo, 1980). This type of analysis provides upper bounds on the errors due to quantization effects, but it is not suitable for extending the results already known for constrained, continuous-time linear systems.

In this paper we present basic results on the controllability of constrained discrete time systems using quantized controls and an application of these results to optimal control problems. It will be shown that, for controllable linear systems, there exist regions of the state space containing initial conditions which can be steered to a neighborhood of the origin. This neighborhood will be characterized in terms of the singular values of the controllability matrix of the system and the norm of the quantization (to be defined).

The main motivation for this paper is to provide a theoretical framework for recently suggested real-time suboptimal controllers (Guez, 1986), but we believe that the results presented here are also valuable for the analysis and design of digital control systems. For instance they can be used as guidelines to select the appropriate hardware for a microprocessor controlled system.

2. *Theoretical results*

In this section we present the basic results on the controllability of constrained discrete time systems using quantized controls. In order to present these results the following definitions are introduced.

Consider the linear time invariant discrete system

$$x(k+1) = Ax(k) + Bu(k) \quad x \in R^n,\ u \in \Omega \subseteq R^m \tag{1}$$

with $x(0) = x_0$, $k = 0, 1, \ldots$, and Ω convex, containing the origin in its interior.

Definition 2.1. The Origin Attainable domain of (1) is the set of all possible end points $x(k)$, $x(k) \in R^n$, $k = 0, 1, \ldots$, for trajectories starting at the origin, i.e. $x_0 = 0$, with $u(k) \in \Omega \subseteq R^m$.

Definition 2.2. The Null Controllable domain of (1) is the set of all points $x \in R^n$ that can be steered to the origin by applying a sequence of admissible controls $u(k) \in \Omega \subseteq R^m$, $k = 0, 1, \ldots$ The Null Controllable domain of (1) will be denoted as C_∞. The Null Controllable domain in j or fewer steps will be denoted as $C_j \subseteq C_\infty$.

The following lemmas characterize the Origin Attainable and Null Controllable domains. Their proofs are a direct extension to the discrete case of the results presented in Lee and Marcus (1967).

Lemma 2.1. Consider the systems

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

and

$$x(j+1) = A^{-1}x(j) - A^{-1}Bu(j) \tag{2}$$

where $x \in R^n$, $u \in \Omega \subseteq R^m$, Ω is convex and contains the origin in its interior and where A^{-1} exists. Then, the Null Controllable domain of (1) coincides with the Origin Attainable domain of (2).

Lemma 2.2. Consider the Null Controllable domains of (1), C_{n+k}, where $k = 0, 1, \ldots$, and where n is the dimension of the system. If A^{-1} exists then the origin is an interior point of C_{n+k} and C_∞ is open iff the pair (A, B) is controllable, that is: $\operatorname{rank}(M) = n$, where $M = [B, AB, \ldots, A^{n-1}B]$ (controllability matrix).

Definition 2.3. A quantization Ω_s of a given set $\Omega \subseteq R^m$ is the set

$$\Omega_s = \{u : u \in \Omega,\ u_i = n_i/s, \text{ where } u_i \text{ is the } i\text{th coordinate of } u, \ n_i \text{ is an integer and } s \text{ is a scaling factor}\}.$$

The quantity $1/s$ will be called the norm of the quantization. In this paper we will restrict ourselves to "quantizable" sets, i.e. sets Ω that verify the following condition: there exists

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$s_0 \in R$ such that

$$\min_{u_s \in \Omega_s} \|u - u_s\|_\infty \leq 1/s$$

for all $u \in \Omega$† and for all $s \geq s_0$.

Definition 2.4. The system (1) is Quantized Null Controllable in a region $C \subseteq R^n$ if, for any open set $O \subseteq R^n$ containing the origin in its interior, there exist a number $s_0(C, O) \in R$ such that for all the quantizations Ω_s of Ω with $s \geq s_0$, there exists a sequence of admissible quantized controls $u(k) \in \Omega_s$ such that the system can be steered from any initial condition $x_0 \in C$ to O.

In the following definition a concept closely related to Quantized Null Controllability is introduced.

Definition 2.5. Consider the set

$$X_s = \{x \in R^n : x = My \text{ for } y \in R^{nm} \text{ and } \|y\|_\infty \leq 1/s\}$$

where M is the controllability matrix of the system (1). The system (1) is X_s controllable in a region $C \subseteq R^n$ if there exists a number $s_0(C) \in R$ such that for all the quantizations Ω_s of Ω with $s \geq s_0$, there exists a sequence of admissible quantized controls $u(k) \in \Omega_s$ such that the system can be steered from any initial condition $x_0 \in C$ to the set X_s.

The set X_s can be characterized in terms of the singular values of the matrix M as follows. Let E and E_s be the hyperellipsoids defined as

$$E = \{x \in R^n : x = My \text{ for } y \in R^{nm} \text{ and } \|y\|_2 \leq 1\}$$

$$E_s = \{x \in R^n : x = My \text{ for } y \in R^{nm} \text{ and } \|y\|_2 \leq \sqrt{(mn)}/s\}.$$

Note that the singular values, σ_i, of the matrix M are the lengths of the semi-axes of E (Golub and Van Loan, 1983) and that $(\sqrt{(mn)}/s)\sigma_i$ are the lengths of the semi-axes of E_s. Since $\|y\|_2 \leq \sqrt{(mn)}\,\|y\|_\infty$ we have that $X_s \subseteq E_s$. Hence we have the following lemma.

Lemma 2.3. If the system (1) is X_s controllable in a region C, then it is Quantized Null Controllable in the same region.

The proof follows from the fact that since $X_s \subseteq E_s$ then, given an open set $O \subseteq R^n$ containing the origin in its interior, it is possible to find a suitable s_0 such that $X_{s_0} \subseteq E_{s_0} \subseteq O$. Therefore, $X_s \subseteq O$ for all $s \geq s_0$ and (1) is Quantized Null Controllable.

The following theorems show that a linear time invariant system is Quantized Null Controllable in the region $C_k \subseteq C_\infty \subseteq R^n$ for all finite k.

Theorem 2.1. The system (1) is Quantized Null Controllable in the region C_n (Null Controllable domain in n steps, with n the dimension of the system); moreover, for any $s > 0$, the region X_s may be reached in n steps starting from any $x_0 \in C_n$ and using controls in Ω_s.

Proof. Since $x_0 \in C_n$ there exists a sequence $v(i) \in \Omega$ such that

$$0 = x(n) = A^n x_0 + \sum_0^{n-1} A^{n-i-1} Bv(i). \tag{3}$$

Let $u(i) = v(i) + \delta v(i)$ where $u(i) \in \Omega_s$, $\delta v(i) \in R^m$ and

$$\|\delta v(i)\|_\infty = \min_{u_s \in \Omega_s} \|v(i) - u_s\|_\infty \leq 1/s.$$

Then

$$0 = A^n x_0 + \sum_0^{n-1} A^{n-i-1} B(u(i) - \delta v(i)) \tag{4}$$

and

$$x_q(n) = A^n x_0 + \sum_0^{n-1} A^{n-i-1} Bu(i) = \sum_0^{n-1} A^{n-i-1} B\,\delta v(i) = My \tag{5}$$

† An example of such a set is $|u_i| \leq k_i$, $i = 1, n$, where k_i are given constants.

where $x_q(n)$ is the final state using quantized controls, $y \in R^{nm}$ and $\|y\|_\infty \leq 1/s$, hence $x_q(n) \in X_s$. Therefore, the system is X_s controllable in C_n and by Lemma 2.3 it is Quantized Null Controllable in C_n.

Lemma 2.4. Let $x \in C_k$ (Null Controllable domain in k steps) and $\delta x \in R^n$. If $y = A^k \delta x \in C_l$, then

$$x + \delta x \in C_{k+l}.$$

Proof. Since $x \in C_k$, it is attainable from the origin in k steps, hence

$$x = -\sum_0^{k-1} A^{-(k-i)} Bu(i),\ u(i) \in \Omega \tag{6}$$

similarly

$$A^k \delta x = -\sum_0^{l-1} A^{-(l-i)} Bv(i),\ v(i) \in \Omega. \tag{7}$$

Hence

$$\delta x = -\sum_0^{l-1} A^{-(l+k-i)} Bv(i). \tag{8}$$

Adding (6) and (8)

$$x + \delta x = -\sum_0^{k-1} A^{-(k-i)} Bu(i) - \sum_0^{l-1} A^{-(l+k-i)} Bv(i)$$

$$= -\sum_0^{l+k-1} A^{-(l+k-i)} Bw(i) \tag{9}$$

where $w(i) = v(i)$ for $i = 0, \ldots, l-1$ and $w(i) = u(i-l)$ for $i = l, \ldots, l+k-1$, so $w(i) \in \Omega$. Hence $x + \delta x \in C_{k+l}$.

Theorem 2.2. If the system (1) is controllable, then it is Quantized Null Controllable in C_{n+k+1} for all $k = 0, 1, \ldots$

Proof. (By induction.) Define

$$M_{n+1} = [B, AB, \ldots, A^n B],\ M_{n+1} : R^{m(n+1)} \to R^n$$

$$Y_s = \{y \in R^n : y = M_{n+1} z \text{ for } z \in R^{m(n+1)} \text{ and } \|z\|_\infty \leq 1/s\}.$$

Let x_0 be the initial condition of (1) and, for a given k, let $r(k)$ be a number such that $A^l y \in C_n$ for all $y \in Y_r$ and all $l = 0, 1, \ldots, k$. Note that since (1) is controllable, the origin is an interior point of C_n (Lemma 2.2) and therefore r exists.

(a) For $k = 0$ we have that

$$x_0 \in C_{n+1} \text{ and } y \in C_n \text{ for all } y \in Y_r. \tag{10}$$

Since $x_0 \in C_{n+1}$ then there exists a sequence $v(i) \in \Omega$ such that

$$0 = x(n+1) = A^{n+1} x_0 + \sum_0^n A^{n-i} Bv(i). \tag{11}$$

Let $u(i) = v(i) + \delta v(i)$ where $u(i) \in \Omega_r$ and

$$\|\delta v(i)\|_\infty = \min_{u_r \in \Omega_r} \|v(i) - u_r\|_\infty.$$

Then

$$0 = A^{n+1} x_0 + \sum_0^n A^{n-i} B(u(i) - \delta v(i)) \tag{12}$$

$$x_q(n+1) = A^{n+1} x_0 + \sum_0^n A^{n-i} Bu(i)$$

$$= \sum_0^n A^{n-i} B\,\delta v(i) = M_{n+1} z \tag{13}$$

where $x_q(n+1)$ is the final state using quantized controls, $z \in R^{m(n+1)}$ and $\|z\|_\infty \leq 1/r$. Hence, $x_q(n+1) \in Y_r$. By hypothesis $Y_r \subseteq C_n$, therefore $x_q(n+1) \in C_n$ and by Theorem 2.1, it can be steered to X_r in n steps. Hence $x_q(2n+1) \in X_r$. For a given set O, let t be a number such that $X_t \subseteq O$. Then, by selecting $s_0 = \max\{r, t\}$ we have that $x_q(2n+1) \in O$ for all $s \geq s_0$ and therefore (1) is Quantized Null Controllable in C_{n+1}.

(b) Assume now that the theorem is true for

$0, 1, 2, \ldots, k-1$. We have to prove that $x_0 \in C_{n+k+1}$ can be steered to X_s (with s large enough). Since $x_0 \in C_{n+k+1}$ there exists a sequence $v(i) \in \Omega$ such that

$$x(n+1) = A^{n+1}x_0 + \sum_0^{n-1} A^{n-i}Bv(i) \in C_k \quad (14)$$

(after $n+1$ steps we need only k more to get to the origin).

Let $u(i) = v(i) + \delta v(i)$ where $u(i) \in \Omega_r$ and

$$\|\delta v(i)\|_\infty = \min_{u_r \in \Omega_r} \|v(i) - u_r\|_\infty.$$

Then

$$x(n+1) = A^{n+1}x_0 + \sum_0^{n-1} A^{n-i}B(u(i) - \delta v(i))$$
$$= x_q(n+1) - M_{n+1}z \quad (15)$$

where $x_q(n+1)$ is the final state achieved using quantized controls, $z \in R^{m(n+1)}$ and $\|z\|_\infty \le 1/r$.

Since $x(n+1) \in C_k$ and $A^k y = A^k M_{n+1} z \in C_n$, we have, by Lemma 2.4, that $x_q(n+1) \in C_{n+k}$ and therefore, by the induction hypothesis, the system can be steered from C_{n+k+1} to X_r. Again, by selecting $s_0 = \max\{r, t\}$ we have that x_0 can be steered to the set O for all $s \ge s_0$ and therefore (1) is Quantized Null Controllable in C_{n+k+1}.

Theorems 2.1 and 2.2 show that the system (1) is Quantized Null Controllable in any region $C_k \subset C_\infty$. However, note that the choice of s_0 in Theorem 2.2 is quite restrictive since, for a system starting out in the region C_{n+k+1}, it requires that $A^l Y_{s_0} \subseteq C_n$ for all $l = 0, 1, \ldots, k$. For an unstable system $A^k Y_{s_0}$ is an expansion of Y_{s_0}. In this case, when the system starts "further" from the origin, the norm of the partition must be smaller, to drive the system to the neighborhood of the origin. Hence s_0 must be selected sufficiently large for each k and the existence of s_0 such that we have Quantized Null Controllability in the union of all the sets C_{n+k} (namely C_∞) is not guaranteed.

3. *Application to optimal control*

In this section we present an application to Optimal Control. In this case the quantization of the control space is introduced as an artifact to simplify the search for an optimal trajectory. Hence we will assume that there are no hardware imposed constraints on the controls. It will be shown that, using quantized controls, the system can be steered to a neighborhood of the origin where the problem reduces to the standard linear quadratic formulation. Once this region is reached it is no longer necessary to use quantized controls since the optimal trajectory is given by a simple linear feedback law of the form $u = -Kx$.

Consider the following optimization problem:

$$\min_U \sum_{n=0}^{\infty} L_n(x(n), u(n)) \quad (16)$$

subject to

$$x(k+1) = Ax(k) + Bu(k), \quad x(k) \in R^n, \quad u(k) \in \Omega \subseteq R^m,$$

Ω compact, convex, containing the origin in its interior, (17)

$$x(0) = x_0$$

where

$L_n(x, u) = 0.5(x^T(n)Qx(n) + u^T(n)Ru(n)) + f(x, u)$,

$U = \{u(k), k = 0, 1, \ldots\}$,

Q positive semidefinite, R positive definite, and

$f(x, u): R^{n+m} \to R$, $f(x, u) \ge 0$, $f(x, u) = 0$ for all $u \in \Omega$, $x \in G \subseteq R^n$; G, Ω compact, convex, containing the origin in their interior.

Note that $f(x, u)$ may represent constraints. "Forbidden zones" may be represented by regions where $f(x, u) \to \infty$.

Let the optimal cost to go from a given point, $x(j)$, to the origin be

$$J(x) = 0.5x^T Sx + h(x) = \sum_{n=j}^{\infty} L_n(x(n), u^*(n)) \quad (18)$$

where $u^*(.)$ is the optimal control and S is the solution to the Algebraic Riccati Equation associated with the unconstrained Linear Quadratic problem obtained when $f(x, u) \equiv 0$ and $\Omega \equiv R^m$. Note that $h(x) \ge 0$ since $0.5x^T Sx$ is the cost to go for the unconstrained problem.

Finally, let $X_{K_0} \subseteq G \subseteq R^n$ be the region defined as

$X_{K_0} = \{x$: for the optimal trajectory starting at x then:
(1) the feedback law $u = -K_0 x(k)$ generates a control $u(k) \in \Omega$ for every k, where K_0 is the optimal gain for the system when $f(x, u) \equiv 0$ and the control is unconstrained;
(2) the states $x(j)$ of the closed-loop system never leave the region G;
(3) $\lim_{j\to\infty} x(j) = 0\}$.

Note that:

(1) if $x(0) \in X_{K_0}$ then the solution to the constrained optimization problem coincides with the solution to the unconstrained Linear Quadratic problem;

(2) if $x(0) \in X_{K_0}$ then $f(x(k), -K_0 x(k)) = 0$ for all k since $x(k) \in G$ and the feedback law $u = -K_0 x(k)$ generates a control $u \in \Omega$;

(3) $h(x) = 0$ for all $x \in X_{K_0}$.

The following theorems characterize the set X_{K_0}. Their proofs, sketched in Appendix A, follow from the behavior of linear systems and from convexity and continuity arguments.

Theorem 3.1. There exists an open ball $B(0, r) \subseteq X_{K_0}$.

Theorem 3.2. Let $Y \subseteq G$ be a convex polyhedron given by its vertices y_i, $i = 1, 2, \ldots$ Then $Y \subseteq X_{K_0}$ iff $y_i \in X_{K_0}$.

The relationship between the different sets defined is illustrated in Fig. 1.

The optimal control law can be found using standard mathematical programming techniques. Usually, the amount of computational time required prevents their application in a real-time feedback-controller although we have suggested an approach for a suboptimal controller for the case of linear inequality constraints (Sznaier and Damborg, 1987). Another drawback of these techniques is that they do not leave room for the incorporation of any knowledge that the designer may have or can guess about the solution.

These difficulties can be solved by the use of a Heuristically Enhanced Optimal Controller (Guez, 1986). A brief description of this technique, based upon casting the optimization problem into a tree search form by partitioning the control space, is presented in Appendix B. However, Guez (1986) gives no clues to the size of the partition or to the effects of such partition on the controllability of the system. Based on our work relating to these concepts, we propose the following suboptimal algorithm.

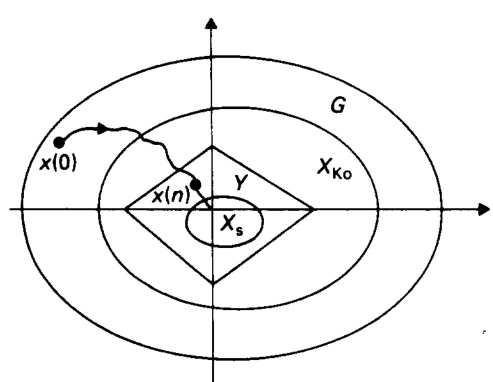

FIG. 1. Relationship between the sets G, X_s, X_{K_0} and Y.

(1) Determine a function $g(x): R^n \to R$, $0 \le g(x) \le h(x)$ for all $x \in G \subseteq R^n$. The function $g(x)$ will be used as a heuristical guideline for estimating the cost to go in the search for optimal trajectories, hence it should be very close to $h(x)$. In the event that $h(x)$ is completely unknown, $g(x) = 0$ may yield acceptable results. We are continuing to investigate how heuristic information can be incorporated at this stage of the algorithm.

(2) Determine a region $Y \subseteq X_{K_0}$ by finding points $y_i \in X_{K_0}$ and by applying Theorem 3.2.

(3) Determine the scaling factor s for the partition of Ω. A lower bound of s can be determined by requiring that $X_s \subseteq Y \subseteq X_{K_0}$ to assure that the origin can be reached using the quantized controls. Since $X_s \subseteq E_s$, s can be easily determined using the singular value decomposition of the controllability matrix M. The upper bound of s depends on the amount of time available for computation. There is a trade-off between computation time and the proximity of the quantized trajectory to the true minimum.

Note that steps 1–3 can be performed off-line, prior to switching on the controller.

(4) Let $x(k)$ be the current state of the system, k the current time instant and ΔT the sampling interval.

(i) If $x(k) \in Y$ the solution coincides with that of the unconstrained LQ problem: $u(k) = -K_0 x(k)$.

(ii) If $x(k) \notin Y$, generate a tree by considering all the possible trajectories starting at $x(k)$, with controls $u(k) \in \Omega_s$. Note that this is a finite tree since we are dealing with a finite set Ω_s. Search the tree for a minimum cost trajectory to the origin using heuristic search algorithms. Note that once a path reaches the region Y, then the cost to proceed from the first point of the trajectory interior to Y, $x(n)$, to the origin is given by $0.5x(n)^T S x(n)$, where S is the Riccati matrix associated with the unconstrained LQ problem.

(iii) If there is no more computation time available for searching and the region Y has not been reached, use the minimum partial cost trajectory that has been found.

(5) Repeat Step 4 until the origin is reached.

It should be remarked that the proposed controller is a feedback-controller and therefore can respond to the present condition.

4. *A simple example*

Consider the test system utilized in Sznaier and Damborg (1987), given by

$$x(k+1) = Ax(k) + Bu(k)$$

with

$$A = \begin{bmatrix} 1.0 & 0.2212 \\ 0.0 & 0.7788 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0288 \\ 0.2212 \end{bmatrix} \tag{19}$$

$$\Omega = \{u \in R,\ |u| \le 0.5\}, \quad G = \{x \in R^2,\ |x_1| \le 1.5,\ |x_2| \le 0.3\},$$

a sampling time of 0.25 s and initial condition $x_0 = (1.0, 0.3)$.

The objective is to drive the system to the origin with unspecified final time and with minimum energy. The matrices Q and R are selected to be the identity of appropriate dimensions. The unconstrained LQ solution is given by

$$u = -K_0 x, \qquad K_0 = [0.8831 \quad 0.8811]. \tag{20}$$

It is easily verified that the points

$$(0.5, 0);\ (0, 0.3);\ (-0.5, 0);\ \text{and}\ (0, -0.3)$$

belong to the region X_{K_0}. Hence the polygon Y that has these points as vertices is entirely contained in X_{K_0}. By construction, it can be shown that the region C_2 is the convex hull of the points

$$(-0.074, 0.324);\ (0.040, -0.040);\ (0.074, -0.324);\ (-0.040, 0.040) \tag{21}$$

and that

$$B(0, 0.0264) \subset C_2 \tag{22}$$

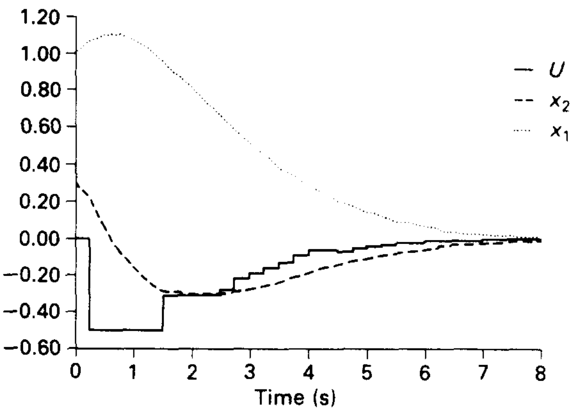

FIG. 2. Results of the HEOC algorithm applied to the example of Section 4.

where $B(0, 0.0264)$ is the ball centered in the origin with radius 0.0264.

By applying Lemma 2.3 with $O \equiv Y$ we have

$$X_s \subseteq E_s \subseteq B(0, \sigma_{max}(M)\sqrt{2}/s) \subseteq B(0, 0.0264) \subset C_2$$

if s is selected such that

$$1/s \le 0.0264/(\sigma_{max}(M)\sqrt{2}) = 0.6288 \tag{23}$$

where $\sigma_{max}(M)$ is the maximum singular value of M.

It can be shown that the initial condition $x_0 \in C_{28}$. Hence, application of Theorem 2.2 with $k = 25$ yields

$$\|A^l y\|_2 = \|A^l M_3 z\|_2 \le \sigma_{max}(A^l)\sigma_{max}(M_3)\sqrt{3}/s \tag{24}$$

for all $y \in Y_s$ and $l = 0, 1, \ldots, 25$. Hence, by selecting s such that

$$1/s \le \min_l \{0.0264/(\sigma_{max}(A^l)\sigma_{max}(M_3)\sqrt{3})\} = 0.0324 \tag{25}$$

we have

$$A^l Y_s \subseteq B(0, 0.0264) \subset C_2 \tag{26}$$

from (23) and (26) we have

$$1/s_0 \le 0.0324. \tag{27}$$

Figure 2 shows the response of the controller with initial conditions (1.0, 0.3), quantization 0.03125, and computation time restricted to 0.20 s. Note that the control action is kept to 0 during the first sampling interval which is used to measure the state of the system (the initial conditions are assumed to be unknown). Afterwards, the control constraint is active from $t = 0$ to 1.5 and the state constraint is active from $t = 1.5$ to 2.75. It is our experience that, for this system, the HEOC algorithm is faster than the On-line Quadratic Programming employed in Sznaier and Damborg (1987), allowing for a solution that takes more terms of the expansion into account. Note that the choice of s_0 in Theorem 2.2 is overly conservative. Experimenting with this problem, we have obtained convergence to the region X_{K_0} with quantizations of up to 0.5.

5. *Conclusions*

In this paper we presented basic results on the controllability of constrained, discrete time linear systems when the controls are limited to a finite (or countably infinite) set of the form $u_i = n_i/s$ where n_i is an integer and s is a scaling factor. These results are important for the analysis of digital controllers. They also provide a much needed theoretical framework for some new developments in real-time optimal controllers. We believe that Heuristically Enhanced Optimal Control is a valuable alternative to mathematical programming, especially for cases where it is neither feasible to compute and store a family of extremal curves nor to solve a Hamilton–Jacobi type equation in real time. An example of such a situation could be a microprocessor controlled robotic system. A future article is planned to discuss applications of heuristically enhanced optimal controllers and an analysis of their performance compared to the performance of real-time quadratic programming controllers which were the subject of Sznaier and Damborg (1987).

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Appendix A. Proofs of Theorems 3.1 *and* 3.2

Proof of Theorem 3.1. By hypothesis the origin is an interior point of G and Ω, hence there exist r_x and $r_u > 0$ such that: $B(0, r_x) \subset G$ and $B(0, r_u) \subset \Omega$.

Consider the mapping $f : R^n \to R^m$ defined by the feedback law $u = -K_0 x$.

Since f is continuous there exists $r > 0$ such that if

$$x \in B(0, r) \text{ then } u \in B(0, r_u) \subset \Omega. \quad \text{(A1)}$$

Let $t = \min\{r, r_x\}$.

Since Q, R are positive definite the closed-loop system

$$x(j+1) = (A - BK_0)x(j) \quad \text{(A2)}$$

is asymptotically stable. Hence

$$\lim_{j\to\infty} x(j) = 0$$

and there exists $\delta > 0$ such that: if $x(0) \in B(0, \delta)$ then $x(j) \in B(0, t) \subset G$ and by (A1) $u(j) \in B(0, r_u) \subset \Omega$. Therefore, $B(0, \delta) \subset X_{K_0}$.

Proof of Theorem 3.2. The direct portion of the proof is immediate since $Y \subseteq X_{K_0}$ implies that $y_i \in X_{K_0}$. Converse.

Let $x(0) = y_i$. Since $y_i \in X_{K_0}$, $i = 1, n$, then by definition of X_{K_0} the control law $u_i(t) = -K_0 y_i(t) \in \Omega$ and the states $y_i(t) \in X_{K_0}$.

Let y be an arbitrary point of Y. Since Y is convex then

$$y = \sum_1^n a_i y_i, \text{ where } 0 \leq a_i \leq 1 \text{ and } \sum_1^n a_i = 1. \quad \text{(A3)}$$

Consider the closed-loop system

$$y(j+1) = (A - BK_0)y(j) \text{ with initial condition } y(0) = y \quad \text{(A4)}$$

then

$$y(m) = (A - BK_0)^m y(0) = \sum_1^n a_i (A - BK_0)^m y_i(0) = \sum_1^n a_i y_i(m) \quad \text{(A5)}$$

where $y_i(j)$ is the trajectory that starts at y_i and

$$\lim_{j\to\infty} y(j) = 0.$$

Since G is convex then $y(j) \in G$ for all j.

Consider now the control law

$$u(m) = -K_0 y(m) = -K_0 \sum_1^n a_i y_i(m) = \sum_1^n a_i u_i(m). \quad \text{(A6)}$$

Since $y_i \in X_{K_0}$, $u_i(m) \in \Omega$ for all m and therefore, by the convexity of Ω, $u(m) \in \Omega$. Hence $y \in X_{K_0}$ and, since y is an arbitrary point of Y, we can conclude that $Y \subseteq X_{K_0}$.

Appendix B. A brief description of Heuristically Enhanced Optimal Control

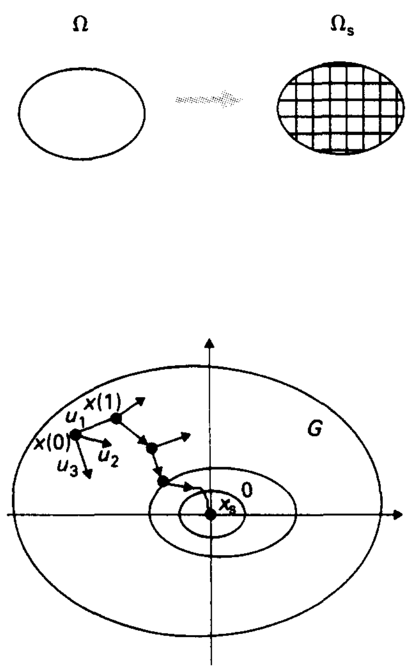
Heuristically Enhanced Optimal Control is an alternative to mathematical programming techniques for approximately solving constrained optimal control problems. In this approach, suggested by Guez (1986), the control space Ω is partitioned into a finite set Ω_s as shown in Fig. 3. The attainable domain from the initial condition, using controls in Ω_s, can be represented now as a tree, with each node corresponding to one of the attainable states. Hence the original optimal control problem is recast as a tree search, with the approximation resulting from the control quantization. The resulting tree can be scanned efficiently for minimum cost paths using artificial intelligence techniques, based upon an under-estimate of the cost to go, as follows (Winston, 1984).

(1) Form a queue of partial paths, with the initial queue consisting of the zero-cost, zero-step path from the root node to nowhere.

(2) Until the queue is empty or the goal has been reached determine if the first path in the queue reaches the goal.

(a) If the first path reaches the goal, do nothing.

(b) If the first path does not reach the goal:



Estimated cost to go for the trajectory $x(0) - x(n)$:

$$J_n(x,u) = \underbrace{\sum_{k=0}^{n-1} L(x(k), u(k))}_{\text{actual cost}} + \underbrace{g(x(n))}_{\text{heuristic approximation}}$$

FIG. 3. Diagram of the HEOC algorithm.

(i) remove the first path from the queue;

(ii) if possible, form new paths by extending the removed path one step with all permissible controls;

(iii) add the new paths to the queue;

(iv) sort the queue by the sum of the actual cost accumulated so far and a lower bound of the cost to go, with least-cost paths in front.

(3) If the goal has been reached announce success; otherwise announce failure.

Note that since the algorithm uses an under-estimate of the cost, the correct path will not be overlooked.

By using the under-estimate of the cost, this approach has the advantage of providing an easy way of incorporating knowledge available about the solution into the controller, thus presenting the potential for a significant reduction of the computation time.