Mixed $l_1/\mathcal{H}_\infty$ Control of MIMO Systems via Convex Optimization

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***Abstract*— Mixed performance control problems have been the object of much attention lately. These problems allow for capturing different performance specifications without resorting to approximations or the use of weighting functions, thus eliminating the need for trial-and-error-type iterations. In this paper we present a methodology for designing mixed $l_1/\mathcal{H}_\infty$ controllers for MIMO systems. These controllers allow for minimizing the worst case peak output due to persistent disturbances, while at the same time satisfying an $\mathcal{H}_\infty$-norm constraint upon a given closed-loop transfer function. Therefore, they are of particular interest for applications dealing with multiple performance specifications given in terms of the worst case peak values, both in the time and frequency domains. The main results of the paper show that 1) contrary to the $\mathcal{H}_2/\mathcal{H}_\infty$ case, the $l_1/\mathcal{H}_\infty$ problem admits a solution in l_1, and 2) rational suboptimal controllers can be obtained by solving a sequence of problems, each one consisting of a finite-dimensional convex optimization and a four-block $\mathcal{H}_\infty$ problem. Moreover, this sequence of controllers converges in the l_1 topology to an optimum.**

***Index Terms*—$l_1, /\mathcal{H}_\infty$, multiobjective control.**

I. Introduction

DURING THE last decade a powerful robust control framework has been developed addressing issues of stability and performance in the presence of norm-bounded model uncertainties. Robust stability and performance are achieved by minimizing a suitably weighted norm (either $\|\cdot\|_\infty$ [12], [15], [34] or $\|\cdot\|_1$ [9], [11], [17], [32]) of a closed-loop transfer function. This framework has gained wide acceptance among control engineers since it embodies many desirable design objectives. Furthermore, the $\mathcal{H}_\infty$ framework, in conjunction with μ-analysis [21], has been successfully applied to a number of hard practical control problems (see for instance [26]).

However, despite its significance, this framework is limited by the fact that in its context, performance must be measured in the same norm used to assess stability. Clearly, a single norm is usually not enough to capture different, and often conflicting, design specifications, such as simultaneous rejection of disturbances having different characteristics (white noise, bounded energy, persistent); good tracking of classes of inputs; satisfaction of bounds on peak values of some outputs; closed-loop bandwidth, etc. Thus, designers are forced to use weighting functions and similarity scaling of appropriate closed-loop transfer functions, in an attempt to cast the specifications into a single norm form, amenable to tools currently available ($\mathcal{H}_2$, $\mathcal{H}_\infty$, l_1). Although there exist some guidelines relating time specifications to the selection of weighting functions [19], this process remains essentially an art. Hence, we can expect at best a complex design procedure requiring considerable expertise and numerous trial-and-error-type iterations.

Multiple performance control problems have been the object of much attention lately (see [30] for references on recent work on multiobjective control). In particular, $\mathcal{H}_2/\mathcal{H}_\infty$ mixed control has been extensively investigated since its introduction (see for instance [1], [16], [18], [20] and references therein). More recently $l_1/\mathcal{H}_\infty$ [6], [27] and $l_1/\mathcal{H}_2$ control problems have been formulated [33]. In this paper we concentrate on discrete-time mixed $l_1/\mathcal{H}_\infty$ controllers. These controllers allow for minimizing the worst case peak output due to persistent disturbances, while at the same time satisfying an $\mathcal{H}_\infty$-norm constraint upon a given closed-loop transfer function. Therefore, they are of particular interest for applications dealing with specifications upon the peak admissible values both in the time and frequency domains.

It is well known that for stable systems the l_1 norm is an upper bound of the $\mathcal{H}_\infty$ norm. Thus, in principle, this problem can be recast into a single-norm form, involving only the l_1 norm, and can be solved using the techniques proposed in [11]. However, it has recently been shown through examples in [31] that this approach can introduce a great deal of conservatism. Moreover, in some extreme cases, minimizing the l_1 norm can cause the $\mathcal{H}_\infty$ norm to increase rather than to decrease [31]. Thus, mixed $l_1/\mathcal{H}_\infty$ problems are true multiobjective problems that cannot be recast into a single norm form.

An alternative approach is to use the Youla parameterization to cast the problem into a (infinite-dimensional) convex optimization form [5], [14], [23]. However, in order to obtain a tractable problem, several approximations, such as replacing the infinite-dimensional $\mathcal{H}_\infty$ constraint with a finite number of constraints obtained by sampling the unit circle, are required. This may prevent finding a solution if the performance specifications are tight. Moreover, it has been recently shown that, for a class of problems, the approximations obtained by sampling the unit circle will fail to converge to a solution, even when the number of sampling points tends to ∞. This difficulty can be avoided by using a linear matrix inequality (LMI) characterization of the $\mathcal{H}_\infty$ constraint [6]. This approach leads to tractable problems that can be efficiently solved using LMI tools. However, it requires imposing that the closed-loop

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system must be an finite impulse response (FIR) in order to maintain convexity. Moreover, at the present time there is no known method to prespecify the order of the approximation in order to meet given approximation error bounds.

The approach that we pursue in this paper evolves from the solution to the mixed $l_1/\mathcal{H}_\infty$ problem for single-input/single-output (SISO) systems presented in [27]. The generalization to a multi-input/multi-output (MIMO) four-block problem is not straightforward but can be achieved by using some of the ideas in [27] and the formulas in [24]. As in [27], it will be shown that a *suboptimal* solution to the mixed $l_1/\mathcal{H}_\infty$ problem, i.e., a solution satisfying the $\mathcal{H}_\infty$ constraint and with performance arbitrarily close to the optimal, can be obtained by solving a finite-dimensional convex optimization problem followed by an unconstrained $\mathcal{H}_\infty$ minimization. Furthermore, stronger results will include the existence of an optimal solution in l_1 (contrary to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ case where it has been shown that the optimal closed-loop system does not belong to $\mathcal{A}_o$ [20]) and the convergence of the sequences of controllers and closed-loop systems in the l_1 topology.

The paper is organized as follows: In Section II we introduce the notation to be used and some preliminary results on functional and complex analysis. In Section III we show that the mixed $l_1/\mathcal{H}_\infty$-control problem admits a minimizing solution in l_1 and that the optimal performance level can be approximated arbitrarily close with rational controllers. In Section IV we furnish a method for computing an optimizing sequence of rational controllers such that the resulting closed-loop systems approximate the optimum in the l_1 topology. In Section V we indicate how to compute ϵ suboptimal solutions by solving a finite-dimensional convex optimization problem and a standard $\mathcal{H}_\infty$ problem. In Section VI we briefly indicate how to extend these results to the continuous-time case. Section VII illustrates our results with some simple design examples. Finally, in Section VIII, we summarize our results and we present some concluding remarks.

II. Preliminaries

A. Notation

Given a matrix A, we denote by A_i its ith row. For $x \in R^n$ we define $|x|$ as the vector with components $|x_i|$. We denote the one-norm as $\|x\|_1 \doteq \sum_{i=0}^n |x_i|$ and the infinity norm as $\|x\|_\infty \doteq \max_i |x_i|$. l_1 denotes the space of absolutely summable sequences $h = \{h_i\}$ equipped with the norm $\|h\|_{l_1} \doteq \sum_{i=0}^\infty |h_i| < \infty$. l_∞ denotes the space of bounded sequences $h = \{h_i\}$ equipped with the norm $\|h\|_{l_\infty} \doteq \sup_{i\geq 0} |h_i| < \infty$. We denote by l_∞^p the space of bounded vector sequences $\{h(k) \in R^p\}$. In this space we define the norm $\|h\|_{l_\infty} \doteq \sup_i \|h_i(k)\|_\infty$. Finally, by c_o we denote the subspace of l_∞ formed by sequences $h = \{h_i\}$ such that $h_i \to 0$. Given a sequence $h \in l_1$, its λ-transform is defined as $H(\lambda) = \sum_{i=0}^\infty h_i \lambda^i$. In the sequel we will denote by $\mathcal{A}$ the space of λ transforms of elements in l_1, and by a slight abuse of notation we will sometimes use the notation $\|H\|_1$ to denote $\|h\|_{l_1}$.

Given a bounded linear operator $H: l_\infty^q \to l_\infty^p$ defined by the usual convolution relation $y = H * u$, its induced $l_\infty^q \to l_\infty^p$ norm is given by

$$\|H\|_1 \doteq \max_i \sum_{j=1}^q \|H(i,j)\|_1.$$

Given a sequence $x = \{x^0, x^1, x^2, \cdots\}$ the truncation operator $P_n (n \geq 1)$ is defined as

$$P_n(x) = \{x^0, x^1, x^2, \cdots, x^{n-1}, 0, 0, \cdots\}.$$

$\mathcal{L}_\infty$ denotes the Lebesgue space of complex valued matrix functions which are essentially bounded on the unit circle, equipped with the norm

$$\|G(\lambda)\|_\infty \doteq \operatorname{ess}\sup_{|\lambda|=1} \bar{\sigma}(G(\lambda))$$

where $\bar{\sigma}$ denotes the largest singular value. By $\mathcal{H}_\infty(\mathcal{H}_\infty^\sim)$ we denote the subspace of functions in $\mathcal{L}_\infty$ with a bounded analytic continuation inside (outside) the unit disk. $\mathcal{A}_o$ denotes the subset of $\mathcal{H}_\infty$ functions continuous in the *closed* unit disk. The norm on $\mathcal{H}_\infty$ is defined by $\|G(\lambda)\|_\infty \doteq \operatorname{ess}\sup_{|\lambda|<1} \bar{\sigma}(G(\lambda))$. Also of interest is the space $\mathcal{H}_{\infty,\delta}$ of transfer matrices in $\mathcal{H}_\infty$ which have analytic continuation inside the disk of radius $\delta > 1$ (usually $\delta \approx 1$). When equipped with the norm $\|G(\lambda)\|_{\infty,\delta} \doteq \operatorname{ess}\sup_{|\lambda|<\delta} \bar{\sigma}(G(\lambda))$, $\mathcal{H}_{\infty,\delta}$ becomes a normed Banach space.

Given $G(\lambda) \in \mathcal{H}_\infty$ one can write the formal series $G(\lambda) = \sum_{i=0}^\infty G_i \lambda^i$. The series converges pointwise for each $|\lambda| < 1$ and uniformly inside any disk with radius smaller than one. It is a standard result that $G \in l_1$ if and only if $\sum_{i=0}^\infty |G_i| < \infty$. In this case, the series converges uniformly also on the unit disk.

For a transfer matrix $G(\lambda)$, $G^\sim \doteq G^T(1/\lambda)$. In the sequel, both the dependence on the complex variable λ and the dimensions of the transfer matrices will be omitted unless necessary for clarity.

Finally, throughout the paper we will use the prefix $\mathcal{R}$ to denote real rational transfer matrices, and packed notation to represent their state-space realizations, i.e.,

$$G(\lambda) = \lambda C(I - \lambda A)^{-1} B + D \doteq \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right).$$

B. Background Materials on Functional and Complex Analysis

In this section we present the mathematical background required for establishing the existence of the solution to the mixed $l_1/\mathcal{H}_\infty$ problem and to assess its properties. This material is standard either in functional analysis (such as [13]) or complex analysis textbooks (such as [22]) and is included here for ease of reference.

1) Preliminaries on Functional Analysis: Let X be a normed linear space. The space of all bounded linear functionals on X is denoted by X^*. Given $x \in X$ and $r \in X^*$, $\langle x, r \rangle$ denotes the value of the linear functional r at x. The induced norm on X^* is defined as

$$\|r\| = \sup_{x \in BX} |\langle x, r \rangle|$$

where $BX \doteq \{x \in X: \|x\| \leq 1\}$.

Definition 1: Let X be a Banach space with dual space X^*. A sequence $\{x_n\} \in X$ converges weakly to $x \in X$ if for every $x^* \in X^*$ we have that $\langle x_n, x^*\rangle \to \langle x, x^*\rangle$.

Definition 2: Let X be a Banach space with dual space X^*. A sequence $\{x_n^*\} \in X^*$ converges weak* to $x^* \in X^*$ if for every $x \in X$ we have that $\langle x, x_n^*\rangle \to \langle x, x^*\rangle$.

Definition 3: A set $K \in X^*$ is said to be weak* compact if every infinite sequence in K contains a weak* convergent subsequence.

It is a standard result that strong convergence (or convergence in the norm) implies weak convergence, which itself implies weak* convergence. However, the converse are not true. The importance of weak* convergence is highlighted by the following theorem.

Theorem 1 (Alaoglu–Banach): Let X be a Banach space and denote its dual space by X^*, then the closed balls in X^*

$$\mathcal{B}_{\mathcal{M}}X^* \doteq \{x^* \in X^* : \|x^*\| \leq M\}$$

are weak* compact for any positive M.

This theorem will be used to establish the existence of a solution to the mixed $l_1/\mathcal{H}_\infty$ problem by constructing an optimizing sequence of controllers such that all the closed-loop systems are inside a given ball in $l_1 = (c_o)^*$ and exploiting weak* compactness. However, this result itself is not enough to show that the sequence of suboptimal closed-loop systems converges in the l_1 topology to the optimal. The latter will be established by exploiting the following result.

Theorem 2 [13, p. 296]: Weak and strong convergence of sequences in l_1 are equivalent.

Corollary 1 [8, p. 219]: If $\phi^n \in l_1$ converges weak* to $\phi \in l_1$ and $\|\phi^n\|_1 \to \|\phi\|_1$, then ϕ^n converges strongly to ϕ, i.e., $\|\phi^n - \phi\|_1 \to 0$.

Finally, we recall a theorem about the invertibility of l_1 functions. This theorem will be used to establish that the optimal controller is indeed in l_1.

Theorem 3 (Wiener–Gelfand [7, p. 483]): Let $\mathcal{A}$ denote a commutative Banach algebra with a unit. An element $x_o \in \mathcal{A}$ is invertible in $\mathcal{A}$ if and only if $f(x_o) \neq 0$ for all $f \in S_p(\mathcal{A})$, where

$$S_p(\mathcal{A}) \doteq \{f \in \mathcal{A}^* : \|f\| = 1\}.$$

Corollary 2: Let $l_1(Z)$ denote the Banach space of sequences $\{a_n\}$: $\sum_{n=-\infty}^{\infty} |a_n| < \infty$. Consider $x \in l_1(Z)$ and its bilateral λ-transform $X(\lambda) = \sum_{n=-\infty}^{\infty} x_n\lambda^n$. Then $X(\lambda)^{-1} \in l_1(Z)$ if and only if $X(\lambda) \neq 0$ for all $|\lambda| = 1$.

Corollary 3: Let $l_1^{p\times q}(Z)$ denote the Banach space of sequences of matrices $\{S_n \in R^{p\times q}\}$: $\max_{1\leq i\leq p} \sum_{j=1}^{q} (\sum_{n=-\infty}^{\infty} |S_n(i,j)|) < \infty$. Consider $S \in l_1^{p\times q}(Z)$. If its bilateral λ-transform $S(\lambda)$ has full column rank on $|\lambda| = 1$, then its left inverse $S_L^\dagger \in l_1^{q\times p}(Z)$. Similarly, if $S(\lambda)$ has full row rank in $|\lambda| = 1$, then $S_R^\dagger \in l_1^{q\times p}(Z)$.

Proof: Since $S(\lambda)$ has full column rank on $|\lambda| = 1$, $\det(S^TS)(\lambda) \neq 0$ for all $|\lambda| = 1$. From Corollary 1 we have that $(\det(S^TS)(\lambda))^{-1} \in l_1(Z)$. Thus $(S^TS)^{-1} \in l_1^{q\times q}(Z)$ and $S_L^\dagger(\lambda) = (S^TS)^{-1}S^T \in l_1^{q\times p}(Z)$. The proof for the right inverse follows along the same lines. □

2) Preliminaries on Complex Analysis: Let $\{f_n\}$ denote a sequence of complex-valued functions defined in a subset A of the complex plane. The sequence $\{f_n\}$ *converges pointwise in* A to the limit function f if for each $z \in A$, $f_n(z) \to f(z)$ as a sequence of complex numbers. A sequence $\{f_n\}$ *converges uniformly on* A to f ($f_n \rightrightarrows f$) if for each $\epsilon > 0$ there exists $N(\epsilon)$ such that $|f_n(z) - f(z)| < \epsilon$ for each $n > N(\epsilon)$ for *all* $z \in A$ (i.e., $N(\epsilon)$ does not depend on z). Uniform convergence is a strong property, and it is preferable to deal with a milder convergence criteria. Suppose that each function f_n is defined in an open subset U. The sequence $\{f_n\}$ *converges normally* in U to f if $\{f_n\}$ is pointwise convergent to f in U and this convergence is uniform on each compact subset of U. The relevance of normal convergence is highlighted by the following theorems.

Theorem 4: Suppose that each function in a sequence $\{f_n\}$ is analytic in an open set U and that the sequence converges normally in U to the limit function f. Then f is analytic in U. Moreover, $f_n^{(k)} \to f^{(k)}$[1] normally in U for each positive integer k.

A family $\mathcal{F}$ of functions analytic in U is said to be *normal* if each sequence $\{f_n\}$ from $\mathcal{F}$ contains at least one normally convergent subsequence. Given a sequence of functions $\{f_n\}$, each of whose terms is analytic in an open set U, it is of interest to know whether $\{f_n\}$ is normal, i.e., if it is possible to extract a normally convergent subsequence. An answer to this question is given by Montel's theorem, which requires a certain equi-boundedness assumption. A family $\mathcal{F}$ is said to be *locally bounded in* U if its members are uniformly bounded on each compact set in U.

Theorem 5 (Montel's Theorem): Let $\mathcal{F}$ be a family of functions that are analytic in an open set U. Suppose that $\mathcal{F}$ is locally bounded in U. Then $\mathcal{F}$ is a normal family in this set.

In particular, if $\mathcal{F} \subset \mathcal{H}_\infty$ is such that $f \in \mathcal{F} \Rightarrow \|f\|_\infty \leq 1$, then the theorem implies that $\mathcal{F}$ is normal inside the unit disk. Thus, every sequence $\{f_i\} \in \mathcal{F}$ contains a normally convergence subsequence. This is the key fact that will be exploited in the sequel to establish convergence of the proposed synthesis method.

III. The $l_1/\mathcal{H}_\infty$ Optimal Control Problem

Consider the system shown in Fig. 1, where S represents the plant to be controlled. The signals $w_\infty \in l_2^{n_\infty}$ (a bounded energy signal), $w_1 \in l_\infty^{n_1}$ (a persistent l_∞ signal), and $u \in l_\infty^{n_u}$ represent exogenous disturbances and the control action respectively; and $\zeta_\infty \in l_2^{m_\infty}, \zeta_1 \in l_\infty^{m_1}$, and $y \in l_\infty^{m_y}$ represent the regulated outputs and the measurements, respectively. Then, the mixed $l_1/\mathcal{H}_\infty$ multiobjective control problem consists of finding an internally stabilizing controller K such that worst case peak amplitude of the performance output ζ_1 due to signals w_1 inside the l_∞-unity ball is minimized, subject to the constraint $\|T_{\zeta_\infty w_\infty}\|_\infty \leq \gamma$.

Assume that the system S has the following state-space realization (without loss of generality we assume that all

[1]Here $f^{(k)}$ denotes the kth derivative of f.

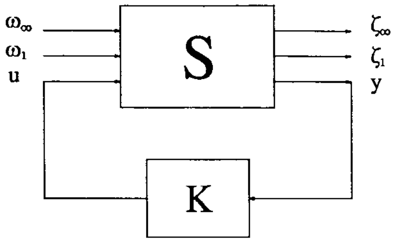

Fig. 1. The plant.

weighting factors have been absorbed into the plant):

$$\left(\begin{array}{c|ccc} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ C_3 & D_{31} & D_{32} & D_{33} \end{array}\right). \tag{S}$$

It is well known (see for instance [12]) that the set of all internally stabilizing controllers can be parameterized in terms of a free parameter $Q \in \mathcal{H}_\infty$ as

$$K = \mathcal{F}_l(J, Q) \tag{1}$$

where J has the following state-space realization:

$$\left(\begin{array}{c|cc} A + B_3F + LC_3 + LD_{33}F & -L & B_3 + LD_{33} \\ \hline F & 0 & I \\ -(C_3 + D_{33}F) & I & -D_{33} \end{array}\right) \tag{J}$$

and where F and L are selected such that $A + B_3F$ and $A + LC_3$ are stable. By using this parameterization, the closed-loop transfer matrices $T_{\zeta_\infty w_\infty}$ and $T_{\zeta_1 w_1}$ can be written as

$$\begin{aligned} \Psi &\doteq T_{\zeta_\infty w_\infty}(\lambda) = T_{11}(\lambda) + T_{12}(\lambda)Q(\lambda)T_{21}(\lambda) \\ \Phi &\doteq T_{\zeta_1 w_1}(\lambda) = S_{11}(\lambda) + S_{12}(\lambda)Q(\lambda)S_{21}(\lambda) \end{aligned} \tag{2}$$

where T_{ij}, S_{ij} are stable rational transfer matrices. In the sequel we will make the following assumptions.

A1) The pairs (A, B_3) and (C_3, A) are stabilizable and detectable, respectively.

A2) D_{13} and D_{31} have full column and row rank, respectively.

A3) $\begin{bmatrix} A - e^{j\theta}I & B_3 \\ C_1 & D_{13} \end{bmatrix}$ and $\begin{bmatrix} A - e^{j\theta}I & B_1 \\ C_3 & D_{31} \end{bmatrix}$ have full column and row rank, respectively, for all $0 \le \theta < 2\pi$.

A4) $\inf_{Q \in \mathcal{RH}_\infty} \|T_{11} + T_{12}QT_{21}\|_\infty \doteq \gamma^* < \gamma$.

Assumptions A1) through A3) are standard in $\mathcal{H}_\infty$ theory [35]: A1) is necessary for the existence of stabilizing controllers; A2) guarantees that the $\mathcal{H}_\infty$ portion of the problem is nonsingular; and A3) guarantees that T_{12} and T_{21} do not have zeros on the unit circle $|\lambda| = 1$ (thus the optimal $\mathcal{H}_\infty$ performance level is achievable). Assumption A4) allows for simplifying the exposition. It guarantees both the existence of suboptimal $\mathcal{H}_\infty$ controllers and nontrivial solutions to the mixed $l_1/\mathcal{H}_\infty$ problem. Moreover, from now on, we will also assume that $\gamma = 1$. This does not entail any loss of generality, since the matrices B_1 and D_{11} can be always scaled down so this assumption holds.

Transformation (1) allows for precisely stating the mixed $l_1/\mathcal{H}_\infty$ problem as follows.

Problem 1 (Mixed $l_1/\mathcal{H}_\infty$ Control Problem): Find the optimal value of the performance measure

$$\mu \doteq \inf_{Q \in l_1} \{\|S_{11} + S_{12}QS_{21}\|_1 : \|T_{11} + T_{12}QT_{21}\|_\infty \le 1\} \tag{3}$$

and a controller Q such that $\|\Phi(Q)\|_1 = \mu$ and $\|\Psi(Q)\|_\infty \le 1$, or establish that none exists.

Problem 1 is a convex infinite-dimensional optimization problem for which no closed-form solution is known to exist. Moreover, experience with similar problems has shown that they may lead to closed-loop systems exhibiting some undesirable properties. Specifically, while mixed $l_1/\mathcal{H}_2$ control problems lead to finite impulse response [33] (and thus exponentially stable) closed-loop systems, it has been recently shown that the optimal mixed $\mathcal{H}_2/\mathcal{H}_\infty$ system does not belong to $\mathcal{A}_o$. Thus, in this later case the resulting closed-loop system is not exponentially stable (albeit it is still bounded-input/bounded-output stable). Moreover, the optimal controller has a nonrational transfer function and cannot be uniformly approximated by a rational (and thus physically implementable) controller. Thus, from an engineering standpoint it is relevant to find out whether or not Problem 1 admits a minimizing solution in l_1. The answer to this question, not trivial since the set $\Xi(Q) \doteq \{Q \in l_1 : \|T_{11} + T_{12}QT_{21}\|_\infty \le 1\}$ is not compact in the $\mathcal{H}_\infty$ topology, is given by the following theorem.

Theorem 6: Assume that $S_{12}(\lambda)$ and $S_{21}(\lambda)$ have full column and row rank on $|\lambda| = 1$, respectively. Then Problem 1 admits a minimizing solution $Q \in l_1$.

Proof: The proof is deferred until Section IV where we give a constructive proof, based upon the construction of a sequence of minimizers that converges to an optimum in the l_1 topology.

Next we restrict our attention to the more meaningful (from an engineering standpoint) space $\mathcal{R}l_1$, and we show that the infimum of the performance index achievable in this space coincides with the minimum achievable over l_1. Notice that $\mathcal{R}l_1 = \mathcal{RH}_\infty$. Hence the infimum over the space $\mathcal{RH}_\infty$ will also be equal to μ.

Lemma 1: Define μ_R as

$$\mu_R \doteq \inf_{Q \in \mathcal{R}l_1} \{\|S_{11} + S_{12}QS_{21}\|_1 : \|T_{11} + T_{12}QT_{21}\|_\infty \le 1\}. \tag{4}$$

Then $\mu_R = \mu$.

Proof: From the definition of μ it follows that given $\epsilon_1 > 0$ there exists $Q_1 \in l_1$ such that

$$\|T_{11} + T_{12}Q_1T_{21}\|_\infty \le 1 \tag{5}$$

and

$$\|S_{11} + S_{12}Q_1S_{21}\|_1 < \mu + \frac{\epsilon_1}{4}. \tag{6}$$

From Assumption A4) there exists $Q_2 \in \mathcal{RH}_\infty$ such that $\|T_{11} + T_{12}Q_2T_{21}\|_\infty < 1$. Let $Q^* \doteq (1 - \eta)Q_1 + \eta Q_2$. From convexity, it follows that $Q^* \in l_1$ satisfies

$$\begin{aligned} \|S_{11} + S_{12}Q^*S_{21}\|_1 &\le \mu + \frac{\epsilon_1}{4} + \eta\|S_{12}(Q_2 - Q_1)S_{21}\|_1 \\ &\le \mu + \frac{\epsilon_1}{2} \\ \|T_{11} + T_{12}Q^*T_{21}\|_\infty &\le 1 - \epsilon_2 \end{aligned} \tag{7}$$

for some $\eta, \epsilon_2 > 0$, small enough. Since $Q^* \in l_1$, for any $\epsilon > 0$, we can find (for instance by truncating the expansion of $Q^*(\lambda)$) $\hat{Q} \in \mathcal{R}l_1$ such that $\|Q^* - \hat{Q}\|_1 \leq \epsilon$. Thus by selecting ϵ such that $\|S_{12}(Q^* - \hat{Q})S_{21}\|_1 \leq \frac{\epsilon_1}{2}$ and $\|T_{12}(Q^* - \hat{Q})T_{21}\|_1 \leq \epsilon_2$, we have

$$\begin{aligned} \|S_{11} + S_{12}\hat{Q}S_{21}\|_1 &\leq \mu + \epsilon_1 \\ \|T_{11} + T_{12}\hat{Q}T_{21}\|_\infty &\leq 1. \end{aligned} \tag{8}$$

Hence $\hat{Q}$ is a feasible solution for (4). It follows that $\mu_R \leq \mu + \epsilon_1$. Since ϵ_1 is arbitrary, the lemma follows. □

IV. Problem Solution

In principle, one can attempt to solve the infinite-dimensional optimization Problem 1 following an approach similar to the one in [14]. This entails a double approximation, since the free parameter Q is approximated by a finite impulse response while the constraint is approximated by computing its value at a finite number of frequency points. Thus, there is no guarantee that the solution obtained is feasible, nor that the actual cost will be bounded above by the objective function. Moreover, it has been recently shown in [31] that for a class of problems the approximations obtained by sampling the unit circle will fail to converge to the solution, even when the number of sampling points tends to ∞. This difficulty can be avoided by using an LMI characterization of the $\mathcal{H}_\infty$ constraint [6]. This approach leads to tractable problems that can be efficiently solved using LMI tools. However, it requires imposing that the closed-loop system must be an FIR in order to maintain convexity. Moreover, at the present time there is no known method to prespecify the order of the approximation in order to meet given approximation error bounds.

In this paper we will pursue a different route, motivated by the earlier results obtained for the simpler SISO case. As in there, we will show that the optimal performance can be found by solving a sequence of modified problems. Additionally, we will show that the sequence of solutions to these problems converges to an optimum, thus proving the existence of a solution to Problem 1. To establish these results we will proceed as follows: 1) introduce a modified $l_1/\mathcal{H}_\infty$ problem; 2) show that the optimal cost μ can be found by solving a sequence of modified problems (Lemma 2); and 3) show that the corresponding sequence of controllers converges to an optimum in the l_1 topology (Theorem 7). To this effect, consider the following modified $l_1/\mathcal{H}_\infty$ problem.

Problem 2 (Problem $l_1/\mathcal{H}_{\infty,\delta}$): Given $\delta > 1$ and $S_{ij}(\lambda)$, $T_{ij}(\lambda) \in \mathcal{RH}_{\infty,\delta}$ such that T_{12}, T_{21} have full column and row rank on the circle $|\lambda| = \delta^2$, find

$$\mu_\delta = \inf_{Q \in \mathcal{RH}_{\infty,\delta}} \|S_{11} + S_{12}QS_{21}\|_1$$

subject to

$$\|T_{11} + T_{12}QT_{21}\|_{\infty,\delta} \leq 1$$

and a controller $Q^* \in \overline{\mathcal{RH}_{\infty,\delta}}$ such that $\|S_{11} + S_{12}Q^*S_{21}\|_1 = \mu_\delta$ and $\|T_{11} + T_{12}Q^*T_{21}\|_{\infty,\delta} \leq 1$.

Remark 1: Under Assumption A3) it can be easily shown, either by a slight extension of [25, Corollary 2] or by constructing an ϵ-net [13], that the set

$$\Xi(Q) \doteq \{Q \in \overline{\mathcal{RH}_{\infty,\delta}} : \|T_{11} + T_{12}QT_{21}\|_{\infty,\delta} \leq 1\} \tag{9}$$

is compact in the $\mathcal{H}_\infty$ topology. Thus Q^* is well defined.

Remark 2: From the Maximum Modulus theorem it follows that any solution Q to Problem 2 is an admissible solution for Problem 1. It follows that μ_δ is an upper bound for μ. In the sequel we will show that $\mu_\delta \downarrow \mu$ and that, under some additional constraints, the sequence of controllers Q_δ converges in the l_1 topology.

Next we show that the sequence of controllers generated in this way converges to an optimal controller. We begin by showing that the sequence of l_1 norms converges to the optimum.

Lemma 2: Consider a decreasing sequence $\delta_i \downarrow 1$. Let μ and μ_{δ_i} denote the solution to Problems 1 and 2, respectively. Then the sequence $\mu_{\delta_i} \downarrow \mu$.

Proof: From the Maximum Modulus theorem it follows that for any $\delta > 1$, any Q_δ feasible for Problem 2 is also feasible for Problem 1. Thus, it follows that $\mu_\delta \geq \mu$. Let $\delta_1 > \delta_2$ and consider the controller Q_{δ_1} that solves Problem 2 for $\delta = \delta_1$. Since Q_{δ_1} is feasible for Problem 2 with $\delta = \delta_2$, it follows that $\mu_{\delta_1} \geq \mu_{\delta_2} \geq \mu$. Therefore $\mu^{\lim} = \lim_{\delta \downarrow 1} \mu_\delta$ exists and $\mu^{\lim} \geq \mu$. Let $\epsilon > 0$ be given; by definition of μ_R and proceeding as in Lemma 1, it is possible to construct $Q^* \in \mathcal{RH}_\infty$ such that

$$\begin{aligned} \|T_{11} + T_{12}Q^*T_{21}\|_\infty &< 1 \\ \|S_{11} + S_{12}Q^*S_{21}\|_1 &\leq \mu_R + \epsilon. \end{aligned} \tag{10}$$

By continuity (recall that all transfer functions involved are now in $\mathcal{RH}_\infty$), it is possible to find $\delta > 1$ such that $Q^* \in \mathcal{RH}_{\infty,\delta}$, $\|T_{11} + T_{12}Q^*T_{21}\|_{\infty,\delta} \leq 1$ and $\|S_{11} + S_{12} Q^*S_{21}\|_1 \leq \mu_R + \epsilon$. It then follows that $\mu^{\lim} \leq \mu_\delta \leq \mu_R + \epsilon$. Since ϵ is arbitrary, $\mu^{\lim} = \mu_R$. From Lemma 1 it follows that $\mu^{\lim} = \mu$. □

While this lemma shows that the sequence of l_1-norms converges to the optimum, it neither establishes that the optimum is achievable, nor does it show that the closed-loop systems (or controllers) approach the optimum. Next we show that the infimum is achievable by showing that there exists a controller $Q^* \in l_1$ such that $\|T_{11} + T_{12}Q^*T_{21}\|_\infty \leq 1$ and $\|S_{11} + S_{12}Q^*S_{21}\|_1 \leq \mu$.

Proof of Theorem 6: Consider a sequence $\delta^j \downarrow 1$ and let $Q^j \in \overline{\mathcal{RH}_{\infty,\delta^j}}$, $\Phi^j = S_{11} + S_{12}Q^jS_{21}$, and $\Psi^j = T_{11} + T_{12}Q^jT_{21}$ denote the optimal controller obtained by solving Problem 2 and the corresponding closed-loop transfer functions. Without loss of generality [by selecting appropriate F and L in (1)], it can be assumed that T_{12} and T_{21} are inner and co-inner, respectively. Then we have that

$$\begin{aligned} \|\Psi^j\|_\infty &\leq 1 \\ \|Q^j\|_\infty &\leq \|T_{11}\|_\infty + 1 \doteq M_Q \\ \|\Phi^j\|_\infty &\leq \|S_{11}\|_\infty + \|S_{12}\|_\infty\|Q^j\|_\infty\|S_{21}\|_\infty \\ &\leq \|S_{11}\|_\infty + \|S_{12}\|_\infty\|S_{21}\|_\infty M_Q \doteq M_\Phi. \end{aligned} \tag{11}$$

From Montel's theorem it follows that both Ψ^j and Q^j are normal families in the open unit disk. Hence Q^j contains a

[2] From Assumption A3) it follows by continuity that this can be accomplished by selecting δ close enough to one.

normally convergent subsequence $\{Q^{\tilde{j}}\}$. Let Q^* denote its limit. Normal convergence implies that Q^* is analytic in the open unit disk and that for any $\rho < 1$

$$\sup_{|\lambda|\le\rho} \bar{\sigma}(T_{11} + T_{12}Q^*(\lambda)T_{21}) \le 1$$

where the last inequality follows from the Maximum Modulus theorem and uniform convergence of $\Psi^{\tilde{j}}$ in $|\lambda| < 1$. This establishes the fact that Q^* is feasible.

Let $M \doteq \|S_{11} + S_{12}Q^0S_{21}\|_1$. From Lemma 2 it follows that $\|\Phi^{\tilde{j}}\|_1 \le M$. Hence, from the Alaoglu–Banach theorem, there exists $\phi^* \in l_1$ and a subsequence $\phi^i \to \phi^*$ weak*, i.e., for every $x \in c_0$, $\langle x, \phi^i\rangle \to \langle x, \phi^*\rangle$. Denote now by $\Phi^*(\lambda)$ the λ-transform of ϕ^*. For every λ with $|\lambda| < 1$, the sequences $\mathrm{Re}\{\lambda^k\}$ and $\mathrm{Im}\{\lambda^k\}$ belong to c_0. Therefore, $\mathrm{Re}\{\Phi^i(\lambda)\} \to \mathrm{Re}\{\Phi^*(\lambda)\}$ and $\mathrm{Im}\{\Phi^i(\lambda)\} \to \mathrm{Im}\{\Phi^*(\lambda)\}$. Thus $\Phi^i(\lambda) \to \Phi^*(\lambda)$ pointwise in the open unit disk. From normal convergence of Q^i to Q^* we have that $\Phi^i \Rightarrow S_{11} + S_{12}Q^*S_{21}$ in $|\lambda| \le \rho < 1$. Thus

$$\Phi^*(\lambda) = S_{11} + S_{12}Q^*S_{21}. \tag{12}$$

Since $\phi^* \in l_1$ and S_{12} and S_{21} have full column and row rank, respectively, on $|\lambda| = 1$, from the Corollaries to Wiener–Gelfand's theorem it follows that $\lambda^{-1}(Q^*) = \lambda^{-1}[S_{12}^{\dagger}(\Phi^* - S_{11})S_{21}^{\dagger}] \in l_1(Z)$. This, combined with the fact that $Q^* \in \mathcal{H}_\infty$ (and hence it admits a Taylor series expansion $Q^*(\lambda) = \sum_{i=0}^{\infty} q_i\lambda^i$ convergent in $|\lambda| < 1$) shows that $Q^* \in l_1$.

To complete the proof we need to show that $\|\phi^*\|_1 = \mu$. Assume that $\|\phi^*\|_1 > \mu$. Then there exist $\epsilon > 0$ and a natural N_0 such that

$$\|P_{N_0}(\phi^*)\|_1 \ge \mu + 2\epsilon. \tag{13}$$

Assume that the norm of $P_{N_0}(\phi^*)$ is achieved by its mth row, $P_{N_0}[(\phi^*)_m]$. Then there exists $x \in c_o$, $\|x\|_\infty = 1$ such that $< x, P_{N_0}[(\phi^*)_m] \ge \mu + 2\epsilon$. From the weak* convergence of ϕ^i it follows that there exists N_1 such that

$$|\langle x, P_{N_0}[(\phi^n)_m]\rangle - \langle x, P_{N_0}[(\phi^*)_m]\rangle| \le \epsilon, \qquad n > N_1.$$

Thus

$$\begin{aligned}\|\phi^n\|_1 &\ge \|P_{N_0}[(\phi^n)_m]\|_1 \ge |\langle x, P_{N_0}[(\phi^n)_m]\rangle| \\ &\ge |\langle x, P_{N_0}[(\phi^*)_m]\rangle| - |\langle x, P_{N_0}[(\phi^n)_m]\rangle \\ &\quad - \langle x, P_{N_0}[(\phi^*)_m]\rangle| \\ &\ge \mu + \epsilon, \qquad \text{for all } n > N_1\end{aligned} \tag{14}$$

against the hypothesis that $\|\phi^n\|_1 \downarrow \mu$. Thus $\|\phi^*\|_1 \le \mu$. Since Q^* is feasible, the fact that $\|\phi^*\|_1 = \mu$ follows now from the definition of μ. □

Lemma 3: Denote by $(\phi^*)_m$ the mth row of ϕ^*. Then

$$\lim_{i\to\infty} \|(\phi^i)_m - (\phi^*)_m\|_1 = 0 \tag{15}$$

for all m such that $\|(\phi^*)_m\|_1 = \mu$.

Proof: The proof follows by extending Corollary 1 to row-vector sequences (see [8, Th. 12.2.5]) and applying it to the rows of ϕ^* satisfying $\|(\phi^*)_m\|_1 = \mu$. □

Next we exploit this result to establish strong convergence of the sequences of closed-loop systems and controllers, under some additional assumptions. To this effect partition S_{12} as

$$S_{12} = \begin{bmatrix} S_{12}^1 \\ \cdots \\ S_{12}^2 \end{bmatrix} \tag{16}$$

where $S_{12}^1 \in {l_1}^{n_u \times n_u}$. With this notation we have the following.

Theorem 7: Assume that $S_{12}^1(\lambda)$ has full rank on $|\lambda| = 1$. If the first n_u rows of ϕ^* satisfy $\|(\phi^*)_i\|_1 = \mu$, $i = 1, \cdots, n_u$ then the sequences ϕ^n and Q^n converge strongly in the l_1 topology to their respective limits ϕ^* and Q^*.

Proof: Let $\phi_{n_u}^*$ and $\phi_{n_u}^n$ denote the submatrices of ϕ^* and ϕ^n formed by the respective first n_u rows. From Lemma 3 we have that $\phi_{n_u}^n$ converges strongly to $\phi_{n_u}^*$. Hence $\|S_{12}^1(Q^* - Q^n)S_{21}\|_1 \to 0$. From Wiener–Gelfand's theorem and the fact that the l_1 norm is submultiplicative we have that $\|Q^* - Q^n\|_1 \to 0$. It follows that $\phi^n = S_{11} + S_{12}Q^nS_{21}$ also converges strongly to $\phi^* = S_{11} + S_{12}Q^*S_{21}$. □

Corollary 4: In the SISO case ϕ^n and Q^n converge strongly to ϕ^* and Q^*, respectively.

V. Computing an Approximate Solution

A. An Upper Bound Leading to a Finite-Dimensional Approximation

In Section IV we have shown (Lemma 2) that μ can be computed by solving a sequence of convex optimization problems (Problem 1 with $\delta_i \downarrow 1$). In principle, these optimization problems are *infinite dimensional*. However, in this section we will show that the solution to Problem 2 can be approximated arbitrarily close by the solution to a *finite-dimensional* convex optimization problem. In order to establish this result, we need the following two results showing that, given $\epsilon > 0$, an ϵ-suboptimal solution to Problem 2 can be obtained by approximating the objective with a function that depends *only* on the first $n(\epsilon)$ Markov parameters of Q.

It is well known (see for instance [35]) that it is possible to select F and L in such a way that $T_{12}(\lambda)$ and $T_{21}(\lambda)$ are inner and co-inner, respectively, over $|\lambda| = \delta$. Moreover, if T_{12} (T_{21}) is not square, it is possible to choose $T_{12\perp}$ ($T_{21\perp}$) such that $T_{12a} \doteq [T_{12}\ \ T_{12\perp}]$ (${T_{21a}}^{\sim} \doteq [{T_{21}}^{\sim}\ \ {T_{21\perp}}^{\sim}]$) is a unitary matrix. With this notation we have the following.

Lemma 4: For every $\epsilon > 0$, there exists $n(\epsilon, \delta)$ such that if $Q \in \mathcal{H}_{\infty,\delta}$ satisfies the constraint

$$\left\| T_{11}(\lambda) + T_{12a}(\lambda)\begin{bmatrix} Q(\lambda) & 0 \\ 0 & 0\end{bmatrix} T_{21a}(\lambda)\right\|_{\infty,\delta} \le 1 \tag{17}$$

and it also satisfies $\|(I - P_{n(\epsilon,\delta)})(\Phi)\|_1 \le \epsilon$.

Proof: Since $Q \in \mathcal{H}_{\infty,\delta}$, Φ is analytic in $|\lambda| \le \delta$ and

$$\phi_k = \frac{1}{2\pi j}\oint_{|\lambda|=\delta} \Phi(\lambda)\lambda^{-(k+1)}d\lambda \tag{18}$$

where ϕ_k denotes the Markov parameters of Φ. Hence

$$\|\phi_k\|_1 \leq \sqrt{n_1}\bar{\sigma}(\phi_k) \leq \sqrt{n_1}\|\Phi\|_{\infty,\delta}\delta^{-k}$$

$$\max_i\left\{\sum_{j=1}^{n_1}\sum_{k=n}^{\infty}|\phi_k(i,j)|\right\} \leq \sum_{k=n}^{\infty}\|\phi_k\|_1 \leq \sqrt{n_1}\frac{\|\Phi\|_{\infty,\delta}\delta^{-n}}{1-\delta^{-1}}. \tag{19}$$

Since T_{12a} and T_{21a} are unitary over $|\lambda| = \delta$, it follows that $\|T_{12}\|_{\infty,\delta} = \|T_{21}\|_{\infty,\delta} = 1$ and from (17) that

$$\|Q\|_{\infty,\delta} \leq 1 + \|T_{11}\|_{\infty,\delta}. \tag{20}$$

Since $\|.\|_{\infty,\delta}$ is submultiplicative, we have

$$\begin{aligned}&\|\Phi\|_{\infty,\delta}\\ &\quad\leq \|S_{11}\|_{\infty,\delta} + \|S_{12}\|_{\infty,\delta}\|Q\|_{\infty,\delta}\|S_{21}\|_{\infty,\delta}\\ &\quad\leq \|S_{11}\|_{\infty,\delta} + \|S_{12}\|_{\infty,\delta}\|S_{21}\|_{\infty,\delta}(1+\|T_{11}\|_{\infty,\delta}) \doteq K.\end{aligned} \tag{21}$$

The desired result follows by selecting

$$n \geq n_o = \left\lceil \frac{\log\sqrt{n_1}K - \log\epsilon(1-\delta^{-1})}{\log\delta}\right\rceil. \tag{22}$$

□

Theorem 8: Consider the following optimization problem.

Problem 3:

$$\begin{aligned}&\min_{Q\in\mathcal{RH}_{\infty,\delta}} \|P_{n(\epsilon,\delta)}(S_{11}+S_{12}QS_{21})\|_1\\ &\quad= \min_{\substack{Q_i\in R^{n_u\times m_y}\\ 0\leq i\leq n-1}} \|\underline{\mathcal{S}}_1 + \underline{\mathcal{S}_{12}}\underline{Q}\underline{\mathcal{S}_{21}}\|_1\end{aligned} \tag{23}$$

subject to

$$\left\|T_{11} + T_{12a}\begin{bmatrix}Q & 0\\ 0 & 0\end{bmatrix}T_{21a}\right\|_{\infty,\delta} \leq 1 \tag{24}$$

where

$$\begin{aligned}\underline{\mathcal{S}}_1 &= \begin{bmatrix}s_{n-1}^{11} & s_{n-2}^{11} & \cdots & s_0^{11}\end{bmatrix}\\ \underline{\mathcal{S}_{12}} &= \begin{bmatrix}s_{n-1}^{12} & s_{n-2}^{12} & \cdots & s_0^{12}\end{bmatrix}\\ \underline{Q} &= \begin{bmatrix}Q_0 & 0 & \cdots & 0\\ Q_1 & Q_0 & \cdots & 0\\ \vdots & & \ddots & \\ Q_{n-1} & \cdots & & Q_0\end{bmatrix}\\ \underline{\mathcal{S}_{21}} &= \begin{bmatrix}s_0^{21} & 0 & \cdots & 0\\ s_1^{21} & s_0^{21} & \cdots & 0\\ \vdots & & \ddots & \\ s_{n-1}^{21} & \cdots & & s_0^{21}\end{bmatrix}\end{aligned} \tag{25}$$

$n(\epsilon,\delta)$ is selected according to (22), and where Q_k, s_k^{ij} denote the kth element of the impulse response of $Q(\lambda), S_{ij}(\lambda)$ respectively. Let Q_δ^n denote the optimal solution and define $\Phi_\delta^n \doteq S_{11} + S_{12}Q_\delta^n S_{21}$, $\mu_\delta^n = \|\Phi_\delta^n\|_1$. Then the following properties hold.

1) $\mu_\delta \leq \mu_\delta^n \leq \mu_\delta + \epsilon$.
2) $\|Q_\delta^n - Q_\delta^*\|_1 \to 0$ as $\epsilon \to 0$ (hence $\|\Phi_\delta^n - \Phi_\delta^*\|_1 \to 0$), where Q_δ^* is an optimal solution to Problem 2, and $\Phi_\delta^* \doteq S_{11} + S_{12}Q_\delta S_{21}$.

Proof: $\mu_\delta \leq \mu_\delta^n$ is immediate from the definition of μ_δ. From the definition of μ_δ^n we have

$$\begin{aligned}\mu_\delta^n = \left\|\Phi_\delta^n\right\|_1 &= \max_i\left\{\sum_{j=1}^{n_1}\sum_{k=0}^{\infty}\left|\phi_k^n(i,j)\right|\right\}\\ &\leq \max_i\left\{\sum_{j=1}^{n_1}\sum_{k=0}^{n-1}\left|\phi_k^n(i,j)\right|\right\} + \max_i\left\{\sum_{j=1}^{n_1}\sum_{k=n}^{\infty}\left|\phi_k^n(i,j)\right|\right\}\\ &\leq \max_i\left\{\sum_{j=1}^{n_1}\sum_{k=0}^{n-1}\left|\phi_k^n(i,j)\right|\right\} + \epsilon\\ &\leq \max_i\left\{\sum_{j=1}^{n_1}\sum_{k=0}^{n-1}\left|\phi_k^\delta(i,j)\right|\right\} + \epsilon\\ &\leq \max_i\left\{\sum_{j=1}^{n_1}\sum_{k=0}^{\infty}\left|\phi_k^\delta(i,j)\right|\right\} + \epsilon = \mu_\delta + \epsilon.\end{aligned} \tag{26}$$

This also shows that as $\epsilon \to 0$, then $\mu_\delta^n \to \mu_\delta$. From (20) it follows that Q^n is a normal family in $|\lambda| < \delta$, and therefore it has a subsequence $Q^{\tilde{n}}$ normally convergent to some $Q_\delta^* \in \mathcal{H}_{\infty,\delta}$. Moreover, it can be easily shown that $\|T_{11}+T_{12}Q_\delta^*T_{21}\|_{\infty,\delta} \leq 1$ and that $\|S_{11}+S_{12}Q_\delta^*S_{21}\|_1 = \mu_\delta$. Thus Q_δ^* is an optimal solution to Problem 2. Since Q_δ^*, $Q_\delta^{\tilde{n}} \in \mathcal{H}_{\infty,\delta}$, from (20) we have that, for any $N > 0$

$$\begin{aligned}\left\|Q_\delta^{\tilde{n}} - Q_\delta^*\right\|_1 \leq \sup_i\Bigg\{&\sum_{j=1}^{m_y}\sum_{k=0}^{N-1}\left|Q_{k\delta}^{\tilde{n}}(i,j) - Q_{k\delta}^*(i,j)\right|\\ &+ 2\sqrt{m_y}(1+\|T_{11}\|_{\infty,\delta})\frac{\delta^{-N}}{1-\delta^{-1}}\Bigg\}.\end{aligned} \tag{27}$$

From the normal convergence of $Q_\delta^{\tilde{n}}$ it follows that, given $\eta > 0$, there exists N_1 such that for $\tilde{n} > N_1$, $|Q_{k\delta}^{\tilde{n}}(i,j) - Q_{k\delta}^*(i,j)| \leq \frac{\eta}{2Nm_y}$, $k = 0,\cdots,N-1, 1 \leq i \leq n_u, 1 \leq j \leq m_y$. Therefore, by selecting N and N_1 large enough, it follows that $\|Q_\delta^{\tilde{n}} - Q_\delta^*\|_1 \leq \eta$. □

Finally, we show that Problem 3 can be decoupled into a *finite-dimensional* convex optimization and an *unconstrained* $\mathcal{H}_\infty$ problem. To this effect we recall a necessary and sufficient condition for the feasibility of the $\mathcal{H}_\infty$ constraint when the first n Markov parameters in the expansion $Q(\lambda) = Q_o + Q_1\lambda + \cdots + Q_{n-1}\lambda^{(n-1)}$ are fixed.

Consider again Problem 1. By choosing F, L, $T_{12\perp}$, and $T_{21\perp}$ such that $T_{12a} \doteq [T_{12}\ \ T_{12\perp}]$ and $T_{21a}{}^\sim \doteq [T_{21}{}^\sim\ \ T_{21\perp}{}^\sim]$ are unitary, $\|\Psi\|_\infty$ can be reduced to

$$\|\Psi\|_\infty = \left\|T_{11} + T_{12a}\begin{bmatrix}Q & 0\\ 0 & 0\end{bmatrix}T_{21a}\right\|_\infty = \left\|G + \begin{bmatrix}Q^\sim & 0\\ 0 & 0\end{bmatrix}\right\|_\infty \tag{28}$$

where $G \doteq T_{21a}T_{11}{}^\sim T_{12a} \in \mathcal{RH}_\infty$ has a state-space realization

$$\left(\begin{array}{c|cc}\hat{A} & B_a & B_b\\ \hline C_a & D_{aa} & D_{ab}\\ C_b & D_{ba} & D_{bb}\end{array}\right) \tag{29}$$

In the sequel, for notational simplicity we will call

$$B_e = [B_a \quad B_b], \qquad D_{er} = [D_{aa} \quad D_{ab}], \qquad C_e = \begin{bmatrix} C_a \\ C_b \end{bmatrix}$$
$$D_{ec} = \begin{bmatrix} D_{aa} \\ D_{ba} \end{bmatrix}. \tag{30}$$

With these definitions, Problem 1 can be reformulated as follows.

Problem 4: Compute $Q \in l_1$ such that $\left\| \begin{matrix} G_{11}+Q^{\sim} & G_{12} \\ G_{21} & G_{22} \end{matrix} \right\|_\infty \leq 1$ and $\|S\|_1$ is minimized.

Consider now the following Riccati equations:

$$\begin{aligned} \hat{X} &= A_e \hat{X} A_e^T + B_e B_e^T + R_x \left(I - D_{er} D_{er}^T - C_a \hat{X} C_a^T\right)^{-1} R_x^T \\ \hat{Y} &= A_e^T \hat{Y} A_e + C_e^T C_e + R_y \left(I - D_{ec}^T D_{ec} - B_a^T \hat{Y} B_a\right)^{-1} R_y^T \end{aligned} \tag{31}$$

where

$$R_x = A_e \hat{X} C_a^T + B_e D_{er}^T, \qquad R_y = A_e^T \hat{Y} B_a + C_e^T D_{ec}.$$

From [24], there exists a Q satisfying the *strict* $\mathcal{H}_\infty$ constraint if and only if there exist positive-definite solutions $\hat{X}$ and $\hat{Y}$ to these Riccati equations such that $\rho(\hat{X}\hat{Y}) < 1$. This will be assumed in what follows. For ease of notation, let $x \doteq \hat{X}^{1/2}$, $y \doteq \hat{Y}^{1/2}$.

Lemma 5: Let G have a state-space realization as in (29), and let $Q_{\text{FIR}}^n(\lambda) = \sum_{i=0}^{n-1} Q_i \lambda^i$. Then there exists $Q_{\text{tail}}^n(\lambda) \in \mathcal{H}_\infty$ such that

$$\left\| \begin{matrix} G_{11} + \sum_{i=0}^{n-1} Q_i^T \lambda^{-i} + \lambda^{-n} Q_{\text{tail}}^n(\lambda)^{\sim} & G_{12} \\ G_{21} & G_{22} \end{matrix} \right\|_\infty \leq 1 \tag{32}$$

if and only if $\bar{\sigma}(W(\mathbf{Q_n})) \leq 1$, where we have (33), as shown at the bottom of the page.

Proof: This is [24, Th. 8]. □

Theorem 9: Given $\epsilon > 0$ and $\delta > 1$, an ϵ-suboptimal solution to Problem 2 is given by $Q(\lambda) = \sum_{i=0}^{\infty} Q_i \lambda^i \doteq Q_{\text{FIR}} + \lambda^n Q_{\text{tail}}$ where Q_{FIR} solves the following finite dimensional convex optimization problem:

$$\min_{Q_i} \|P_{n(\epsilon,\delta)}(S_{11} + S_{12} Q_{\text{FIR}} S_{21})\|_1 \ \text{ s.t. } (W_\delta(\mathbf{Q_n})) \leq 1 \tag{34}$$

and Q_{tail} solves the unconstrained $\mathcal{H}_\infty$ optimization problem (35), as shown at the bottom of the next page, where $n(\epsilon, \delta)$ is given by (22) and where $W_\delta(\mathbf{Q_n})$ is obtained from $W(\mathbf{Q_n})$ in (33) by using the change of variable $\lambda \to \delta^{-1}\lambda$.

Proof: The proof follows from combining Lemmas 4 and 5 with Theorem 8. □

Remark 3: It can be easily shown that the change of variable $\lambda \to \delta^{-1}\lambda$ is equivalent to the following transformation on the state-space realizations: $A_e \to A_\delta \doteq \delta A_e$, $[B_e \ \ B_a] \to B_\delta \doteq [\delta B_e \ \ \delta B_a]$, $X \to X_\delta$, and $Y \to Y_\delta$, where X_δ and Y_δ denote the solutions to the Riccati equations (31) after the transformation $A \to A_\delta$ and $B \to B_\delta$.

B. Computing a Lower Bound of the Cost

In the last section we have shown that an ϵ-suboptimal solution to Problem 2 can be obtained by solving a finite-dimensional convex optimization problem of size $n(\epsilon, \delta)$. However, the estimate of n provided by (22) can be very conservative, leading to large optimization problems. Additionally, while this approach guarantees that the corresponding suboptimal solution achieves a cost $\|\phi_\delta^n\|_1 \leq \mu_\delta + \epsilon$, it does not provide any information on its distance to μ, the optimal solution to Problem 1. These difficulties can be circumvented by simply solving (34) for increasing values of n (obtaining a decreasing sequence of suboptimal solutions) until the approximation error e_n falls below a given threshold. Clearly, this requires the ability to compute an upper bound on e_n. To this effect in this section we introduce a procedure for computing a lower bound of the cost, μ^n, and a sequence of superoptimal closed-loop systems with increasing l_1 norms $\mu^{n_i} \uparrow \mu$. By combining this lower bound with the upper bound derived in the last section, we can obtain sequences of suboptimal and superoptimal solutions and stop the optimization when the difference between the upper and lower bounds, $\mu_\delta^n - \mu_n$, is smaller than a prescribed tolerance.

Theorem 10: Consider the following optimization problem.

Problem 5:

$$\mu^n = \min_{Q_i} \|P_n(S_{11} + S_{12} P_n(Q^n) S_{21})\|_1 \ \text{ s.t. } (W(\mathbf{Q_n})) \leq 1 \tag{36}$$

where

$$Q^n(\lambda) = \sum_{i=0}^{n-1} Q_i^n \lambda^i + \lambda^n Q_{\text{tail}}^n \doteq Q_{\text{FIR}}^n + \lambda^n Q_{\text{tail}}^n \tag{37}$$

$$W(\mathbf{Q_n}) = \begin{bmatrix} yA_e^n x & yA_e^{n-1}B_a & \cdots & yA_eB_a & yB_a & yA_e^{n-1}B_b & \cdots & yA_eB_b & yB_b \\ C_aA_e^{n-1}x & C_aA_e^{n-2}B_a & \cdots & C_aB_a & D_{aa} & C_aA_e^{n-2}B_b & \cdots & C_aB_b & D_{ab} \\ C_aA_e^{n-2}x & C_aA_e^{n-3}B_a & \cdots & D_{aa} & 0 & C_aA_e^{n-3}B_b & \cdots & D_{ab} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ C_ax & D_{aa} & 0 & \cdots & 0 & D_{ab} & 0 & \cdots & 0 \\ C_bA_e^{n-1}x & C_bA_e^{n-2}B_a & \cdots & C_bB_a & D_{ba} & C_bA_e^{n-2}B_b & \cdots & C_bB_b & Q_0^t \\ C_bA_e^{n-2}x & C_bA_e^{n-3}B_a & \cdots & D_{ba} & 0 & C_bA_e^{n-3}B_b & \cdots & Q_0^t & Q_1^t \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ C_bx & D_{ba} & 0 & \cdots & 0 & Q_0^T & Q_1^T & \cdots & Q_{n-1}^T \end{bmatrix} \tag{33}$$

and Q^n_{tail} solves the unconstrained $\mathcal{H}_\infty$ approximation problem

$$\left\| \begin{matrix} G_{11} + \sum_{i=0}^{n-1} \left(Q_i^n\right)^T \lambda^{-i} + \lambda^{-n} Q^n_{\text{tail}}(\lambda)^\sim & G_{12} \\ G_{21} & G_{22} \end{matrix} \right\|_\infty \leq 1. \tag{38}$$

Assume that S_{12} and S_{21} have full column and row rank on $|\lambda| = 1$. Then the following properties hold.

1) $\mu^n \to \mu$.
2) $P_n(Q^n) \to Q^*$ normally in $|\lambda| < 1$, where Q^* is an optimal solution to Problem 1.
3) $\Phi^n \doteq P_n(S_{11} + S_{12}P_n(Q^n)S_{21}) \to \Phi^* \in l_1$ weak*, where $\Phi^* \doteq S_{11} + S_{12}Q^*S_{21}$.

Proof: We begin by showing that $\|\phi^n\|_1 \to \mu$. Suppose that some Q solves Problem 5 with horizon $\tilde{n}+1$. Then clearly Q is a feasible solution for the same problem with horizon $\tilde{n}$. It follows that $\mu^{\tilde{n}} \leq \mu^{\tilde{n}+1}$. Moreover, from the definition of μ_R and μ^n and the fact that $\mu_R = \mu$ it follows that $\mu^{\tilde{n}} \leq \mu$. Thus $\mu^{\tilde{n}}$ is a nondecreasing sequence, bounded above, and therefore has a limit $\mu^{\lim} \leq \mu$. Next we will show that $\mu^{\lim} = \mu$. From (20) we have that $\|Q^{\tilde{n}}\|_\infty \leq \|T_{11}\|_\infty + 1 \doteq M_Q$. Thus, $Q^{\tilde{n}}$ is a normal family in $|\lambda| < 1$ and by Montel's theorem has a subsequence $\{Q^n\}$ that converges normally to $Q^* \in \mathcal{H}_\infty$. It can be easily shown that $\|T_{11} + T_{12}Q^*T_{21}\|_\infty \leq 1$. Thus, Q^* is a feasible solution for Problem 1. Normal convergence of Q^n in $|\lambda| < 1$ implies uniform convergence in any closed disk $|\lambda| \leq \lambda_o < 1$. Thus, given $\epsilon > 0$, there exists $N > 0$ such that for $n > N$

$$\bar{\sigma}[Q^n(\lambda) - Q^*(\lambda)] < \epsilon, \qquad \forall |\lambda| \leq \lambda_o < 1.$$

Let Q_i^n denote the Markov parameters of Q^n. Proceeding as in (18) it can be easily seen that $\bar{\sigma}(Q_i^n) \leq \|Q^n\|_\infty$. Hence, for $|\lambda| \leq \lambda_o$ we have

$$\begin{aligned} \bar{\sigma}[P_n(Q^n)(\lambda) - Q^*(\lambda)] &< \epsilon + \bar{\sigma}\left[\sum_{i=n}^{\infty} Q_i^n \lambda^i\right] \\ &< \epsilon + \|Q^n\|_\infty \frac{\lambda_o^n}{1-\lambda_o} \\ &\leq \epsilon + M_Q \frac{\lambda_o^n}{1-\lambda_o.} \end{aligned}$$

Thus, for $n > \max\{N, \lceil \frac{\log \epsilon(1-\lambda_o) - \log M_Q}{\log \lambda_o} \rceil\}$

$$\bar{\sigma}\left[P_n(Q^n)(\lambda) - Q^*(\lambda)\right] < 2\epsilon, \qquad \forall |\lambda| \leq \lambda_o < 1.$$

This implies that $P_n(Q^n)$ converges normally to Q^*. Consider now the corresponding sequence of Φ^n's. Since $\|\Phi^n\|_1 \leq \mu$, it follows from Alaoglu–Banach's theorem that there exist $\phi^* \in l_1$ and a subsequence $\phi^{\tilde{j}} \to \phi^*$ weak*. Proceeding as in the proof of Theorem 6 it can be easily shown that $\|\phi^*\|_1 \leq \mu^{\lim}$ and that

$$\Phi^*(\lambda) = S_{11} + S_{12}Q^*S_{21}. \tag{39}$$

As in Theorem 6 this implies that $Q^* \in l_1$. Since $\mu^{\lim} \leq \mu$ and Q^* is feasible for Problem 1 it follows that $\mu^{\lim} = \mu$ and $\|\phi^*\|_1 = \mu$. □

Corollary 5: Assume that $S_{12}^1(\lambda)$ has full rank on $|\lambda| = 1$. If the first n_u rows of ϕ^* satisfy $\|(\phi^*)_i\|_1 = \mu, i = 1, \cdots, n_u$, then the sequences ϕ^n and $P_n(Q^n)$ converge strongly in the l_1 topology to their respective limits ϕ^* and Q^*.

VI. The Continuous-Time Case

In the previous sections we have shown that a discrete-time mixed $l_1/\mathcal{H}_\infty$ problem can be solved by solving a sequence of convex optimization problems. In this section we will briefly address the continuous-time counterpart of the problem. The main result of this section shows that suboptimal $\mathcal{L}^1/\mathcal{H}_\infty$ controllers can be synthesized by solving a discrete-time mixed $l_1/\mathcal{H}_\infty$ for an auxiliary discrete-time system. To this effect we introduce the discrete-time Euler Approximating System (EAS) [2], [3] and explore some of its properties.

Definition 4: Consider the continuous time system (S). EAS is defined as the following discrete time system:

$$\left(\begin{array}{c|ccc} I + \tau A & \tau B_1 & \tau B_2 & \tau B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ C_3 & D_{31} & D_{32} & D_{33} \end{array}\right). \tag{EAS}$$

where $\tau > 0$.

Next we recall some of the properties of the EAS, showing that the l_1 and $\mathcal{H}_\infty$ norms of the EAS are upper bounds of the corresponding continuous-time quantities. Moreover, these upper bounds are nonincreasing with τ and converge to the exact value as $\tau \to 0$.

Lemma 6 [3]: Assume that (S) is asymptotically stable and consider a strictly decreasing sequence $\tau_i \to 0$. Let $T_{\zeta w}(s)$ denote the transfer function of (S) and $T_{\zeta w}^{\text{EAS}}(\lambda, \tau_i)$, the transfer function of the EAS corresponding to τ_i. Then

$$\begin{aligned} \|T_{\zeta w}(s)\|_1 &\leq \left\|T_{\zeta w}^{\text{EAS}}(\lambda, \tau_i)\right\|_1 \quad \forall i \\ \left\|T_{\zeta w}^{\text{EAS}}(\lambda, \tau_i)\right\|_1 &\leq \left\|T_{\zeta w}^{\text{EAS}}(\lambda, \tau_j)\right\|_1 \quad i > j \\ \lim_{\tau_i \to 0} \left\|T_{\zeta w}^{\text{EAS}}(\lambda, \tau_i)\right\|_1 &= \|T_{\zeta w}(s)\|_1. \end{aligned} \tag{40}$$

Lemma 7 [28]: Assume that (S) is asymptotically stable and consider a strictly decreasing sequence $\tau_i \to 0$. Let $T_{\zeta w}(s)$ denote the transfer function of (S) and $T_{\zeta w}^{\text{EAS}}(\lambda, \tau_i)$ the transfer function of the EAS corresponding to τ_i. Then

$$\begin{aligned} \|T_{\zeta w}(s)\|_\infty &\leq \left\|T_{\zeta w}^{\text{EAS}}(\lambda, \tau_i)\right\|_\infty \quad \forall i \\ \left\|T_{\zeta w}^{\text{EAS}}(\lambda, \tau_i)\right\|_\infty &\leq \left\|T_{\zeta w}^{\text{EAS}}(\lambda, \tau_j)\right\|_\infty \quad i > j \\ \lim_{\tau_i \to 0} \left\|T_{\zeta w}^{\text{EAS}}(\lambda, \tau_i)\right\|_\infty &= \|T_{\zeta w}(s)\|_\infty. \end{aligned} \tag{41}$$

Combining the results of Lemmas 6 and 7 we have the following result.

$$Q_{\text{tail}} = \arg \min_{Q \in \mathcal{H}_{\infty,\delta}} \left\| \begin{matrix} G_{11} + \sum_{i=0}^{n-1} Q_i^T \lambda^{-i} + \lambda^{-n} Q(\lambda)^\sim & G_{12} \\ G_{21} & G_{22} \end{matrix} \right\|_{\infty\delta} \leq 1 \tag{35}$$

Lemma 8: Assume that $\inf_{Q\in\mathcal{RH}_\infty} \|T_{\zeta_\infty\omega_\infty}(s)\|_\infty = \gamma_o < \gamma$. Consider a strictly decreasing sequence $\tau_i \to 0$ and the corresponding EAS(τ_i). Let

$$\mu_i = \inf_{\substack{Q\in\mathcal{RH}_\infty(T)\\ \|T_{\zeta_\infty w_\infty}\|_\infty\le\gamma}} \left\|T^{(\mathrm{EAS})}_{\zeta_1 w_1}(\lambda,\tau_i)\right\|_1$$
$$\mu_o = \inf_{\substack{Q\in\mathcal{RH}_\infty(j\Re)\\ \|T_{\zeta_\infty w_\infty}\|_\infty\le\gamma}} \|T_{\zeta_1 w_1}(s)\|_1. \tag{42}$$

Then the sequence μ_i is nonincreasing and such that $\mu_i \to \mu^o$.

Finally, we note that from the definition of the EAS it is easily seen that the closed-loop transfer function obtained by applying the rational controller $K(s)$ to (S) is the same as the closed-loop transfer function obtained by applying the controller $K(\frac{\lambda^{-1}-1}{\tau})$ to the EAS, up to the complex transformation $\lambda = \frac{1}{\tau s+1}$. Therefore, if a rational compensator $K(\lambda)$ yielding an $l_1/\mathcal{H}_\infty$ cost μ_d is found for the EAS, then $K((\tau s+1)^{-1})$ internally stabilizes (S) and yields an $\mathcal{L}^1/\mathcal{H}_\infty$ cost $\mu_c \le \mu_d$. It follows that a rational compensator can be synthesized using the EAS with a suitably small τ. These observations are formalized in the following lemma.

Lemma 9: Consider the mixed $\mathcal{L}^1/\mathcal{H}_\infty$ control problem for continuous time-systems. A suboptimal rational solution can be obtained by solving a discrete-time mixed $l_1/\mathcal{H}_{\infty,\delta}$ control problem for the corresponding EAS, with $\delta = (1-\tau^2)^{-1}$. Moreover, if $K(\lambda)$ denotes the $l_1/\mathcal{H}_\infty$ controller for the EAS, the suboptimal $\mathcal{L}^1/\mathcal{H}_\infty$ controller is given by $K((\tau s+1)^{-1})$.

Finally, we show that by taking $\tau \to 0$, the proposed design method yields controllers with cost arbitrarily close to the optimal $\mathcal{L}^1/\mathcal{H}_\infty$ cost.

Theorem 11: Let $\tau_i \to 0$ be a strictly decreasing sequence. Denote by K_i the controller obtained using the design procedure of Lemma 4 with $\tau = \tau_i$ and by $T_{z_1 w_1}(s, K_i)$ the corresponding closed-loop transfer function. Then the sequence $\mu_i \doteq \|T_{z_1 w_1}(s,K_i)\|_1$ is nonincreasing and such that $\lim_{i\to\infty}\mu_i = \mu_o$.

Proof: The proof, omitted for space reasons, follows along the same lines of the proof of [28, Th. 4]. □

VII. Some Simple Examples

Example 1: Consider the four-block unstable, nonminimum phase MIMO system shown in Fig. 2 where

$$P = \frac{\lambda(5-10\lambda)}{(1-10\lambda)(1-0.5\lambda)}, \qquad W_1 = \frac{0.4}{1-0.6\lambda}$$
$$W_2 = \frac{1-0.75\lambda}{1-0.25\lambda}, \qquad W_3 = \frac{0.02}{1-0.2\lambda}$$

and K is the controller transfer function. Define the transfer matrices Φ and Ψ as follows:

$$\begin{pmatrix} y_1\\ y_2\end{pmatrix} = \Phi\begin{pmatrix} n_1\\ n_2\end{pmatrix}, \qquad \begin{pmatrix} y\\ u\end{pmatrix} = \Psi\begin{pmatrix} r\\ d\end{pmatrix}$$

where

$$\Phi = \begin{pmatrix} (1-PK)^{-1}W_1 & PK(1-PK)^{-1}W_2\\ 0.1K(1-PK)^{-1}W_1 & 0.1K(1-PK)^{-1}W_2\end{pmatrix}$$
$$\Psi = \begin{pmatrix} PK(1-PK)^{-1} & P(1-PK)^{-1}W_3\\ K(1-PK)^{-1} & PK(1-PK)^{-1}W_3\end{pmatrix}.$$

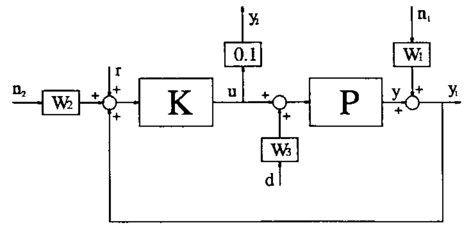

Fig. 2. Block diagram of the closed-loop system (Example 1).

The four-block MIMO l_1/H_∞ control problem of interest is

$$\text{Minimize } \|\Phi\|_{l_1}$$
$$\text{subject to } \|\Psi\|_{H_\infty} \le 37.$$

For this problem, the proposed synthesis procedure, described in Sections V-A and V-B, yields $\|\Phi\|_1 = 72.6418$ and $\|\Psi\|_\infty = 37.00$. Table I shows a comparison of the optimal l_1 norms corresponding to several values of δ, with the corresponding closed-loop impulse responses shown in Fig. 3. Here $n(\epsilon,\delta)$ is calculated from (22) with the error bound $\epsilon = 0.001$. Since as δ approaches one $n(\epsilon,\delta)$ gets rather large, the controller synthesis was followed by a model reduction step. The last column in Table I, N_d, shows the order of the resulting controller.

For comparison, Table II lists the lower bounds of the cost, obtained by solving Problem 5 in Section V-B for increasing values of n. As n gets larger, μ^n approaches the optimal value from below.

Example 2: Consider now the continuous-time SISO plant used in [10] and [3]

$$P(s) = \frac{s-1}{s-2}. \tag{43}$$

The controller that minimizes $\|T\|_1 \doteq \|PC(1+PC)^{-1}\|_1$ is given by

$$K_{\mathcal{L}^1} = \frac{(s-2)(1.7071 - 4.1213e^{-0.8814s})}{(s-1)(-0.7071 + 4.1213e^{-0.8814s})} \tag{44}$$

and yields $T(s) = 1.7071 - 4.1213e^{-0.8814s}$, with $\|T\|_1 = 5.8284$. It is easily seen that $S(s) \doteq (1+PC)^{-1} = 0.7071 + 4.1213e^{-0.8814s}$, with $\|S\|_\infty = 4.8284$. Given the difficulty of physically implementing a nonrational controller, in [3] we developed a method for synthesizing rational approximations to the optimal $\mathcal{L}^1$ controller. The rational approximation proposed there yields

$$T(s) = 1.8414 - 4.3423\frac{1}{(1+0.1s)^9}$$
$$S(s) = -0.8414 + 4.3423\frac{1}{(1+0.1s)^9} \tag{45}$$

with $\|S\|_\infty = 3.9$ and $\|T\|_1 = 6.18$. The $\mathcal{H}_\infty$ controller that minimizes $\|S\|_\infty$ is given by $C(s) = -\frac{4}{3}$ and yields $\|S\|_\infty = 3$ and $\|T\|_1 = 10$. Finally, a mixed $\mathcal{L}^1/\mathcal{H}_\infty$ design yields

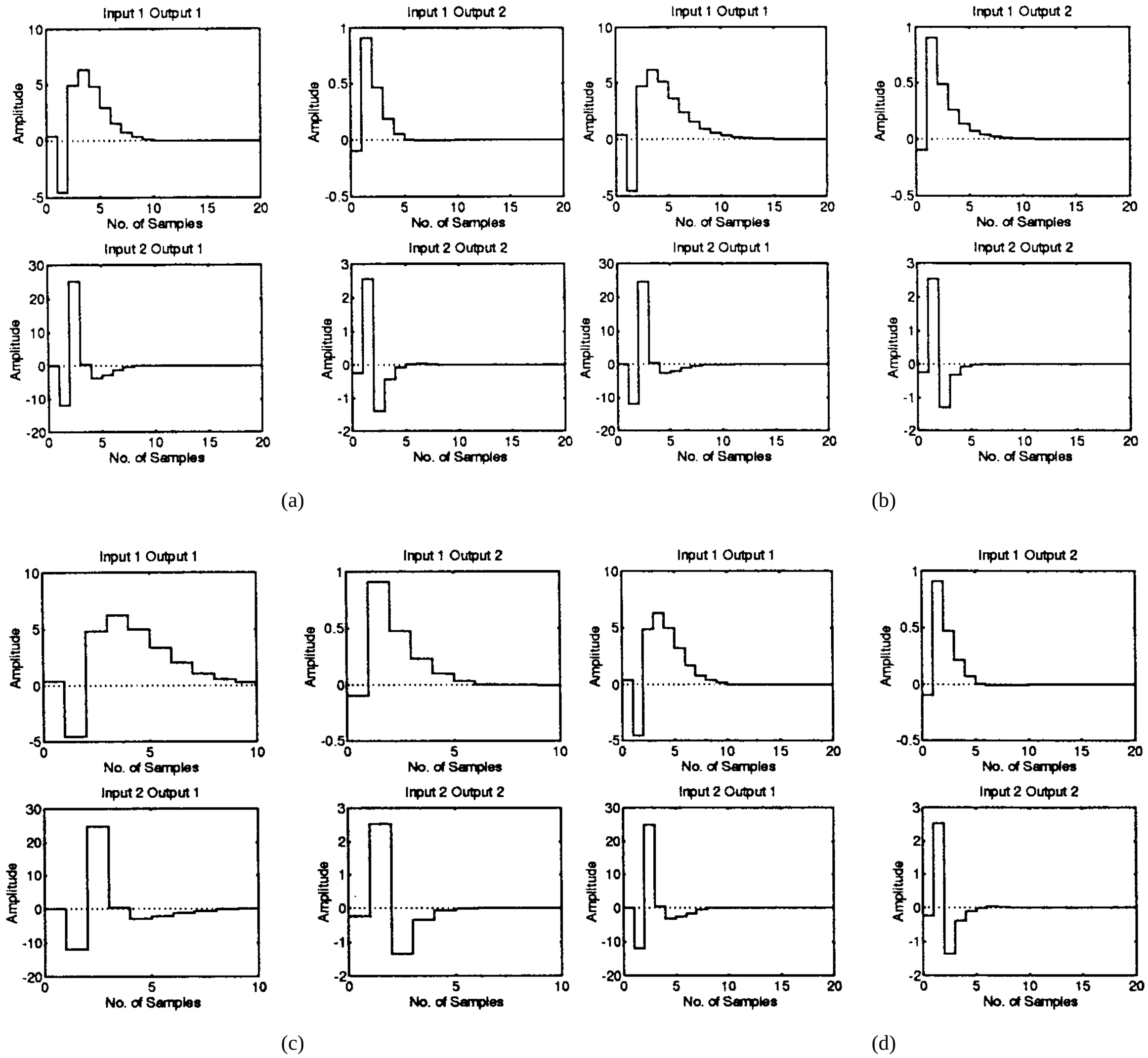

Fig. 3. Impulse responses of resulting systems (Example 1): (a) optimal controller; (b) $\delta = 1.1765$; (c) $\delta = 1.1111$; and (d) $\delta = 1.0526$.

TABLE I
COMPARISON OF THE OPTIMAL SOLUTIONS CORRESPONDING TO DIFFERENT δ'S

δ	$\|\Phi\|_1$	$\|\Psi\|_\infty$	$n(\epsilon, \delta)$	N_d
1.1765	73.9412	36.96	87	21
1.1111	73.0260	36.95	136	14
1.0526	72.7859	36.93	288	14
optimal	72.6418	37.00	-	14

TABLE II
LOWER BOUNDS FOR DIFFERENT VALUES OF THE HORIZON n

n	50	100	150	200	250
μ^n	71.9572	72.1890	72.3827	72.5256	72.6145

$\|T\|_1 = 6.41$ and $\|S\|_\infty = 3.45$. The different frequency responses for S and the corresponding impulse responses for T are shown in Fig. 4.

VIII. CONCLUSION

In this paper we present an iterative algorithm for solving a general mixed $l_1/\mathcal{H}_\infty$ control problem. The main idea is to construct a sequence of optimization problems and then show that the sequence of solutions thus generated converges, in the l_1 topology, to a solution of the original problem. At each step, the optimization problems are convex and have a structure which allows for efficient computations. Additionally, our approach provides new insights into some properties of the optimal solutions, in particular the facts that the problem admits a minimizing solution in l_1 and, more importantly from an engineering standpoint, that the optimal performance can be approached arbitrarily close by a real-rational controller. Moreover, from a practical standpoint, our approach allows for finding exponentially stable suboptimal solutions with a prescribed degree of stability, by selecting $\delta > 1$ in Problem 2.

Finally, we want to point out that, although these results deal with mixed $l_1/\mathcal{H}_\infty$ control problems, they also provide an alternative to the delay-augmentation [8] method for solving MIMO pure l_1 problems. This approach, based upon recasting the l_1 problem into a mixed $l_1/\mathcal{H}_\infty$ problem by adding a nonbinding artificial $\mathcal{H}_\infty$ constraint (see [29] for details), does not necessitate obtaining the zero structure of S_{12} and S_{21} and

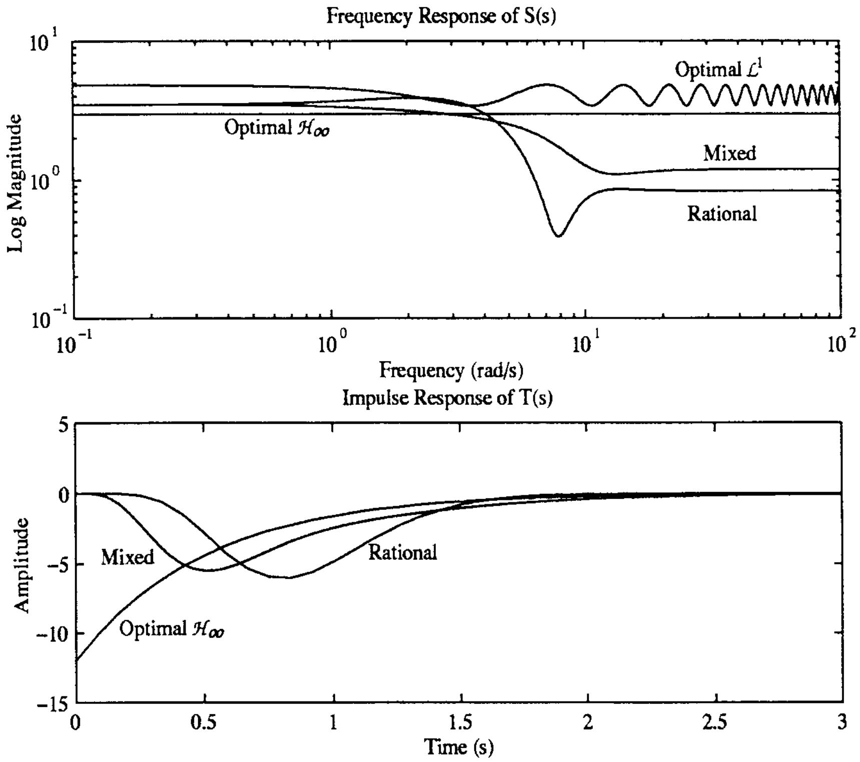

Fig. 4. Impulse and frequency responses for different designs (Example 2).

computing the zero interpolation and the rank interpolation conditions. Thus, it may provide a useful alternative to delay augmentation, especially for cases where the number of inputs or outputs is not small. In these cases, delay augmentation will tend to result in larger linear programming problems, and it may require a large number of trial-and-error-type iterations (reordering inputs and outputs) in order to satisfy the sufficient conditions for convergence of the upper bound.

Perhaps the most severe limitation of the proposed method is that it may result in very large-order controllers (roughly N), necessitating some type of model reduction. Note, however, that this disadvantage is shared by some widely used design methods, such as μ-synthesis or l_1 optimal control theory, that will also produce controllers with no guaranteed complexity bound. Application of some well-established methods in order reduction (noteworthy, weighted balanced truncation) usually succeed in producing controllers of manageable order.

Recent results show that in the state-feedback case, the same l^1 cost achieved with a linear dynamical controller, can be achieved with nonlinear memoryless feedback [4]. Since it is well known that the same results hold for $\mathcal{H}_\infty$ controllers, this raises the issue of using static nonlinear controllers, rather than high-order dynamical controllers, to solve the mixed $l_1/\mathcal{H}_\infty$ problem. Research is currently under way addressing this issue and the issue of model reduction in the presence of mixed performance objectives.

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