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A convex approach to robust $\mathscr{H}_2$ performance analysis ☆

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Abstract

In this paper, we consider the problem of assessing worst-case $\mathscr{H}_2$ performance for MIMO systems and we give an LMI based sufficient condition for robust performance under LTI (not necessarily causal) model uncertainty, having the same complexity as $\mathscr{H}_\infty$ conditions for the same problem. In addition, we show that this condition is indeed necessary and sufficient for MISO and SIMO systems under a class of LTI uncertainty, and for MIMO plants under (arbitrarily slow) LTV uncertainty. © 2002 Published by Elsevier Science Ltd.

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1. Introduction

$\mathscr{H}_2$ control theory is appealing since there is a well established connection between the performance index being optimized and performance requirements encountered in practical situations. Moreover, the resulting controllers are easily found by solving two Riccati equations, and in the state-feedback case exhibit good robustness properties (Anderson & Moore, 1990). However, as the classical paper (Doyle, 1978) established, these margins vanish in the output feedback case.

Following this paper, several attempts were made to incorporate robustness into the $\mathscr{H}_2$ framework (Stein & Athans, 1987; Zhang & Freudenberg, 1990). More recently these efforts led to the mixed $\mathscr{H}_2/\mathscr{H}_\infty$ problem (Bernstein & Haddad, 1989; Zhou, Glover, Bodenheimer, & Doyle, 1994; Kaminer, Khargonekar, & Rotea, 1993; Sznaier, 1994; Scherer, 1995; Chen & Wen, 1995), where the resulting controller guarantees optimal performance for the *nominal* plant and stability against LTI dynamic uncertainty. While these results represent significant progress towards obtaining robust $\mathscr{H}_2$ controllers, they suffer from the fact that only nominal performance is guaranteed. Moreover, the resulting controllers have potentially high order.

Robust $\mathscr{H}_2$ performance under non-causal, non-linear time varying perturbations was analyzed in (Stoorvogel, 1993). More recently, both state-space (Feron, 1997) and frequency domain (Paganini, 1995a, b) convex upper bounds on the worst case $\mathscr{H}_2$ norm have been proposed. The state-space based bound, obtained using dynamic stability multipliers), is appealing since it takes into account, to some extent, causality. However, in order to obtain tractable problems, these multipliers must be restricted to the span of some basis, selected a-priori. Moreover, the complexity of this basis is limited by the fact that the computational complexity of the resulting LMI problem grows roughly as the 10th power of the state dimension (Paganini & Feron, 1999). On the other hand, while the frequency-domain based methods (Paganini, 1995a, b) cannot impose causality, they lead to simple LMI based conditions. Unfortunately, as shown by Sznaier and Tierno (2000) both the time and frequency-domain based bounds can be conservative by a factor of $\sqrt{m}$, where m denotes the dimensions of the exogenous input, even for very simple plants.

In this paper, we consider the problem of assessing worst case $\mathscr{H}_2$ performance under both LTI and LTV uncertainty. The main result of the paper provides sufficient conditions for robust $\mathscr{H}_2$ performance in the presence of LTI uncertainty. Further, these conditions are necessary and sufficient

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for a class of SIMO and MISO systems or for MIMO systems subject to (arbitrarily slowly) time varying perturbations.

For simplicity, in the sequel all the derivations are carried out in the discrete-time case. However, the formulae apply as well to continuous-time systems with minimal modifications.

2. Preliminaries

2.1. Notation and definitions

By $\mathscr{H}_2$ we denote the space of complex valued matrix functions $G(\lambda)$ with analytic continuation in $|\lambda| < 1$ and square integrable on the unit disk, equipped with the usual $\mathscr{H}_2$ norm

$$\|G\|_2^2 \doteq \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Trace}[G(e^{j\omega})G(e^{j\omega})^*]\,d\omega.$$

Given two matrices M and Δ of compatible dimensions we denote by $M \star \Delta$ the upper LFT $\mathscr{F}_u(M,\Delta)$, i.e.

$$M \star \Delta = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}.$$

Let $\mathscr{L}(\ell^2)$ denote the set of linear bounded operators in ℓ^2. In the sequel we will consider the following set of structured bounded operators in $\mathscr{L}(\ell^2)$:

$$\mathscr{B}\mathbf{\Delta} = \{\Delta \in \mathscr{L}(\ell^2)\colon \Delta = \mathrm{diag}[\delta_1 I_{r_1},\ldots,\delta_S I_{r_S}, \Delta_{S+1},\ldots,\Delta_{S+F}],\ \|\Delta\|_{\ell^2\to\ell^2} \leqslant 1\}.$$

The subsets of $\mathscr{B}\mathbf{\Delta}$ formed by linear time invariant, causal linear time invariant, linear time varying and (arbitrarily) slowly linear time varying operators will be denoted by $\mathscr{B}\mathbf{\Delta}^{\mathrm{LTI}}$, $\mathscr{B}\mathbf{\Delta}^{\mathrm{LTI}}_{\mathrm{causal}}$, $\mathscr{B}\mathbf{\Delta}^{\mathrm{LTV}}$ and $\mathscr{B}\mathbf{\Delta}^{\mathrm{SLTV}}$,[1] respectively. For ease of notation we also introduce a set of *constant* complex matrices having a structure similar to that of the operators in $\mathscr{B}\mathbf{\Delta}$:

$$\mathscr{B}\mathbf{\Delta}_m = \{\Delta \in C^{n\times n}\colon \Delta = \mathrm{diag}[\delta_1 I_{r_1},\ldots,\delta_S I_{r_S}, \Delta_{S+1},\ldots,\Delta_{S+F}],\ \bar{\sigma}(\Delta) \leqslant 1\},$$

where $\bar{\sigma}(\cdot)$ denotes the largest singular value. Finally, we will also make use of the following set of scaling matrices which commute with the elements in $\mathscr{B}\mathbf{\Delta}$:

$$\mathbf{X} = \{X\colon X = \mathrm{diag}[X_1,\ldots,X_S, x_{S+1}I_{m_1},\ldots,x_{S+F}I_{m_F}],\ X = X^*\}.$$

2.2. The $\mathscr{H}_2$ norm for LTV systems

In this paper, we are interested in analyzing the worst case $\mathscr{H}_2$ norm of the interconnection shown in Fig. 1,

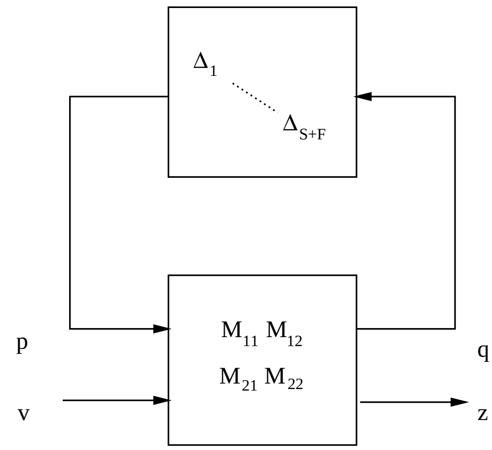

Fig. 1. Setup for robust $\mathscr{H}_2$ analysis.

where the nominal plant M is finite-dimensional, linear time invariant (FDLTI) and where the structure uncertainty $\mathbf{\Delta} = \mathrm{diag}\{\Delta_1, \Delta_2, \ldots, \Delta_{S+F}\}$ belongs to one of the classes discussed above. While this problem is well defined if $\Delta \in \mathscr{B}\mathbf{\Delta}^{\mathrm{LTI}}_{\mathrm{causal}}$, extending it to the other cases requires an appropriate definition of the $\mathscr{H}_2$ norm for LTV systems.

Several such definitions have been proposed (see for instance Paganini & Feron, 1999). These definitions can essentially be divided into the following three groups: (i) energy of the impulse response, added (or averaged) over the input direction, (ii) a stochastic interpretation based on the covariance of the output due to Gaussian white noise, and (iii) a deterministic approach based on considering the $\mathscr{H}_2$ norm as an induced norm from a subset of ℓ^2 to ℓ^2, considering for instance either the subset of ℓ^2 formed by signals with unity spectral density or by signal "white up to a small quantity η" (Paganini, 1995a). It is well known that these definitions coincide for LTI systems, but do not do so in the LTV case. In pursuing the extension of (ii) to the LTV case, care must be exercised since the output to stationary noise may no longer be stationary. This leads to two different interpretations based on whether the average or worst case output variance are considered (Paganini & Feron, 1999). Both approaches (i) and (iii) extend naturally from the LTI to the (N)LTV case. Note in passing that for the case of LTI systems (i) and (iii) actually coincide, since the impulse is the worst case signal among both sets. The motivation for using the "energy of the impulse response" definition is less clear in the case of (N)LTV systems, since here this may no longer be the case. On the other hand, as pointed out by Paganini (1995a), using an induced norm approach allows for bringing to bear to the problem powerful methods originally developed in the context of $\mathscr{H}_\infty$ control. Moreover, as we show in the sequel, surprisingly, approaches (i) and (iii) also coincide in the LTV case, i.e. the worst-case signal over the set of signals in the unit spectral density ball can always be taken to be an impulse.

[1] In rigor $\mathscr{B}\mathbf{\Delta}^{\mathrm{SLTV}}$ is a class containing all the LTV operators with variation slower than a given $v > 0$, i.e. $\mathscr{B}\mathbf{\Delta}^{\mathrm{SLTV}}_v \doteq \{\Delta \in \mathscr{B}\mathbf{\Delta}^{\mathrm{LTV}}\colon \|\lambda\Delta - \Delta\lambda\| \leqslant v\}$, where λ denotes the unit delay operator. In the sequel, for notational simplicity and with a slight abuse of notation we will drop the subscript v.

Based on these considerations, in this paper we consider the following definition of the $\mathscr{H}_2$ norm:

Definition 1 (H_2 norm for LTV systems). Given an exponentially stable, LTV system M with n_v inputs and n_o outputs, its $\mathscr{H}_2$ norm is defined as

$$\|M\|_{\mathscr{H}_2}^2 \doteq \sum_{i=1}^{n_v} \sup_{v_i \in \mathscr{BS}} \|M_{zv_i} v_i\|_2^2, \tag{1}$$

where $\mathscr{BS} \doteq \{v \in \mathscr{H}_2 \colon v(e^{j\omega})^* v(e^{j\omega}) \leqslant 1,\ \omega \in [0, 2\pi)\}$ and where M_{zv_i} denotes the operator that maps the ith input to the output z.

Using Definition 1, we can formalize now the concept of *Robust $\mathscr{H}_2$ Performance* and state the problem under consideration.

Definition 2 (Robust H_2 performance). The uncertain system $M \star \Delta$ has robust $\mathscr{H}_2$ performance against perturbations in the set $\mathscr{B}\mathbf{\Delta}$ (which in the sequel will designate one of the sets $\mathscr{B}\mathbf{\Delta}^{\mathrm{LTI}}$, $\mathscr{B}\mathbf{\Delta}^{\mathrm{SLTV}}$ or $\mathscr{B}\mathbf{\Delta}^{\mathrm{LTV}}$) if it is robustly exponentially stable and

$$\sup_{\Delta \in \mathscr{B}\mathbf{\Delta}} \|M \star \Delta\|_{\mathscr{H}_2} < 1, \tag{2}$$

where the $\|.\|_{\mathscr{H}_2}$ is taken in the sense of Definition 1.

Remark 1. In the case of LTI uncertainty we have that

$$\begin{aligned}
&\|M \star \Delta\|_{\mathscr{H}_2}^2 \\
&= \sum_{i=1}^{n_v} \sup_{v_i \in \mathscr{BS}} \frac{1}{2\pi} \int_0^{2\pi} \|(M \star \Delta)_{zv_i}(e^{j\omega}) v_i(e^{j\omega})\|^2 \, d\omega \\
&= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Trace}[(M \star \Delta)(e^{j\omega})(M \star \Delta)(e^{j\omega})^*] \, d\omega
\end{aligned}$$

and thus robust $\mathscr{H}_2$ performance in the sense of Definition 2 is equivalent to robust $\mathscr{H}_2$ performance in the usual sense.

3. Necessary and sufficient conditions for robust $\mathscr{H}_2$ performance

The inequalities that we obtain in this paper can be motivated by looking first into the case of MISO or SIMO systems subject to LTI uncertainty. In this case simple μ-analysis arguments can be used to derive the following sufficient robust performance conditions:

Lemma 1. *Consider a MISO system with input $v \in R^{n_v}$ and a scalar output z. If there exist a positive definite hermitian matrix $X(\omega) \in \mathbf{X}$ and a real transfer function $y(\omega) > 0$, such that*

$$M(e^{j\omega})^* \begin{bmatrix} X(\omega) & 0 \\ 0 & 1 \end{bmatrix} M(e^{j\omega}) - \begin{bmatrix} X(\omega) & 0 \\ 0 & y(\omega) I_{n_v \times n_v} \end{bmatrix} < 0 \tag{3}$$

holds for all $\omega \in [0, 2\pi)$, and

$$\int_0^{2\pi} y(\omega) \frac{d\omega}{2\pi} < 1 \tag{4}$$

then the interconnection $M \star \Delta$ achieves robust $\mathscr{H}_2$ performance against LTI (not necessarily causal) perturbations $\Delta \in \mathscr{B}\mathbf{\Delta}^{\mathrm{LTI}}$.

Proof. Consider the worst-case value of $\|M \star \Delta\|_{\mathscr{H}_2}$. Using standard singular value inequalities we have that:

$$\begin{aligned}
&\sup_{\Delta \in \mathscr{B}\mathbf{\Delta}^{\mathrm{LTI}}} \|M \star \Delta\|_{\mathscr{H}_2}^2 \\
&= \sup_{\Delta \in \mathscr{B}\mathbf{\Delta}^{\mathrm{LTI}}} \frac{1}{2\pi} \left\{ \int_0^{2\pi} \mathrm{Trace}[(M \star \Delta)(e^{j\omega}) (M \star \Delta)(e^{j\omega})^*] \, d\omega \right\} \\
&= \sup_{\Delta \in \mathscr{B}\mathbf{\Delta}^{\mathrm{LTI}}} \frac{1}{2\pi} \int_0^{2\pi} \bar{\sigma}[(M \star \Delta)(e^{j\omega})]^2 \, d\omega \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sup_{\Delta(e^{j\omega}) \in \mathscr{B}\mathbf{\Delta}_m} \{\bar{\sigma}[(M \star \Delta)(e^{j\omega})]^2\} \, d\omega,
\end{aligned} \tag{5}$$

where the last equality follows from the fact that the supremum of the integral (under possibly non causal perturbations) is achieved by maximizing the integrand frequency by frequency. From the main loop theorem (Packard & Doyle, 1993) it follows that if there exist a positive definite, hermitian matrix $X(\omega) \in \mathbf{X}$, and a real function $y(\omega) > 0$ such that

$$M(e^{j\omega})^* \begin{bmatrix} X(\omega) & 0 \\ 0 & 1 \end{bmatrix} M(e^{j\omega}) - \begin{bmatrix} X(\omega) & 0 \\ 0 & y(\omega) I_{n_v \times n_v} \end{bmatrix} < 0 \tag{6}$$

then

$$\sup_{\Delta \in \mathscr{B}\mathbf{\Delta}_m} \{\bar{\sigma}[(M \star \Delta)(e^{j\omega})]^2\} < y(\omega). \tag{7}$$

The proof follows now by noting that (5) together with (7) and (4) imply that

$$\sup_{\Delta \in \mathscr{B}\mathbf{\Delta}^{\mathrm{LTI}}} \|M \star \Delta\|_{\mathscr{H}_2} < \frac{1}{2\pi} \int_0^{2\pi} y(\omega) \, d\omega < 1. \quad \square \tag{8}$$

Clearly, a similar sufficient condition holds for SIMO systems, i.e.

Lemma 2. *Consider a SIMO system with a scalar input v and n_z outputs. If there exist a positive definite hermitian matrix $X(\omega)\in\mathbf{X}$ and a real transfer function $y(\omega)>0$, such that*

$$M(e^{j\omega})^*\begin{bmatrix}X(\omega) & 0\\ 0 & I_{n_z\times n_z}\end{bmatrix}M(e^{j\omega})-\begin{bmatrix}X(\omega) & 0\\ 0 & y(\omega)\end{bmatrix}<0 \quad (9)$$

holds for all $\omega\in[0,2\pi)$, and

$$\int_0^{2\pi} y(\omega)\frac{d\omega}{2\pi}<1 \quad (10)$$

then the interconnection $M\star\Delta$ achieves robust $\mathscr{H}_2$ performance against LTI (not necessarily causal) perturbations $\Delta\in\mathscr{B}\mathbf{\Delta}^{\mathrm{LTI}}$.

Corollary 1. *Conditions (3) and (4) or (9) and (10) are indeed necessary and sufficient for robust $\mathscr{H}_2$ performance against $\Delta\in\mathscr{B}\mathbf{\Delta}^{\mathrm{LTI}}$ when the uncertainty structure $\mathbf{\Delta}$ satisfies $2S+F\leqslant 2$.*

Proof. Follows immediately from the fact that in this case the uncertainty structure is μ-simple and thus the LMI (6) is indeed necessary and sufficient for (7) to hold (Packard & Doyle, 1993). □

3.1. Necessary and sufficient robust performance conditions

Lemma 3. *Consider a SIMO system with a scalar input v and n_z outputs. Then the interconnection $M\star\Delta$ achieves robust $\mathscr{H}_2$ performance against arbitrarily slowly time varying perturbations $\Delta\in\mathscr{B}\mathbf{\Delta}^{\mathrm{SLTV}}$ if and only if there exist a positive definite hermitian matrix $X(\omega)\in\mathbf{X}$, and a real transfer function $y(\omega)>0$, such that conditions (9) and (10) hold. Moreover, if (9) is satisfied by a constant matrix $X\in\mathbf{X}$ then the interconnection achieves robust performance against arbitrarily fast LTV uncertainty $\Delta\in\mathscr{B}\mathbf{\Delta}^{\mathrm{LTV}}$.*

Proof (*Sufficiency*). Proceeding as in the proof of Theorem 6.5 in (Paganini, 1995a, p. 91), it can be shown that Eq. (9) implies that for any $\Delta\in\mathscr{B}\mathbf{\Delta}^{\mathrm{SLTV}}$ the following holds:

$$\|z\|_2^2<\frac{1}{2\pi}\int_0^{2\pi} y(\omega)v^*(\omega)v(\omega)\,d\omega. \quad (11)$$

Thus

$$\sup_{v\in\mathscr{B}\mathscr{S}}\|z\|_2^2\leqslant\frac{1}{2\pi}\int_0^{2\pi}y(\omega)\,d\omega<1, \quad (12)$$

which establishes the desired result.

Necessity: Given in the appendix. □

Next we obtain necessary and sufficient conditions for robust $\mathscr{H}_2$ performance of a MIMO system, by decomposing it into a collection of SIMO systems.

Theorem 1. *Consider a MIMO system having n_v inputs and n_z outputs. Let $M_{ij}^{(i)}$ denote the ith column of the operator M_{ij}. Then the interconnection $M\star\Delta$ achieves robust $\mathscr{H}_2$ performance against arbitrarily slowly time varying perturbations $\Delta\in\mathscr{B}\mathbf{\Delta}^{\mathrm{SLTV}}$ if and only if there exist n_v positive definite hermitian matrices $X^{(i)}(\omega)\in\mathbf{X}$, and n_v real transfer functions $y_i(\omega)\geqslant 0$, such that the following conditions hold:*

$$\begin{bmatrix}M_{11} & M_{12}^{(i)}\\ M_{21} & M_{22}^{(i)}\end{bmatrix}^*\begin{bmatrix}X^{(i)} & 0\\ 0 & I_{n_z\times n_z}\end{bmatrix}\begin{bmatrix}M_{11} & M_{12}^{(i)}\\ M_{21} & M_{22}^{(i)}\end{bmatrix}-\begin{bmatrix}X^{(i)} & 0\\ 0 & y_i\end{bmatrix}<0,\quad i=1,\dots,n_v,$$

$$\int_0^{2\pi}\sum_{i=1}^{n_v}y_i(\omega)\frac{d\omega}{2\pi}<1. \quad (13)$$

Moreover, if the inequality (13) holds for constant scales $X^{(i)}$ then the interconnection $M\star\Delta$ achieves robust performance against $\Delta\in\mathscr{B}\mathbf{\Delta}^{\mathrm{LTV}}$.

Proof. Given in the appendix. □

Remark 2. It is interesting to compare this condition with the one obtained in Paganini (1995a). It can be easily shown that if the LMI proposed there admits a solution X,Y, then the set of LMIs (13) admits the solution $X^{(i)}=X$, $y_i=[Y]_{ii}$. Thus the bound obtained from Theorem 1 is always tighter than the one obtained in Paganini (1995a). This points out to a potential source of conservatism in the latter condition, since it uses a single scale X, rather than allowing for different scales in different channels.

4. Refinements for LTI uncertainty

To obtain tighter sufficient conditions for robust $\mathscr{H}_2$ performance under LTI uncertainty, begin by noting that in the LTI case the $\mathscr{H}_2$ norm can be found by computing the $\mathscr{H}_2$ norm of the system obtained by stacking all the individual transfer functions M_{ij} in a single vector. This leads to the following sufficient condition:

Theorem 2. *Consider a MIMO system having n_v inputs and n_z outputs. As before, let $M_{ij}^{(i)}$ denote the ith column of the operator M_{ij} and define the following operator*

$M_{\mathrm{col}}:\ell^2\to\ell^2$:

$$M_{\mathrm{col}}=\left[\begin{array}{cccc|c} M_{11} & 0 & \dots & 0 & M_{12}^{(1)} \\ 0 & M_{11} & \dots & 0 & M_{12}^{(2)} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & M_{11} & M_{12}^{(n_v)} \\ \hline M_{21} & 0 & \dots & 0 & M_{22}^{(1)} \\ 0 & M_{21} & \dots & 0 & M_{22}^{(2)} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & M_{21} & M_{22}^{(n_v)} \end{array}\right]. \tag{14}$$

If there exists $\tilde X(\omega)\in\tilde{\mathbf{X}}_{n_v}$, *and a function* $y(\omega)=y^*(\omega)\in\mathscr{L}^2$, *such that*

$$M_{\mathrm{col}}(\mathrm{e}^{\mathrm{j}\omega})^*\begin{bmatrix}\tilde X(\omega) & 0\\ 0 & I\end{bmatrix}M_{\mathrm{col}}(\mathrm{e}^{\mathrm{j}\omega})-\begin{bmatrix}\tilde X(\omega) & 0\\ 0 & y(\omega)\end{bmatrix}<0 \tag{15}$$

and

$$\int_0^{2\pi} y(\omega)\frac{\mathrm{d}\omega}{2\pi}<1,$$

where

$$\tilde{\mathbf{X}}_{\mathbf{n}}\doteq\left\{\begin{array}{l}\tilde X\colon \tilde X(\omega)=\begin{bmatrix}X_{11} & \dots & X_{1n}\\ \vdots & \dots & \vdots\\ X_{n1} & \dots & X_{nn}\end{bmatrix},\\ X_{ij}\in\mathbf{X},\ \tilde X(\omega)=\tilde X^*(\omega)>0\end{array}\right\} \tag{16}$$

then the interconnection $M\star\Delta$ *achieves robust* $\mathscr{H}_2$ *performance against* $\Delta\in\mathscr{B}\mathbf{\Delta}^{\mathrm{LTI}}$.

Proof.

$$\begin{aligned}\|M\star\Delta\|_{\mathscr{H}_2}^2&=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Trace}[(M\star\Delta)^*(M\star\Delta)]\,\mathrm{d}\omega\\ &=\frac{1}{2\pi}\int_0^{2\pi}\sum_i[(M\star\Delta)^{(i)}]^*(M\star\Delta)^{(i)}\,\mathrm{d}\omega\\ &=\frac{1}{2\pi}\int_0^{2\pi}\|M_{\mathrm{col}}\star\tilde\Delta\|^2\,\mathrm{d}\omega,\end{aligned} \tag{17}$$

where

$$\tilde\Delta=\begin{bmatrix}\Delta & 0 & \dots & 0\\ 0 & \Delta & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \Delta\end{bmatrix},\quad \Delta\in\mathscr{B}\mathbf{\Delta}^{\mathrm{LTI}}. \tag{18}$$

The proof is completed by noting that if (15) holds then for all $\Delta\in\mathscr{B}\mathbf{\Delta}^{\mathrm{LTI}}$, $\|M_{\mathrm{col}}\star\tilde\Delta\|^2<y(\omega)$ and hence

$$\sup_{\Delta\in\mathscr{B}\mathbf{\Delta}^{\mathrm{LTI}}}\|M\star\Delta\|_{\mathscr{H}_2}^2<\frac{1}{2\pi}\int_0^{2\pi}y(\omega)\,\mathrm{d}\omega<1.\qquad\square \tag{19}$$

A similar condition can be achieved by decomposing the operator M by rows, rather than columns, leading to the following result:

Theorem 3. *Consider a MIMO system having* n_v *inputs and* n_z *outputs. Let* $M_{ij}^{(i,r)}$ *denote the ith row of the operator* M_{ij} *and define the following operator* $M_{\mathrm{row}}:\ell^2\to\ell^2$:

$$M_{\mathrm{row}}=\left[\begin{array}{cccc|cccc} M_{11} & 0 & \dots & 0 & M_{12} & 0 & \dots & 0\\ 0 & M_{11} & \dots & 0 & 0 & M_{12} & \dots & 0\\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots\\ 0 & 0 & \dots & M_{11} & 0 & \dots & \dots & M_{12}\\ \hline M_{21}^{(1,r)} & M_{21}^{(2,r)} & \dots & M_{21}^{(n_z,r)} & M_{22}^{(1,r)} & M_{22}^{(2,r)} & \dots & M_{22}^{(n_z,r)}\end{array}\right]. \tag{20}$$

Then the interconnection $M\star\Delta$ *achieves robust* $\mathscr{H}_2$ *performance against* $\Delta\in\mathscr{B}\mathbf{\Delta}^{\mathrm{LTI}}$ *if there exists* $\tilde X(\omega)\in\tilde{\mathbf{X}}_{\mathbf{n_z}}$, *and* n_z $\mathscr{L}^2$ *functions* $y_i(\omega)$, $y_i=y_i^*\geqslant 0$ *such that*

$$M_{\mathrm{row}}(\mathrm{e}^{\mathrm{j}\omega})^*\begin{bmatrix}\tilde X(\omega) & 0\\ 0 & 1\end{bmatrix}M_{\mathrm{row}}(\mathrm{e}^{\mathrm{j}\omega})-\begin{bmatrix}\tilde X(\omega) & 0\\ 0 & Y(\omega)\end{bmatrix}<0 \tag{21}$$

and

$$\int_0^{2\pi}\sum_{i=1}^{n_z}y_i(\omega)\frac{\mathrm{d}\omega}{2\pi}<1,$$

where

$$Y(\omega)=\mathrm{diag}\{y_1I_{n_v\times n_v},y_2I_{n_v\times n_v},\dots,y_{n_z}I_{n_v\times n_v}\}.$$

5. Analysis of the conditions

The operator M_{col} introduced in Section 4 can be interpreted as decomposing the original system M into a collection of SIMO subsystems that are then "stacked" together (see Fig. 2). In order for this approach to be non-conservative, we need to enforce the additional constraint that the same uncertainty must act on all the subsystems, leading to the uncertainty structure (18). In this context the scales $\tilde X$ defined in (16) and the LMI (15) are the equivalent of the well known upper bound of μ for an uncertainty structure having repeated (not necessarily scalar) blocks.

We consider now briefly the issue of LTI vs. SLTV uncertainty and the relationship between Theorems 1 and 2.

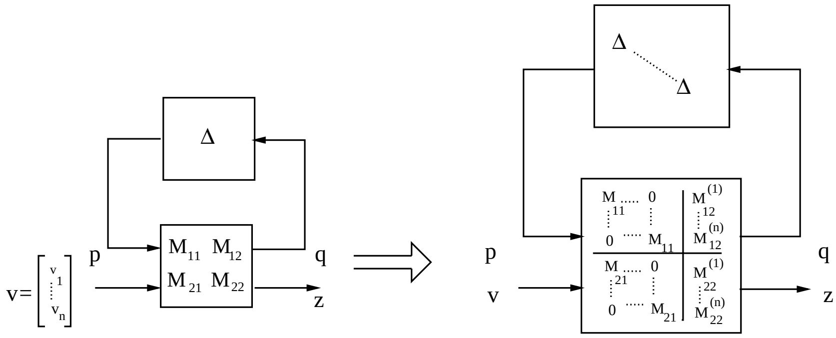

Fig. 2. System decomposition for Robust $\mathscr{H}_2$ analysis with LTI uncertainty.

Let

$$\tilde{M} = \left[\begin{array}{cccc|cccc} M_{11} & 0 & \dots & 0 & M_{12}^{(1)} & 0 & \dots & 0 \\ 0 & M_{11} & \dots & 0 & 0 & M_{12}^{(2)} & \dots & 0 \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & M_{11} & 0 & 0 & \dots & M_{12}^{(n_v)} \\ \hline M_{21} & 0 & \dots & 0 & M_{22}^{(1)} & 0 & \dots & 0 \\ 0 & M_{21} & \dots & 0 & 0 & M_{22}^{(2)} & \dots & 0 \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & M_{21} & 0 & 0 & \dots & M_{22}^{(n_v)} \end{array}\right]. \tag{22}$$

It can be easily shown (by interchanging rows and columns) that the set of LMIs (13) can be rewritten in terms of $\tilde{M}$ as

$$\tilde{M}^* \begin{bmatrix} \text{diag}\{X^{(i)}\} & 0 \\ 0 & I \end{bmatrix} \tilde{M} - \begin{bmatrix} \text{diag}\{X^{(i)}\} & 0 \\ 0 & \text{diag}\{y_i\} \end{bmatrix} < 0. \tag{23}$$

Moreover, since

$$M_{\text{col}} = \tilde{M} \left[\begin{array}{c|c} I & 0 \\ \hline & 1 \\ 0 & \vdots \\ & 1 \end{array}\right]$$

it follows that if the set (13) is satisfied by some $X^{(i)}$, y_i then (15) is satisfied by $\tilde{X} = \text{diag}\{X^{(i)}\}$, $\tilde{y} \doteq \sum_i y_i$. Thus, for LTI uncertainty condition (15) is always tighter than condition (13). Further, given the diagonal structure of the operator $\tilde{M}$, it can be shown that the LMI (23) can be relaxed to

$$\tilde{M}^* \begin{bmatrix} \tilde{X}(\omega) & 0 \\ 0 & I \end{bmatrix} \tilde{M} - \begin{bmatrix} \tilde{X} & 0 \\ 0 & Y(\omega) \end{bmatrix} < 0; \quad \tilde{X} \in \tilde{\mathbf{X}}_{\mathbf{n}}, \tag{24}$$

where

$Y(\omega) = \text{diag}\{y_1, y_2, \dots, y_{n_v}\}$

in the sense that this last LMI is feasible if and only if (23) is feasible.

The LMI (24) is precisely the condition that one would get by decomposing a MIMO system into a collection of smaller SIMO subsystems and constraining each of these subsystems to have the same uncertainty Δ. The fact that (24) admits a diagonal solution, hence reducing to n_v smaller, uncoupled LMIs indicates that for time varying uncertainty (even arbitrarily slowly varying) the worst case $\mathscr{H}_2$ norm over $\Delta \in \mathscr{B}\mathbf{\Delta}^{\text{SLTV}}$ is exactly the same that one would obtain if *different* uncertainties were allowed to act on different channels. Roughly speaking this is due to the fact that in the SLTV case an admissible perturbation can be always constructed by finding the worst case perturbation channel by channel and then combining these perturbations through time-delaying and adding (see the appendix for details).

6. Examples and comparisons

In this section, we illustrate our approach with some simple examples.

Example 1. Consider the following discrete-time system, used by Sznaier and Tierno (2000) to illustrate the gap between W^η and $\mathscr{H}_2$ performance. The plant is given by

$$M = \left[\begin{array}{c|c} 0_{m\times m} & I_{m\times m} \\ \hline e_1 & 0_{1\times m} \end{array}\right], \quad e_1 = [1 \quad 0 \quad \dots \quad 0],$$

$$\Delta_u = \begin{bmatrix} \Delta_{11} & \dots & \Delta_{1m} \\ \vdots & & \vdots \\ \Delta_{m1} & \dots & \Delta_{mm} \end{bmatrix} \in \mathscr{B}\mathbf{\Delta}^{\text{LTI}}_{\text{causal}}. \tag{25}$$

Simple algebra shows that $T_{zw} = M \star \Delta = [\Delta_{11} \;\; \dots \;\; \Delta_{1m}]$, from where it follows that $\sup_{\|\Delta_u\|_\infty < 1} \|T_{zw}\|^2_{\mathscr{H}_2} = 1$. In this case Theorem 3 reduces to Lemma 1, leading to the following LMI

$$\left[\begin{array}{cccc|c} 1 & & & & \\ & 0 & & & \\ & & \ddots & & 0_{m\times m} \\ & & & 0 & \\ \hline 0_{m\times m} & & & & xI_{m\times m} \end{array}\right] - \left[\begin{array}{cccc|c} x & & & & \\ & x & & & \\ & & \ddots & & 0_{m\times m} \\ & & & x & \\ \hline 0_{m\times m} & & & & yI_{m\times m} \end{array}\right] < 0. \tag{26}$$

Eq. (26) implies the following inequalities

$$1 - x < 0, \quad x - y < 0$$

from where it follows that

$$\int_0^{2\pi} y(\omega) \frac{d\omega}{2\pi} > 1$$

and the inequality can be achieved up to arbitrarily small ε. It follows that the worst case $\mathscr{H}_2$ norm is 1. On the other hand, the approach proposed in Paganini (1995a) leads to a similar LMI, with $yI_{m\times m}$ replaced by $Y \in R^{m\times m}$. The corresponding inequalities in this case are:

$$1 - x <, \quad x - y_{ii} < 0, \quad i = 1,\dots,m \Rightarrow y_{ii} > 1.$$

Thus $\int_0^{2\pi} \mathrm{Trace}\{Y(\omega)\}\, d\omega/2\pi \geqslant m$ and one can only conclude that the worst case $\mathscr{H}_2$ norm is no larger than $m^{1/2}$. It is worth noticing that the alternative LMI proposed in Theorem 2 also yields the conservative value $m^{1/2}$, due to the gap between μ and its upper bound (see Sznaier & Parrilo, 1999 for details). Finally, for completeness we also consider the impulse response approach proposed by Feron (1997). Here, an upper bound on the worst case $\mathscr{H}_2$ norm is found by solving the optimization problem:

$$\sup_{\Delta\in\mathscr{B}\mathbf{\Delta}} \|M \star \Delta\|^2_{\mathscr{H}_2} \leqslant J_s$$
$$\stackrel{\mathrm{def}}{=} \sup_{p\in\ell^2[0,\infty),\, \|q_i\|_2^2 \geqslant \|p_i\|_2^2} \sum_{i=1}^m \left\| [M_{21} \quad M_{22}] \begin{bmatrix} p_i \\ e_i\delta(k) \end{bmatrix} \right\|_2. \quad (27)$$

In the special case of system (25) this reduces to

$$\sup_{\Delta\in\mathscr{B}\mathbf{\Delta}} \|M \star \Delta\|^2_{\mathscr{H}_2} \leqslant J_s$$
$$= \sup_{p\in\ell^2[0,\infty),\, \|p_i\|_2^2 \leqslant 1} \sum_{i=1}^m \|p_i\|_2^2 = m. \quad (28)$$

Example 2 illustrates the relative weight of the ability to incorporate causality constraints on the bound against the ability to use high-order dynamic multipliers. [2]

Example 2. Consider the following 2 inputs and 2 outputs continuous-time system:

$$\begin{bmatrix} q \\ z_1 \\ z_2 \end{bmatrix} = M \begin{bmatrix} p \\ v_1 \\ v_2 \end{bmatrix} \text{ where } M \doteq \begin{bmatrix} M_{11} & M_{12}^{11} & 0 \\ M_{21}^{11} & 0 & 0 \\ 0 & 0 & M_{22}^{22} \end{bmatrix}$$
$$= \begin{bmatrix} -0.845\dfrac{s+1.5}{s+2} & \dfrac{s-1.5}{(s+2)^2} & 0 \\ \dfrac{s-1.5}{(s+2)^2} & 0 & 0 \\ 0 & 0 & \frac{1}{s+1} \end{bmatrix}. \quad (29)$$

[2] Since the LMI (13) is solved frequency by frequency and no explicit state space realization of $X(\omega)$ is required, there is no constraint on the order of the multipliers.

Table 1
$\|T_{zw}\|^2_{\mathscr{H}_2}$ vs. the order N of the dynamic multipliers

N	0	2	4	6
Bound	0.5383	0.5300	0.5300	0.5300
n_{dec}	4	61	135	241

Using Theorem 3 we have that [3]

$$M_{\mathrm{row}} = \left[\begin{array}{cc|cccc} M_{11} & 0 & M_{12}^{11} & 0 & 0 & 0 \\ 0 & M_{11} & 0 & 0 & M_{12}^{11} & 0 \\ \hline M_{21}^{11} & 0 & 0 & 0 & 0 & M_{22}^{22} \end{array}\right], \quad (30)$$

$$X_{\mathrm{row}} = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}, \quad Y_{\mathrm{row}} = \begin{bmatrix} y_1 I_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & y_2 I_{2\times 2} \end{bmatrix}. \quad (31)$$

Solving the corresponding LMI and integrating over ω we get $\|T_{zw}\|^2_{\mathscr{H}_2} = 0.5304$. Table 1 shows the results of applying the method proposed in Feron (1997) to this problem. Here N and n_{dec} denote the order of the dynamic multipliers and the number of decision variables in the associated LMI problem. [4] From Table 1 it follows that the causality constraints embedded in this formulation outweighs the ability to use arbitrarily high order multipliers when $N \geqslant 2$. However, note that the rather modest cost improvement is achieved at the price of a substantial increase in computational complexity.

7. Conclusions

In this paper, we use simple μ-analysis techniques to obtain sufficient (in fact necessary and sufficient in the MISO and MISO cases when the uncertainty structure satisfies $2S + F \leqslant 2$) conditions for robust $\mathscr{H}_2$ performance under LTI or slowly LTV structured perturbations. Since these conditions are essentially the integral of μ over the frequency, their complexity is the same as that of $\mathscr{H}_\infty$ analysis for the same problem. As a corollary, it follows that robust $\mathscr{H}_2$ analysis using set modelling of white noise (Paganini, 1995a) is indeed exact in the SISO case, and that the worst possible case in the MIMO case is $\sqrt{n_w}$.

A potentially serious drawback of the proposed approach is that in its present form it cannot impose causality constraints on the model uncertainty. However, this drawback is balanced by its ability to handle MIMO systems in a less conservative fashion, the lack of restrictions on the order of the multipliers $X(\omega)$, and the modest computational complexity increase with the order of the plant. As discussed by Paganini and Feron (1999) the relative weight of the last two effects is problem dependent, with the ability to use high order multipliers tending to outweight the causality constraints as the order of the plant M grows.

[3] For continuous-time systems the condition equivalent to (21) is $\int_{-\infty}^{\infty} \sum_{i=1}^{n_z} y_i(\omega)\, d\omega/2\pi < 1$.

[4] Recall that the computational time grows as n_{dec}^5 (Paganini & Feron, 1999).

Research is currently underway seeking to develop a frequency-domain method capable to incorporate the effects of causality.

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Appendix A. Proofs of Lemma 3 and Theorem 1

A.1. Preliminary results

In order to prove necessity in Lemma 3, we need the following preliminary result. For notational simplicity, we assume that all the blocks in the uncertainty structure are full (i.e., $S=0$).

Lemma A.1. *Consider a system having a transfer matrix* $M(\mathrm{e}^{\mathrm{j}\omega})$:

$$M(\mathrm{e}^{\mathrm{j}\omega}) = \begin{cases} M_o, & \omega \in [\omega_o, \omega_o + h), \\ 0 & \textit{otherwise}. \end{cases} \tag{A.1}$$

If the following LMI

$$M_o^* \begin{bmatrix} X & 0 \\ 0 & I_{n_z \times n_z} \end{bmatrix} M_o - \begin{bmatrix} X & 0 \\ 0 & 1 \end{bmatrix} < 0 \tag{A.2}$$

does not have a solution, then there exist signals $r = \binom{p}{v}$, $s = \binom{q}{z}$ *supported in* $[\omega_o, \omega_o + h)$ *such that*:

$$s = Mr, \quad \|q_i\|_2^2 \geqslant \|p_i\|_2^2, \quad \|z\|_2^2 \geqslant \|v\|_2^2,$$

$$|v(\omega)| = 1, \ \ \omega \in [\omega_o, \omega_o + h), \tag{A.3}$$

where the inputs and outputs have been partitioned according to the uncertainty structure.

Proof. Let $n := F+1$, and P_k, Q_k, $k = 1, \ldots, n$ be matrices of the form $[0 \ \cdots \ 0 \ I \ 0 \ \cdots \ 0]$, such that $p_k = P_k r$, $q_k = Q_k s$, $k = 1, \ldots, F$ and $v = P_n r$, $z = Q_n s$. Given a matrix A, denote by A_i its ith column. If the LMI (A.2) is not feasible, then its dual LMI:

$$\mathrm{Trace}\, Q_k M_o Z M_o^* Q_k^* - \mathrm{Trace}\, P_k Z P_k^* \geqslant 0, \quad k = 1, \ldots, n,$$
$$Z \geqslant 0, \quad Z \neq 0 \tag{A.4}$$

always has a solution (Meinsma, Shrivastava, & Fu, 1997).

If $P_n Z P_n^* = 0$, then the interconnection is not robustly stable against the family of perturbations Δ. Therefore, we can always scale Z such that $P_n Z P_n^* = 1$. Let the rank of Z be m. Factor Z as $Z = RR^*$ (R is in $\mathscr{C}^{n \times m}$), to obtain:

$$\mathrm{Trace}\, Q_k M_o R R^* M_o^* Q_k^* - \mathrm{Trace}\, P_k R R^* P_k^* \geqslant 0,$$
$$k = 1, \ldots, n. \tag{A.5}$$

Right multiplying by a unitary matrix if necessary, we can always choose R such that $P_n R$ has no zero component. The normalization condition $P_n Z P_n^* = 1$ implies that $\|P_n R\|^2 = \sum_{i=1}^m |(P_n R)_i|^2 = 1$. To convert these constant vectors into L_2 signals we make them orthogonal, putting every vector in a different frequency interval $[\omega_{i-1}, \omega_i)$. To this effect, consider the signal:

$$r(\omega) = \frac{1}{(P_n R)_i} R_i, \quad i = 1, \ldots, m,$$

$$\omega \in [\omega_{i-1}, \omega_i), \quad \omega_i = \omega_o + h \sum_{j=1}^{i} |(P_n R)_j|^2. \tag{A.6}$$

Notice that $v(\omega) = P_n r(\omega)$ is identically one in $[\omega_o, \omega_o + h)$ and

$$\int_0^{2\pi} r r^* \, \mathrm{d}\omega = h R R^*. \tag{A.7}$$

Eq. (A.3) follows now from (A.5).

A.2. Proof of Necessity in Lemma 3

For each ω, define

$$y(\omega) \doteq \inf\{y\colon \mathbf{L}MI \ (9) \text{ has a solution}\},$$

$$y_k \doteq \min_{\omega \in [kh, (k+1)h]} y(\omega). \tag{A.8}$$

Assume that condition (10) fails. Then, given any $\varepsilon > 0$, there exists h_1 small enough such that $(1/2\pi) \sum_k y_k h > 1 - \varepsilon/3$, $\forall h \leqslant h_1$. Define $M_k = diag\{I, 1/\sqrt{y_k - \varepsilon/3}\} * M(\mathrm{e}^{\mathrm{j}kh})$. From the choice of y_k it can be easily shown that the following LMI:

$$M_k^* \begin{bmatrix} X & 0 \\ 0 & I_{n_z \times n_z} \end{bmatrix} M_k - \begin{bmatrix} X & 0 \\ 0 & 1 \end{bmatrix} < 0 \tag{A.9}$$

does not have a solution. Hence, from Lemma A.1 we have that there exist an input r^k supported in $[kh, (k+1)h]$ such that the corresponding output $s^k = \binom{q^k}{z^k} \doteq M(\mathrm{e}^{\mathrm{j}kh}) r^k$ satisfies:

$$|v^k| = 1, \quad \|q_i^k\|_2^2 \geqslant \|p_i^k\|_2^2, \quad \|z^k\|_2^2 > \frac{h}{2\pi}\left(y_k - \frac{\varepsilon}{3}\right). \tag{A.10}$$

Consider the perturbation $\tilde{\Delta} = \mathrm{diag}\{\tilde{\Delta}_i\}$, where $\tilde{\Delta}_i$ is defined by

$$\tilde{\Delta}_i u \doteq \sum_k \frac{p_i^k \langle q_i^k, u_i \rangle}{\|q_i^k\|_2^2}. \tag{A.11}$$

Finally, define

$$\tilde{v}(\omega) \doteq v^k, \quad \omega \in [kh, (k+1)h), \quad \tilde{z}(\omega) \doteq z^k,$$

$$\omega \in [kh, (k+1)h),$$

$$\tilde{M}(\omega) \doteq M(\mathrm{e}^{\mathrm{j}kh}), \quad \omega \in [kh, (k+1)h). \tag{A.12}$$

By construction, $p^k = \tilde{\Delta} q^k$ and $\tilde{z} = (\tilde{M} \star \tilde{\Delta})\tilde{v}$. Moreover, it can be easily shown that $\|\lambda\tilde{\Delta} - \tilde{\Delta}\lambda\|_2 \leqslant 2\sin h/2 \doteq v$ and thus $\tilde{\Delta} \in \mathscr{B}\mathbf{\Delta}_v^{\mathrm{SLTV}}$.

From Corollary B.5 in Paganini (1995a) we have that there exists some $\beta < \infty$ such that

$$\|(I - M_{11}\Delta)^{-1}\|_{\ell^2 \to \ell^2} < \beta, \quad \forall \Delta \in \mathscr{B}\mathbf{\Delta}^{\mathrm{SLTV}}. \tag{A.13}$$

Since $M(\mathrm{e}^{\mathrm{j}\theta})$ is continuous in $[0, 2\pi]$ it follows that

$$\|(I - \tilde{M}_{11}\Delta)^{-1}\|_{\ell^2 \to \ell^2} < \tilde{\beta}, \quad \forall \Delta \in \mathscr{B}\mathbf{\Delta}^{\mathrm{SLTV}} \tag{A.14}$$

for some $\tilde{\beta} < \infty$. Eqs. (A.13) and (A.14), combined with the continuity of M imply that there exists h_2 such that for all $\Delta \in \mathscr{B}\mathbf{\Delta}^{\mathrm{SLTV}}$ and all $h \leqslant h_2$

$$\|(M \star \Delta) - (\tilde{M} \star \Delta)\|^2_{\ell^2 \to \ell^2} \leqslant \frac{\varepsilon}{3}. \tag{A.15}$$

Thus,

$$\|z\|_2^2 \doteq \|(M \star \tilde{\Delta})\tilde{v}\|_2^2 \geqslant \|(\tilde{M} \star \tilde{\Delta})\tilde{v}\|_2^2 - \frac{\varepsilon}{3} > 1 - \varepsilon. \tag{A.16}$$

Since ε is arbitrary, this last equation implies that robust $\mathscr{H}_2$ performance is violated. Note in passing that from the proof it follows that the worst case signal can be always taken to be an impulse. This observation extends to the case $\Delta \in \mathscr{B}\mathbf{\Delta}^{\mathrm{SLTV}}$ a fact proved in Paganini and Feron (1999, p. 145) for the case of $\Delta \in \mathscr{B}\mathbf{\Delta}^{\mathrm{LTV}}$.

A.3. Proof of Theorem 1

Sufficiency: From Definition 1 we have that

$$\sup_{\Delta} \|M \star \Delta\|^2_{\mathscr{H}_2} = \sup_{\Delta} \sum_{i=1}^{n_v} \left\{ \sup_{v_i \in \mathscr{B}S} \|(M^{(i)} \star \Delta)v_i\|_2^2 \right\}, \tag{A.17}$$

where

$$M^{(i)} = \begin{bmatrix} M_{11} & M_{12}^{(i)} \\ M_{21} & M_{22}^{(i)} \end{bmatrix}. \tag{A.18}$$

If (13) holds, then for any $\Delta \in \mathscr{B}\mathbf{\Delta}$ and any $v, v_i \in \mathscr{B}S$ we have that

$$\sum_{i=1}^{n_v} \|(M^{(i)} \star \Delta)v_i\|_2^2 \leqslant \frac{1}{2\pi} \sum_i \int_0^{2\pi} y_i(\omega) v_i^*(\omega) v_i(\omega)\,\mathrm{d}\omega \leqslant \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=1}^{n_v} y_i(\omega)\,\mathrm{d}\omega < 1 \tag{A.19}$$

from where robust $\mathscr{H}_2$ performance follows.

Necessity: Proceeding as in the proof of Lemma 3, define

$$\tilde{y}^{(j)}(\omega) = \inf\{y\colon \text{ the } j\text{th LMI in (13) has a solution}\}. \tag{A.20}$$

If the conditions in the Theorem fail, it follows that $\sum_{i=1}^{n_v} \int_0^{2\pi} \tilde{y}^{(i)} \mathrm{d}\omega / 2\pi \geqslant 1$. From Lemmas 3 and A.1 it follows that there exist signals $v^{(j)}, p^{(j)}, q^{(j)}, z^{(j)}$ such that $|v^{(j)}(\omega)| = 1$ and

$$\begin{pmatrix} q^{(j)} \\ z^{(j)} \end{pmatrix} = M^{(j)} \begin{pmatrix} p^{(j)} \\ v^{(j)} \end{pmatrix},$$

$$\int_{s_k}^{s_k+h} \|p_i^{(j)}\|^2\,\mathrm{d}\omega \leqslant \int_{s_k}^{s_k+h} \|q_i^{(j)}\|^2\,\mathrm{d}\omega$$

for some partition $\{s_k\}$ of $[0, 2\pi]$,

$$\|z^{(j)}\|_2^2 \geqslant \int_0^{2\pi} \tilde{y}^{(j)}(\omega) \frac{\omega}{2\pi}, \tag{A.21}$$

where the input p and output q have been partitioned according to the uncertainty structure. Moreover, without loss of generality, $v^{(j)}$ can be taken to be an impulse. Let P_N denote the truncation operator $P_N\{q_o, q_1, \ldots,\} = \{q_o, \ldots, q_{N-1}, 0, \ldots\}$ and let $p_M \doteq P_M p$, $q_N \doteq P_N q$. Since $p^{(j)}, q^{(j)} \in \ell^2$ it follows that there exist M, N large enough, $N > M$ such that:

$$\|q_N^{(j)}\|_2 \geqslant \|q^{(j)}\|_2 - \varepsilon,$$

$$\|p_M^{(j)}\|_2 \geqslant \|p^{(j)}\|_2 - \varepsilon,$$

$$\int_{s_k}^{s_k+h} \|p_M^{(j)}\|^2\,\mathrm{d}\omega \leqslant \int_{s_k}^{s_k+h} \|q_N^{(j)}\|^2\,\mathrm{d}\omega. \tag{A.22}$$

Consider now the perturbation $\tilde{\Delta} \doteq \sum_{i=1}^{n_v} \lambda^{iN} \Delta_i \lambda^{-iN}$, where λ denotes the unit delay operator and where

$$(\Delta_i u) = p_M^{(i)} \frac{\langle q_N^{(i)}, u \rangle}{\|q_N^{(i)}\|^2}. \tag{A.23}$$

Since by construction the signals$\{\lambda^{Ni} p_M^{(i)}, \lambda^{Nj} p_M^{(j)}\}$ and $\{\lambda^{Ni} q_N^{(i)}, \lambda^{Nj} q_N^{(j)}\}$ are orthogonal, it follows that $\tilde{\Delta} q_N^{(i)} = p_M^{(i)}$. From Bessel's inequality we have:

$$\|\tilde{\Delta} u\|^2 \leqslant \sum_{i=1}^{n_v} \left| \left\langle \frac{q_N^{(i)}}{\|q_N^{(i)}\|}, \lambda^{-Ni} u \right\rangle \right|^2 \leqslant \|u\|^2. \tag{A.24}$$

Thus $\|\tilde{\Delta}\|_{\ell^2 \to \ell^2} \leqslant 1$. By construction, $\Delta_i \in \mathscr{B}_v^{\mathrm{SLTV}}$. Hence

$$\|\lambda\tilde{\Delta} - \tilde{\Delta}\lambda\| \leqslant \sum_{i=1}^{n_v} \|\lambda\Delta_i - \Delta_i\lambda\| \leqslant n_v v \doteq \tilde{v}. \tag{A.25}$$

Hence $\tilde{\Delta} \in \mathscr{B}\mathbf{\Delta}_{\tilde{v}}^{\mathrm{SLTV}}$. Since $\|(I - M_{11}\Delta)^{-1}\|_{\ell^2 \to \ell^2}$ is uniformly bounded over $\mathscr{B}\mathbf{\Delta}^{\mathrm{SLTV}}$, from (A.22) it follows that $\|M^i \star \tilde{\Delta}\lambda^{Ni} v^{(i)}\|_2^2 = \|z^{(i)}\|_2^2 + \mathscr{O}(\varepsilon)$. Thus

$$\sum_{i=1}^{n_v} \|(M^{(i)} \star \tilde{\Delta})\lambda^{Ni} v^{(i)}\|_2^2 \geqslant \sum_{i=1}^{n_v} \|z^{(j)}\|_2^2 - \mathscr{O}(\varepsilon) \geqslant \int_0^{2\pi} \tilde{y}^{(j)}(\omega) \frac{\omega}{2\pi} - \mathscr{O}(\varepsilon) \geqslant 1 - \mathscr{O}(\varepsilon). \tag{A.26}$$

Since ε is arbitrary, this implies that robust $\mathscr{H}_2$ performance is violated.

References

Anderson, B. D. O., & Moore, J. B. (1990). *Optimal control: linear quadratic methods*. Englewood Cliffs, NJ: Prentice-Hall.

Bernstein, D., & Haddad, W. (1989). LQG control with an $\mathscr{H}_\infty$ performance bound: A Riccati equation approach. *IEEE Transactions on Automatic Control*, *34*(2), 293–305.

Chen, X., & Wen, J. T. (1995). A linear matrix inequality approach to the general mixed $\mathscr{H}_2/\mathscr{H}_\infty$ control problem. In *Proceedings of the 1995 american control conference* (pp. 1443–1447).

Doyle, J. (1978). Guaranteed margins for LQG regulators. *IEEE Transactions on Automatic Control*, *23*(4), 756–757.

Feron, E. (1997). Analysis of robust $\mathscr{H}_2$ performance using multiplier theory. *SIAM Journal on Control and Optimization*, *35*(1), 160–177.

Kaminer, I., Khargonekar, P., & Rotea, M. (1993). Mixed $\mathscr{H}_\infty/\mathscr{H}_2$ control for discrete time systems via convex optimization. *Automatica*, *29*(1), 57–70.

Meinsma, G., Shrivastava, Y., & Fu, M. (1997). A dual formulation of mixed μ and on the losslessness of (D, G) scaling. *IEEE Transactions on Automatic Control*, *42*(7), 1032–1036.

Packard, A., & Doyle, J. (1993). The complex structured singular value. *Automatica*, *29*(1), 71–109.

Paganini, F. (1995a). *Sets and constraints in the analysis of uncertain systems*. Ph.D. Dissertation, Caltech, Pasadena, CA.

Paganini, F. (1995b). Necessary and sufficient conditions for robust $\mathscr{H}_2$ performance. *Proceedings of the 1995 CDC*, New Orleans, LA (pp. 1970–1975).

Paganini, F., & Feron, E. (1999). LMI methods for robust $\mathscr{H}_2$ analysis: A survey with comparisons. In L. El Ghaoui & S. Niculescu (Ed.), *Recent Advances on LMI methods in Control* (pp. 108–133). Philadelphia, PA: SIAM Press (Chapter 7).

Scherer, C. W. (1995). Multiobjective $\mathscr{H}_2/\mathscr{H}_\infty$ control. *IEEE Transactions on Automatic Control*, *40*(6), 1054–1062.

Stein, G., & Athans, M. (1987). The LQG/LTR procedure for multivariable feedback control design. *IEEE Transactions on Automatic Control*, *32*(2), 102–114.

Stoorvogel, A. A. (1993). The robust $\mathscr{H}_2$ problem: A worst-case design. *IEEE Transactions on Automatic Control*, *38*(9), 1358–1370.

Sznaier, M. (1994). An exact solution to general SISO mixed $\mathscr{H}_2/\mathscr{H}_\infty$ problems via convex optimization. *IEEE Transactions on Automatic Control*, *39*(12), 2511–2517.

Sznaier, M., & Parrilo, P. (1999). On the gap between μ and its upper bound for some classes of problems. *Proceedings of the 38th CDC*, December 1999 (pp. 4511–4516).

Sznaier, M., & Tierno, J. (2000). Is set modeling of white noise a good tool for robust $\mathscr{H}_2$ analysis?. *Automatica*, *36*(2), 261–267.

Zhang, Z., & Freudenberg, J. S. (1990). LTR for nonminimum phase plants. *IEEE Transactions on Automatic Control*, *35*(5), 547–553.

Zhou, K., Glover, K., Bodenheimer, B., & Doyle, J. (1994). Mixed $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance objectives I: Robust performance analysis. *IEEE Transactions on Automatic Control*, *39*(8), 1564–1574.

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