$$\underline{B}_1\underline{D}_{21}^* = \int_0^T e^{(T-\tau)A} B_1 [\Psi_0'(\tau) \quad \cdots \quad \Psi_{n-1}'(\tau)]\, d\tau,$$

$$\underline{D}_{21}\underline{D}_{21}^* = \int_0^T \begin{bmatrix} \Psi_0(\tau) \\ \vdots \\ \Psi_{n-1}(\tau) \end{bmatrix} [\Psi_0'(\tau) \quad \cdots \quad \Psi_{n-1}'(\tau)]\, d\tau.$$

With the two symmetric matrices $E_{12}'E_{12}$ and $E_{21}E_{21}'$ computed, there are many choices for E_{12} and E_{21}; for example, we can take them as the square roots or Cholesky factors of the two symmetric matrices respectively.

REFERENCES

[1] H. Al-Rahmani and G. F. Franklin, "A new optimal multirate control of linear periodic and time-varying systems," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 406–415, 1990.

[2] M. Araki and K. Yamamoto, "Multivariable multirate sampled-data systems: state-space description, transfer characteristics, and Nyquist criterion," *IEEE Trans. Automat. Contr.*, vol. 30, pp. 145–154, 1986.

[3] B. Bamieh and J. B. Pearson, "A general framework for linear periodic systems with application to $\mathcal{H}_\infty$ sampled-data control," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 418–435, 1992.

[4] B. Bamieh and J. B. Pearson, "The $\mathcal{H}_2$ problem for sampled-data systems," *Syst. Contr. Lett.*, vol. 19, pp. 1–12, 1992.

[5] B. Bamieh, J. B. Pearson, B. A. Francis, and A. Tannenbaum, "A lifting technique for linear periodic systems with applications to sampled-data control," *Syst. Contr. Lett.*, vol. 17, pp. 79–88, 1991.

[6] M. C. Berg, N. Amit, and J. Powell, "Multirate digital control system design," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 1139–1150, 1988.

[7] T. Chen and B. A. Francis, "$\mathcal{H}_2$-optimal sampled-data control," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 387–397, 1991.

[8] T. Chen and L. Qiu, "$\mathcal{H}_\infty$ design of general multirate sampled-data control systems," *Automatica*, vol. 30, pp. 1139–1152, 1994.

[9] T. Chen and B. A. Francis, "Linear time-varying $\mathcal{H}_2$-optimal control of sampled-data systems," *Automatica*, vol. 27, no. 6, pp. 963–974, 1991.

[10] M. A. Dahleh, P. G. Voulgaris, and L. S. Valavani, "Optimal and robust controllers for periodic and multirate systems," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 90–99, 1992.

[11] B. A. Francis, *A Course in $\mathcal{H}_\infty$ Control Theory.* New York: Springer-Verlag, 1987.

[12] B. A. Francis and T. T. Georgiou, "Stability theory for linear time-invariant plants with periodic digital controllers," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 820–832, 1988.

[13] I. C. Gohberg and M. G. Kreĭn, *Introduction to the Theory of Linear Nonselfadjoint Operators.* Providence, RI: American Mathematical Society, 1969.

[14] T. Hagiwara and M. Araki, "Design of a stable feedback controller based on the multirate sampling of the plant output," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 812–819, 1988.

[15] Y. Hayakawa, Y. Yamamoto, and S. Hara, "$\mathcal{H}_\infty$ type problem for sampled-data control system—A solution via minimum energy characterization," in *Proc. IEEE Conf. Decis. Contr.*, pp. 463–468, 1992.

[16] P. T. Kabamba and S. Hara, "Worst case analysis and design of sampled-data control systems," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 1337–1357, 1993.

[17] R. Kalman, Y. C. Ho, and K. Narendra, "Controllability of linear dynamical systems," in *Contributions to Differential Equations*, vol. 1. New York: Interscience, 1963.

[18] P. P. Khargonekar and N. Sivashankar, "$\mathcal{H}_2$ optimal control for sampled-data systems," *Syst. Contr. Lett.*, vol. 18, no. 3, pp. 627–631, 1992.

[19] P. P. Khargonekar, K. Poolla, and A. Tannenbaum, "Robust control of linear time-invariant plants using periodic compensation," *IEEE Trans. Automat. Contr.*, vol. 30, pp. 1088–1096, 1985.

[20] G. M. Kranc, "Input–output analysis of multirate feedback systems," *IRE Trans. Automat. Control*, vol. 3, pp. 21–28, 1957.

[21] D. G. Meyer, "A parameterization of stabilizing controllers for multirate sampled-data systems," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 233–236, 1990.

[22] ——, "Cost translation and a lifting approach to the multirate LQG problem," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1411–1415, 1992.

[23] R. Ravi, P. P. Khargonekar, K. D. Minto, and C. N. Nett, "Controller parameterization for time-varying multirate plants," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 1259–1262, 1990.

[24] N. Sivashankar and P. P. Khargonekar, "Characterization of the $\mathcal{L}_2$-induced norm for linear systems with jumps with applications to sampled-data systems," preprint, 1991.

[25] W. Sun, K. M. Nagpal, and P. P. Khargonekar, "$\mathcal{H}_\infty$ control and filtering with sampled measurements," in *Proc. Amer. Contr. Conf.*, pp. 1652–1657, 1991.

[26] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space.* Amsterdam: North-Holland, 1970.

[27] G. Tadmor, "$\mathcal{H}_\infty$ optimal sampled-data control in continuous time systems," *Int. J. Contr.*, vol. 56, no. 1, pp. 99–141, 1992.

[28] H. T. Toivonen, "Sampled-data control of continuous-time systems with an $\mathcal{H}_\infty$ optimality criterion," *Automatica*, vol. 28, no. 1, pp. 45–54, 1992.

[29] P. G. Voulgaris and B. Bamieh, "Optimal $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control of hybrid multirate systems," *Syst. Contr. Lett.*, vol. 20, pp. 249–261, 1993.

[30] P. G. Voulgaris, M. A. Dahleh, and L. S. Valavani, "$\mathcal{H}_\infty$ and $\mathcal{H}_2$ optimal controllers for periodic and multi-rate systems,"*Automatica*, vol. 32, pp. 251–264, 1994.

[31] Y. Yamamoto, "A new approach to sampled-data control systems: A function space method," in *Proc. IEEE Conf. Decis. Contr.*, 1990, pp. 1882–1887.

An Exact Solution to General SISO Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Problems via Convex Optimization

Mario Sznaier

Abstract—**The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem can be motivated as a nominal LQG optimal control problem, subject to robust stability constraints, expressed in the form of an $\mathcal{H}_\infty$ norm bound. A related modified problem consisting on minimizing an upper bound of the $\mathcal{H}_2$ cost subject to $\mathcal{H}_\infty$ constraints was introduced in [1]. Although there presently exist efficient methods to solve this modified problem, the original problem remains, to a large extent, still open. In this paper we propose a method for solving general discrete-time SISO $\mathcal{H}_2/\mathcal{H}_\infty$ problems. This method involves solving a sequence of problems, each one consisting of a finite-dimensional convex optimization and an unconstrained Nehari approximation problem.**

I. INTRODUCTION

During the last decade, a large research effort has been devoted to the problem of designing robust controllers capable of guaranteeing stability in the face of plant uncertainty. As a result, a powerful $\mathcal{H}_\infty$ framework has been developed, addressing the issue of robust stability in the presence of norm-bounded plant perturbations. Since its introduction, the original formulation of Zames [2] has been substantially simplified, resulting in efficient computational schemes for finding solutions. Of particular importance is [3] where a state-space approach is developed and an efficient procedure is given to compute suboptimal $\mathcal{H}_\infty$ controllers. Since these controllers are not unique, the extra degrees of freedom available can then be used to

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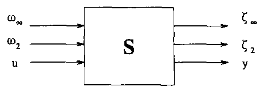

Fig. 1. The generalized plant.

optimize some performance measure. This leads naturally to a robust performance problem: Design a controller guaranteeing a desired level of performance in the face of plant uncertainty. In spite of a large research effort [4], however, this problem has not been completely solved.

Alternatively, the extra degrees of freedom can be used to solve a problem of the form nominal performance with robust stability. In this case the controller yields a desired performance level for the nominal system while guaranteeing stability for all possible plant perturbations. A problem of this form that has been the object of much attention lately is the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem: Given the system represented by the block diagram in Fig. 1, where the scalar signals w_∞ (an l^2 signal) and w_2 (white noise) represent exogenous disturbances, u represents the control action, ζ_∞ and ζ_2 represent regulated outputs, and where y represents the measurements; find an internally stabilizing controller $u(z) = K(z)y(z)$ such that the root mean square (RMS) value of the performance output ζ_2 due to w_2 is minimized, subject to the specification $\|T_{\zeta_\infty w_\infty}(z)\|_\infty \leq \gamma$.

Different versions of this problem have been studied recently. Bernstein and Haddad [1] considered the case where $w_2 = w_\infty$ and obtained necessary conditions for solving the modified problem of minimizing an upper bound of $\|T_{w_2\zeta_2}\|_2$, subject to the $\mathcal{H}_\infty$ constraint. In [5] and [6] the dual problem of minimizing this upper bound for the case $w_2 \neq w_\infty$, $\zeta_2 = \zeta_\infty$ was considered and sufficient conditions for optimality where given. Finally, in [7] these conditions where shown to be necessary and sufficient. These conditions involve solving several coupled Riccati equations, however, and at this point there are no effective procedures for achieving this. In [8], Khargonekar and Rotea (see also [9] for the discrete-time version) showed that the modified problem can be cast into the format of a constrained convex optimization problem over a bounded set of matrices and solved using nondifferentiable optimization techniques.

The approaches mentioned above provide a solution to the modified problem. At this time, however, there is no information regarding the gap between the upper bound minimized in the modified problem and the true $\mathcal{H}_2$ cost. Very little work has been done concerning the original problem, which remains, to a large extent, still open. In [10], Rotea and Khargonekar addressed a simultaneous $\mathcal{H}_2/\mathcal{H}_\infty$ state-feedback control problem and showed that a solution to this problem, when it exists, also solves the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem. Although this provides some insight into the structure of the problem, there are cases (most notably the case where $B_1 = B_2$) where the simultaneous problem provides little help in solving the original problem. Recently, mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control using fixed-order controllers was analyzed using a Lagrange multipliers-based approach and necessary conditions for optimality were obtained [11]. These conditions involve solving coupled nonlinear matrix equations and finding the neutrally stable solution to a Lyapunov equation, which leads to numerical difficulties. Moreover, in [10] it was shown that even in the state-feedback case, the optimal controller must be dynamic, and it is conjectured that in the general case it may have higher order than the plant. This makes a fixed-order approach less attractive, since there is little *a priori* information on the order of the optimal controller.

In this paper we propose a solution to general discrete-time single-input single-output (SISO) mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems. The main result of the paper shows that these problems can be exactly solved by solving a sequence of modified problems, each one requiring the solution of a finite dimensional convex, constrained optimization problem, and an unconstrained Nehari approximation problem. This approach, which follows the spirit of [12], [13], represents a significant departure from other convex optimization-based approaches (e.g., [14]) where several approximations are required to obtain a tractable mathematical problem.

The paper is organized as follows: In Section II we introduce the notation to be used and some preliminary results. Section III contains the bulk of the theoretical results and the proposed solution method. In Section IV we present a simple design example. Finally, in Section V, we summarize our results and indicate directions for future research.

II. Preliminaries

A. Notation

By l^p we denote the space of real sequences $q = \{q_i\}$, equipped with the norm $\|q\|_p = \left(\sum_{k=0}^{\infty} |q_k|^p\right)^{\frac{1}{p}} < \infty$. $\mathcal{L}_\infty$ denotes the Lebesgue space of complex valued transfer functions which are essentially bounded on the unit circle with norm $\|T(z)\|_\infty \triangleq \sup_{|z|=1} |T(z)|$. $\mathcal{H}_\infty$ ($\mathcal{H}_\infty^-$) denotes the set of stable (antistable) complex functions $G(z) \in \mathcal{L}_\infty$, i.e., analytic in $|z| \geq 1$ ($|z| \leq 1$). $\mathcal{H}_2$ denotes the space of complex transfer functions square integrable in the unit circle and analytic in $|z| > 1$, equipped with the norm

$$\|G\|_2^2 = \frac{1}{2\pi} \oint_{|z|=1} \frac{|G(z)|^2}{z} \, dz.$$

Given $R \in \mathcal{L}_\infty$, $\Gamma_H(R)$ denotes its maximum Hankel singular value. Given a sequence $q \in l_1$ we will denote its z-transform by $Q(z)$. It is a standard result that $q \in \mathcal{R}l1$ iff $Q(z) \in \mathcal{RH}_\infty$. Throughout the paper we will use the prefix $\mathcal{R}$ to denote real rational transfer matrices and packed notation to represent their state-space realizations, i.e.,

$$G(z) = C(zI - A)^{-1}B + D \triangleq \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right).$$

Given two transfer matrices $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ and Q with appropriate dimensions, the lower linear fractional transformation is defined as: $\mathcal{F}_l(T, Q) \triangleq T_{11} + T_{12}Q(I - T_{22}Q)^{-1}T_{21}$. Finally, for a transfer matrix $G(z)$, $G^\sim \triangleq G^T((1/z))$.

B. Problem Transformation

Assume that the system S has the following state-space realization (where without loss of generality we assume that all weighting factors have been absorbed into the plant)

$$\left(\begin{array}{c|ccc} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ C_3 & D_{31} & D_{32} & D_{33} \end{array}\right) \qquad \text{(S)}$$

where D_{13} has full column rank, D_{31} has full row rank, and where the pairs (A, B_3) and (C_3, A) are stabilizable and detectable, respectively. It is well known (see, for instance, [4]) that the set of all internally stabilizing controllers can be parameterized in terms of a free parameter $Q \in \mathcal{H}_\infty$ as

$$K = \mathcal{F}_l(J, Q) \qquad (1)$$

where J has the following state-space realization

$$\left(\begin{array}{c|cc} A + B_3F + LC_3 + LD_{33}F & -L & B_3 + LD_{33} \\ \hline F & 0 & I \\ -(C_3 + D_{33}F) & I & -D_{33} \end{array}\right) \qquad \text{(J)}$$

and where F and L are selected such that $A + B_3F$ and $A + LC_3$ are stable. By using this parameterization, the closed-loop transfer functions $T_{\zeta_\infty w_\infty}$ and $T_{\zeta_2 w_2}$ can be written as

$$\begin{aligned} T_{\zeta_\infty w_\infty}(z) &= T_1^\infty(z) + T_2^\infty(z)Q(z) \\ T_{\zeta_2 w_2}(z) &= T_1(z) + T_2(z)Q(z) \end{aligned} \tag{2}$$

where T_i, T_i^∞ are stable transfer functions. Moreover (see, for instance, [4], [13]), it is possible to select F and L in such a way that $T_2^\infty(z)$ is inner (i.e., $T_2^{\infty\sim}T_2^\infty = I$). By using this parameterization the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem can be now precisely stated as the following.

Problem 1: Find the optimal value of the performance measure

$$\mu^o = \inf_{Q \in \mathcal{RH}_\infty} \|T_{\zeta_2 w_2}\|_2 = \inf_{q_i \in l^1} \left(\sum_{i=0}^{\infty} |t_i|^2 \right)^{\frac{1}{2}} \qquad (\mathcal{H}_2/\mathcal{H}_\infty)$$

subject to

$$\|T_1^\infty(z) + T_2^\infty(z)Q(z)\|_\infty \le \gamma \tag{3}$$

where $\{t_i\}$ and $\{q_i\}$ are the coefficients of the impulse responses of $T_{\zeta_2 w_2}$ and Q, respectively.

Remark 1: In the sequel we will assume that $\inf_{Q \in \mathcal{RH}_\infty} \|T_1^\infty + T_2^\infty Q\|_\infty \triangleq \gamma^* < \gamma$. This assumption guarantees both the existence of suboptimal $\mathcal{H}_\infty$ controllers and nontrivial solutions to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem.

III. Problem Solution

In this section we show that the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem can be solved by solving a sequence of problems, each one requiring the solution of a finite dimensional convex optimization problem and an unconstrained Nehari extension problem. To establish this result we will proceed as follows:

i) introduce a modified $\mathcal{H}_2/\mathcal{H}_\infty$ problem,

ii) show that the original problem can be solved by solving a sequence of modified problems (Lemma 1),

iii) show that an approximate solution (arbitrarily close to the optimum) to each modified problem can be found by solving a truncated problem (Lemma 3),

iv) show that solving the truncated problem entails solving a finite dimensional convex optimization problem and a standard, unconstrained, $\mathcal{H}_\infty$ problem (Theorem 2).

A. A Modified $\mathcal{H}_2/\mathcal{H}_\infty$ Problem

Given $\delta < 1$, define the space $\mathcal{RH}_{\infty,\delta} \triangleq \{Q(z) \in \mathcal{RH}_\infty$: $Q(z)$ analytic in $|z| \ge \delta\}$, equipped with the norm $\|Q\|_{\infty,\delta} \triangleq \sup_{|z|=\delta} |Q(z)|$, and consider the following modified $\mathcal{H}_2/\mathcal{H}_\infty$ problem.

Problem $\mathcal{H}_2/\mathcal{H}_{\infty,\delta}$: Given $T_1(z), T_2(z), T_1^\infty(z), T_2^\infty(z) \in \mathcal{RH}_{\infty,\delta}$, find

$$\mu_\delta^o = \inf_{Q \in \mathcal{RH}_{\infty,\delta}} \|T_{\zeta_2 w_2}\|_2 \qquad (\mathcal{H}_2/\mathcal{H}_{\infty,\delta})$$

subject to

$$\|T_1^\infty(z) + T_2^\infty(z)Q(z)\|_{\infty,\delta} \le \gamma.$$

Remark 2: From the maximum modulus theorem, it follows that any controller Q that is admissible for $\mathcal{H}_2/\mathcal{H}_{\infty,\delta}$ is also admissible for $\mathcal{H}_2/\mathcal{H}_\infty$. It follows that μ_δ^o is an upper bound for μ^o.

Remark 3: Problem $\mathcal{H}_2/\mathcal{H}_{\infty,\delta}$ can be thought as solving problem $\mathcal{H}_2/\mathcal{H}_\infty$ with the additional constraint that all the poles of the closed-loop system must be inside the disk of radius δ. A parameterization of all achievable closed-loop transfer functions, such that T_i, T_i^∞ satisfy this additional constraint can be obtained from (1) by simply changing the stability region from the unit-disk to the δ-disk using the transformation $z = \delta\hat{z}$ before performing the factorization. Furthermore, by combining this transformation with the inner factorization, the resulting $T_2^\infty(z)$ satisfies $T_2^\infty(\delta z)T_2^\infty((1/\delta z)) = 1$. It follows then that the constraint $\|T_1^\infty(z) + T_2^\infty(z)Q\|_{\infty,\delta} \le \gamma$ is equivalent to $\|R + Q\|_{\infty,\delta} \le \gamma$ where $R \triangleq T_1^\infty(z)T_2^\infty(z)^\sim$ is analytic in the disk $|z| \le \delta$.

Next we show that a rational suboptimal solution to $\mathcal{H}_2/\mathcal{H}_\infty$, with cost arbitrarily close to the optimum, can be found by solving a sequence of truncated problems, each one requiring consideration of only a finite number of elements of the impulse response of $T_{\zeta_2 w_2}$. To establish this result we will show that: i) $\mathcal{H}_2/\mathcal{H}_\infty$ can be solved by considering a sequence of modified problems $\mathcal{H}_2/\mathcal{H}_{\infty,\delta}$; and ii) given $\epsilon > 0$, a suboptimal solution to $\mathcal{H}_2/\mathcal{H}_{\infty,\delta}$, with cost μ_δ^ϵ such that $\mu_\delta^o \le \mu_\delta^\epsilon \le \mu_\delta^o + \epsilon$ can be found by solving a truncated problem.

Lemma 1: Consider an increasing sequence $\delta_i \to 1$. Let μ^o and μ_i denote the solution to problems $\mathcal{H}_2/\mathcal{H}_\infty$ and $\mathcal{H}_2/\mathcal{H}_{\infty,\delta_i}$, respectively. Then the sequence $\mu_i \to \mu^o$.

Proof: See Appendix A.

Lemma 2: For every $\epsilon > 0$, there exists $N(\epsilon, \delta)$ such that if $Q \in \mathcal{RH}_{\infty,\delta}$ satisfies the constraint $\|R(z) + Q(z)\|_{\infty,\delta} \le \gamma$, it also satisfies $\sum_{i=N}^\infty |t_k|^2 \le \epsilon^2$, where t_k denote the coefficients of the impulse response of $T_{\zeta_2 w_2} = T_1 + T_2 Q$.

Proof: Since $Q \in \mathcal{RH}_{\infty,\delta}$, $T_{\zeta_2 w_2}$ is analytic in $|z| \ge \delta$ and

$$t_k = \frac{1}{2\pi j} \oint_{|z|=\delta} T_{\zeta_2 w_2}(z) z^{k-1}\, dz. \tag{4}$$

Hence

$$\begin{aligned} |t_k| &\le \|T_{\zeta_2 w_2}\|_{\infty,\delta}\delta^k \\ \sum_{i=N}^{\infty} |t_k|^2 &\le \frac{\|T_{\zeta_2 w_2}\|_{\infty,\delta}^2 \delta^{2N}}{1-\delta^2}. \end{aligned} \tag{5}$$

Since $\|\cdot\|_{\infty,\delta}$ is submultiplicative, we have

$$\begin{aligned} \|T_{\zeta_2 w_2}(z)\|_{\infty,\delta} &\le \|T_1\|_{\infty,\delta} + \|T_2\|_{\infty,\delta}\|Q\|_{\infty,\delta} \\ &\le \|T_1\|_{\infty,\delta} + \|T_2\|_{\infty,\delta}(\gamma + \|R\|_{\infty,\delta}) \triangleq K. \end{aligned} \tag{6}$$

The desired result follows by selecting $N \ge N_o = (1/2)((\log \epsilon^2(1-\delta^2) - \log K^2)/\log \delta)$. ◊

Lemma 3: Consider the following optimization problem

$$\min_{Q \in \mathcal{RH}_{\infty,\delta}} \left(\sum_{i=0}^{N(\epsilon,\delta)-1} |t_i|^2 \right)^{1/2} = \min_{q \in R^{N(\epsilon,\delta)}} \|\underline{t}_1 + \tau \underline{q}\|_2^2 \qquad (\mathcal{H}_2/\mathcal{H}_{\infty,\delta}^\epsilon)$$

subject to

$$\|R(z) + Q(z)\|_{\infty,\delta} \le \gamma$$

where

$$\underline{t}_1 \triangleq (t_{1o} \quad \cdots \quad t_{1N-1})'$$

$$\tau = \begin{pmatrix} t_{2o} & 0 & \cdots & 0 \\ t_{21} & t_{20} & \cdots & 0 \\ \vdots & & \ddots & \\ t_{2N-1} & \cdots & & t_{2o} \end{pmatrix} \tag{7}$$

$$\underline{q} \triangleq (q_o \quad \cdots \quad q_{N-1})'$$

and where q_k, t_{k_i} denote the kth element of the impulse response of $Q(z)$, $T_i(z)$ respectively. Let Q^* and $T^*_{\zeta_2 w_2}$ denote the optimal solution and define $\mu^\epsilon_\delta = \|T^*_{\zeta_2 w_2}\|_2$. Then $\mu^o_\delta \le \mu^\epsilon_\delta \le \mu^o_\delta + \epsilon$.

Proof: $\mu^o_\delta \le \mu^\epsilon_\delta$ is immediate from the definition of μ^o_δ. Denote by $T^\epsilon_{\zeta_2 w_2}$ and $T^\delta_{\zeta_2 w_2}$ the solution to problems $\mathcal{H}_2/\mathcal{H}^\epsilon_{\infty,\delta}$ and $\mathcal{H}_2/\mathcal{H}_{\infty,\delta}$ respectively and let t^ϵ_i, t^δ_i be the corresponding impulse responses. Then

$$
\begin{aligned}
(\mu^\epsilon_\delta)^2 = \|T^\epsilon_{\zeta_2 w_2}\|_2^2 &= \sum_{i=0}^{\infty}|t^\epsilon_i|^2 = \sum_{i=0}^{N-1}|t^\epsilon_i|^2 + \sum_{i=N}^{\infty}|t^\epsilon_i|^2 \\
&\le \sum_{i=0}^{N-1}|t^\epsilon_i|^2 + \epsilon^2 \le \sum_{i=0}^{N-1}|t^\delta_i|^2 + \epsilon^2 \le \sum_{i=0}^{\infty}|t^\delta_i|^2 + \epsilon^2 \\
&= (\mu^o_\delta)^2 + \epsilon^2 \le (\mu^o_\delta + \epsilon)^2. \qquad \diamond
\end{aligned}
$$

By combining the results of Lemmas 1, 2, and 3, the following result is now apparent.

Lemma 4: Consider an increasing sequence $\delta_i \to 1$. Let μ^o and $\mu^\epsilon_{\delta_i}$ denote the solution to problems $\mathcal{H}_2/\mathcal{H}_\infty$ and $\mathcal{H}_2/\mathcal{H}^\epsilon_{\infty,\delta_i}$ respectively. Then the sequence $\mu^\epsilon_{\delta_i}$ has an accumulation point $\hat{\mu}_\epsilon$ such that $\mu^o \le \hat{\mu}_\epsilon \le \mu^o + \epsilon$.

B. The $\mathcal{H}_\infty$ Constraint

In the last section we showed that $\mathcal{H}_2/\mathcal{H}_\infty$ can be solved by solving a sequence of truncated problems. In this section we show that each problem $\mathcal{H}_2/\mathcal{H}^\epsilon_{\infty,\delta}$ can be exactly solved by solving a finite dimensional convex optimization problem and an unconstrained Nehari approximation problem. Moreover, since the solution to this Nehari approximation problem is rational, it follows that the solution to $\mathcal{H}_2/\mathcal{H}^\epsilon_{\infty,\delta}$ is also rational. The key to establish this result is to note that: i) the objective function of the truncated problem involves only the first N terms of the impulse response of Q, and ii) if the first N terms of the impulse response of Q are fixed, the existence of Q such that $\|R + Q\|_{\infty,\delta} \le \gamma$ is equivalent to a finite dimensional convex constraint on these elements.

Theorem 1: Let $R(z) \in \mathcal{RH}^-_{\infty,\delta}$ and $Q_F = \sum_{i=0}^{N-1} q_i z^{-i}$ be given. Assume that $G(z) \triangleq R(\delta z)^\sim$ has a minimal state space realization

$$
G = \left(\begin{array}{c|c} A_g & b_g \\ \hline c_g & d_g \end{array}\right)
$$

with controllability and observability gramians L_{cg} and L_{og} respectively. Then, there exist $Q_R \in \mathcal{RH}_{\infty,\delta}$ such that $\|R + Q_F + z^{-N}Q_R\|_{\infty,\delta} \le \gamma$ if and only if $\|\mathcal{Q}\|_2 \le \gamma$ where

$$
\mathcal{Q} = \begin{pmatrix}
L_{og}^{1/2}A_g^N L_{cg}^{1/2} & L_{og}^{1/2}b_g & L_{og}^{1/2}A_g b_g & \cdots & L_{og}^{1/2}A_g^{N-1}b_g \\
c_g A_g^{N-1}L_{cg}^{1/2} & d_g & c_g b_g & \cdots & c_g A_g^{N-2}b_g \\
\vdots & 0 & d_g & \ddots & \vdots \\
c_g A_g L_{cg}^{1/2} & \vdots & \vdots & \ddots & c_g b_g \\
c_g L_{cg}^{1/2} & 0 & 0 & \cdots & d_g
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & \cdots & \cdots & 0 \\
0 & q_o & q_1\delta^{-1} & \cdots & q_{N-1}\delta^{-(N-1)} \\
0 & 0 & q_o & \cdots & q_{N-2}\delta^{-(N-2)} \\
0 & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & q_o
\end{pmatrix}. \tag{8}
$$

Proof: [1]For simplicity, we will consider the case where $\delta = 1$. The case $\delta < 1$ follows by using the transformation $z = \delta\hat{z}$. Given Q_F, there exist $Q_R \in \mathcal{RH}_\infty$ such that $\|R + Q_F + z^{-N}Q_R\|_\infty \le \gamma$ iff the corresponding unconstrained 1 block Nehari approximation problem has a solution, i.e., if

$$
\begin{aligned}
\min_{Q_R\in\mathcal{RH}_\infty}\|R + Q_F + z^{-N}Q_R\|_\infty
&= \min_{Q_R\in\mathcal{RH}_\infty}\|z^N(R + Q_F) + Q_R\|_\infty \\
&= \min_{Q_R\in\mathcal{RH}^-_\infty}\|z^{-N}(G + Q_F^\sim) + Q_R\|_\infty \\
&= \Gamma_H[z^{-N}(G + Q_F^\sim)] \le \gamma
\end{aligned} \tag{9}
$$

where we used the fact that z^N is an inner function. To compute Γ_H we need a space-state realization for the stable part of $z^{-N}(G+Q_F^\sim)$. Standard space-state manipulations [4] yield

$$
F_1 \triangleq Gz^{-N} = \left(\begin{array}{cc|c} A_g & b_g e'_N & 0 \\ 0 & A_q & e_1 \\ \hline c_g & d_g e'_N & 0 \end{array}\right) \tag{10}
$$

where

$$
A_q = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}
$$

$$
e'_1 = (1 \ \cdots \ 0), \qquad e'_N = (0 \ \cdots \ 1).
$$

The similarity transformation

$$
T = \begin{pmatrix} I_n & Y \\ 0 & I_N \end{pmatrix} \tag{11}
$$

where Y is the unique solution to the Sylvester equation

$$
YA_q - A_gY = -b_g e'_N \tag{12}
$$

yields

$$
F_1 = \left(\begin{array}{cc|c} A_g & 0 & Ye_1 \\ 0 & A_q & e_1 \\ \hline c_g & d_g e'_N - c_g Y & 0 \end{array}\right). \tag{13}
$$

Similarly

$$
F_2 \triangleq z^{-N}Q_F^\sim = \sum_{i=1}^{N-1} q_{N-i}z^{-i} = \left(\begin{array}{c|c} A_q & e_1 \\ \hline c_q & 0 \end{array}\right) \tag{14}
$$

where $c_q = (q_{N-1}\cdots q_0)$. Hence

$$
F \triangleq z^{-N}(G + Q_F^\sim) = F_1 + F_2 = \left(\begin{array}{cc|c} A_g & 0 & Ye_1 \\ 0 & A_q & e_1 \\ \hline c_g & H & 0 \end{array}\right) \tag{15}
$$

where $H \triangleq c_q + d_g e'_N - c_g Y \triangleq (h_1 \cdots h_N)$. It can be easily shown, by successive right multiplications by the columns of the identity that the solution to (12) is given by:[2] $Y = (A_g^{-N}b_g \ A_g^{-(N-1)}b_g \cdots A_g^{-1}b)$. Substituting this explicit expression for Y into (15) yields

$$
\begin{aligned}
F &= \left(\begin{array}{cc|c} A_g & 0 & A_g^{-N}b_g \\ 0 & A_q & e_1 \\ \hline c_g & H & 0 \end{array}\right) \\
h_i &= q_{N-i} - c_g A_g^{-(N-i+1)}b_g \qquad 1 \le i \le N-1 \\
h_N &= q_0 + d_g - c_g A_g^{-1}b_g.
\end{aligned} \tag{16}
$$

[1] While this paper was under review, an independent proof and equivalent formulas appeared in [15].

[2] Since $R(z)$ is antistable, its conjugate G is stable and has no poles at the origin. Hence A_g^{-1} is well defined.

To compute the approximation error we need to compute the observability and controllability gramians of F. For the controllability gramian we have

$$\begin{pmatrix} A_g & 0 \\ 0 & A_q \end{pmatrix}\begin{pmatrix} L_{11}^C & L_{12}^C \\ L_{12}'^C & L_{22}^C \end{pmatrix}\begin{pmatrix} A_g & 0 \\ 0 & A_q \end{pmatrix}' - \begin{pmatrix} L_{11}^C & L_{12}^C \\ L_{12}'^C & L_{22}^C \end{pmatrix} = -\begin{pmatrix} A_g^{-N} b_g b_g' A_g^{-N'} & A_g^{-N} b_g e_1' \\ e_1 b_g' A_g^{-N'} & e_1 e_1' \end{pmatrix}. \quad (17)$$

Solving for each of the blocks of the gramian yields $L_{11}^C = L_o^C$, $L_{12}^C = Y$, and $L_{22}^C = I_N$, where L_o^C is the solution of the following Lyapunov equation

$$A_g L_o^C A_g' - L_o^C = -A_g^{-N} b_g b_g' A_g^{-N'} \quad (18)$$

and where the expression for L_{12}^C was obtained from the corresponding equation by successive right multiplications by the columns of the identity. Similarly, for the observability gramian we have

$$\begin{pmatrix} A_g & 0 \\ 0 & A_q \end{pmatrix}'\begin{pmatrix} L_{11}^0 & L_{12}^0 \\ L_{12}'^0 & L_{22}^0 \end{pmatrix}\begin{pmatrix} A_g & 0 \\ 0 & A_q \end{pmatrix} - \begin{pmatrix} L_{11}^0 & L_{12}^0 \\ L_{12}'^0 & L_{22}^0 \end{pmatrix} = -\begin{pmatrix} c_g' c_g & c_g' H \\ H' c_g & H' H \end{pmatrix}. \quad (19)$$

Solving for each of the blocks of the gramian yields $L_{11}^0 = L_{og}$, $L_{12}^0 = \mathcal{A}\mathcal{H}'$, and $L_{22}^0 = \mathcal{H}\mathcal{H}'$ where

$$\mathcal{H} \triangleq \begin{pmatrix} h_N & h_{N-1} & \cdots & \cdots & h_1 \\ & h_N & h_{N-1} & \cdots & h_2 \\ & & \ddots & & \\ & & & h_N & h_{N-1} \\ & & & & h_N \end{pmatrix}. \quad (20)$$

$$\mathcal{A} = (A_g^{N-1'} c_g' \quad A_g^{N-2'} c_g' \cdots c_g').$$

Finally, simple computations show that L_{og} satisfies

$$L_{og} - \mathcal{A}\mathcal{A}' = A_g'^N L_{og} A_g^N \quad (21)$$

and that $L_c^0 - YY' = L_{cg}$, the controllability gramian of G. Using these facts we get the following explicit expressions for the gramians of F

$$L_{oF} = \begin{pmatrix} I_n & 0 \\ 0 & \mathcal{H} \end{pmatrix}\begin{pmatrix} L_{og} & \mathcal{A} \\ \mathcal{A}' & I_N \end{pmatrix}\begin{pmatrix} I_n & 0 \\ 0 & \mathcal{H}' \end{pmatrix} = \begin{pmatrix} A_g'^N & \mathcal{A} \\ 0 & \mathcal{H} \end{pmatrix}\begin{pmatrix} L_{og} & 0 \\ 0 & I_N \end{pmatrix}\begin{pmatrix} A_g^N & 0 \\ \mathcal{A}' & \mathcal{H}' \end{pmatrix}$$

$$L_{cF} = \begin{pmatrix} L_{cg} & Y \\ Y' & I_N \end{pmatrix} = \begin{pmatrix} I_n & Y \\ 0 & I_N \end{pmatrix}\begin{pmatrix} L_{cg} & 0 \\ 0 & I_N \end{pmatrix}\begin{pmatrix} I_n & 0 \\ Y' & I_N \end{pmatrix}. \quad (22)$$

Define

$$\mathcal{M} \triangleq \begin{pmatrix} L_{og}^{1/2} & 0 \\ 0 & I_N \end{pmatrix}\begin{pmatrix} A_g^N & 0 \\ \mathcal{A}' & \mathcal{H}' \end{pmatrix}\begin{pmatrix} L_{cg}^{1/2} & 0 \\ 0 & I_N \end{pmatrix}\begin{pmatrix} I_n & 0 \\ Y' & I_N \end{pmatrix}. \quad (23)$$

From Nehari Theorem ([16]) it follows that

$$\|F+Q_R\|_\infty \le \gamma \Leftrightarrow \rho^{1/2}(L_{cF}^{1/2} L_{oF} L_{cF}^{1/2}) \le \gamma \Leftrightarrow \|\mathcal{M}\|_2 \le \gamma \quad (24)$$

where ρ indicates the spectral radius. Note that since $\mathcal{M}$ is a linear function of the coefficients of Q_F, the constraint (24) is convex in the variables q_i. Although the expression $\mathcal{M}$ can be used to establish the desired result, it may result in numerically ill-conditioned problems, since it involves powers of A_g^{-1}, which has all its eigenvalues outside the unit circle. To obtain an alternative expression that avoids this ill-conditioning, consider again the similarity transformation (11).

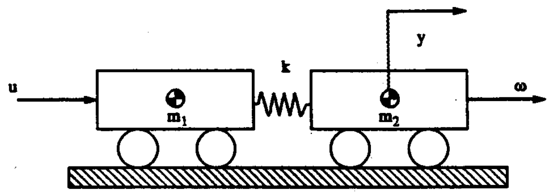

Fig. 2. The ACC robust control benchmark problem.

Since the spectral radius of $L_{oF}L_{cF}$ is invariant under a similarity transformation, it follows that $\mathcal{M}$ can be replaced by

$$\mathcal{Q} = \begin{pmatrix} L_{og}^{1/2} & 0 \\ 0 & I_N \end{pmatrix}\begin{pmatrix} A_g^N & A_g^N Y \\ \mathcal{A}' & \mathcal{H}' - \mathcal{A}'Y \end{pmatrix}\begin{pmatrix} L_{cg}^{1/2} & 0 \\ 0 & I_N \end{pmatrix}$$

$$= \begin{pmatrix} L_{og}^{1/2} A_g^N L_{cg}^{1/2} & L_{og}^{1/2} b_g & L_{og}^{1/2} A_g b_g & \cdots & L_{og}^{1/2} A_g^{N-1} b_g \\ c_g A_g^{N-1} L_{cg}^{1/2} & d_g & c_g b_g & \cdots & c_g A_g^{N-2} b_g \\ \vdots & 0 & d_g & \ddots & \vdots \\ c_g A_g L_{cg}^{1/2} & \vdots & \vdots & \ddots & c_g b_g \\ c_g L_{cg}^{1/2} & 0 & 0 & \cdots & d_g \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & q_o & q_1 & \cdots & q_{N-1} \\ 0 & 0 & q_o & \cdots & q_{N-2} \\ 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & q_o \end{pmatrix}. \quad (25)$$

Hence $\Gamma_H(F) \le \gamma \Leftrightarrow \|\mathcal{Q}\|_2 \le \gamma$. ◊

Combining Lemma 3 and Theorem 1 yields the main result of this section.

Theorem 2: A suboptimal solution to $\mathcal{H}_2/\mathcal{H}_{\infty,\delta}$, with cost $\mu_\delta \le \mu_\delta^\epsilon \le \mu_\delta + \epsilon$ is given by $Q^o = Q_F^o + z^{-N} Q_R^o$ where $Q_F^o = \sum_{i=0}^{N-1} q_i z^{-i}$, $\underline{q}^o = (q_o \cdots q_{N-1})'$ solves the following finite dimensional convex optimization problem

$$\underline{q}^o = \arg\min_{\substack{\underline{q} \in R^N \\ \|\mathcal{Q}\|_2 \le \gamma}} \|\underline{t}_1 + \tau \underline{q}\|_2 \quad (26)$$

and Q_R solves the unconstrained approximation problem

$$Q_R^o(z) = \arg\min_{Q_R \in \mathcal{RH}_{\infty,\delta}} \|R(z) + Q_F^o + z^{-N} Q_R(z)\|_{\infty,\delta} \quad (27)$$

where $N(\epsilon, \delta)$ is selected according to Lemma 2, t_1 and τ are defined in (7), and $\mathcal{Q}$ is defined in (8).

Remark 4: By using the transformation $z = \delta\hat{z}$ we have that

$$\begin{aligned} &\|R(z) + Q_F^o(z) + z^{-N} Q_R(z)\|_{\infty,\delta} \\ &= \|R(\delta\hat{z}) + Q_F^o(\delta\hat{z}) + \delta^{-N}\hat{z}^{-N} Q_R(\delta\hat{z})\|_\infty \\ &\triangleq \|\hat{R}(\hat{z}) + \hat{Q}_F^o(\hat{z}) + \hat{z}^{-N}\hat{Q}_R(\hat{z})\|_\infty \\ &= \|\hat{z}^N(\hat{R}(\hat{z}) + \hat{Q}_F^o(\hat{z})) + Q_R(\hat{z})\|_\infty. \end{aligned}$$

It follows that the approximation problem (27) is equivalent to the following standard Nehari approximation problem

$$\hat{Q}_R^o = \arg\min_{Q_R \in \mathcal{RH}_\infty} \|\hat{z}^N(\hat{R} + \hat{Q}_F^o) + Q_R\|_\infty \quad (28)$$

that can be readily solved (see, for instance, [16, p. 64]).

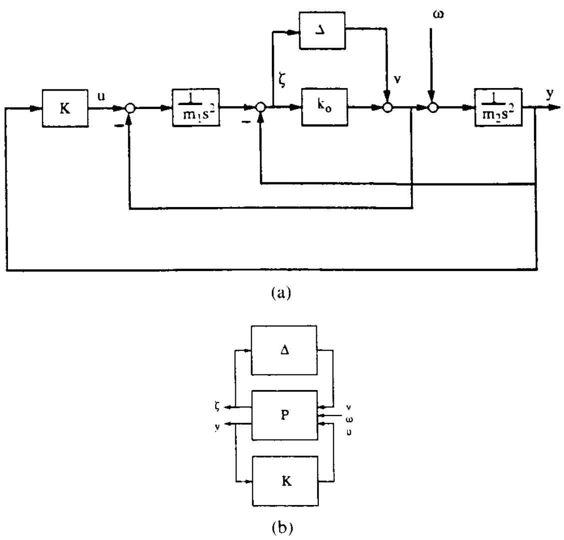

Fig. 3. (a) Block diagram with the uncertainty "pulled out" of the system, (b) the standard form.

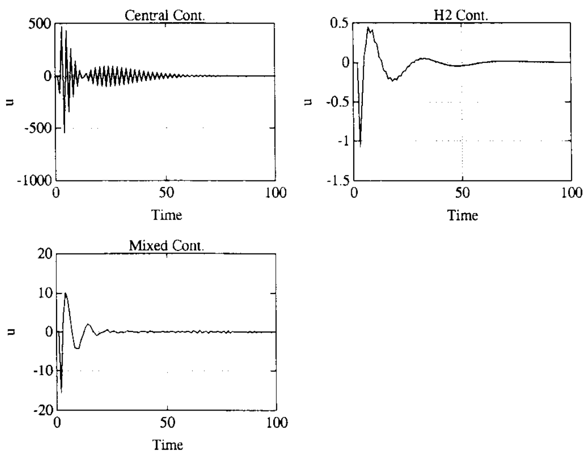

Fig. 4. $T_{\omega u}$ impulse response for the $\mathcal{H}_\infty$, $\mathcal{H}_2$ and $\mathcal{H}_2/\mathcal{H}_\infty$ cont.

TABLE I
$\|T_{\zeta v}\|_\infty$ AND $\|T_{u\omega}\|_2$ FOR THE EXAMPLE

	$\|T_{\zeta v}\|_\infty$	$\|T_{u\omega}\|_2$
$\mathcal{H}_2$	2.604	1.5760
$\mathcal{H}_\infty$	0.9977	1085.2
$\mathcal{H}_2/\mathcal{H}_\infty$	1.292	22.6493

C. Synthesis Algorithm

Combining Theorem 2 and Lemma 4, it follows that a suboptimal solution to $\mathcal{H}_2/\mathcal{H}_\infty$, with cost arbitrarily close to the optimum, can be found using the following iterative algorithm.

1) Data: An increasing sequence $\delta_i \to 1$, $\epsilon > 0$, $\nu > 0$.
2) Solve the unconstrained $\mathcal{H}_2$ problem (using the standard $\mathcal{H}_2$ theory). Compute $\|T_{\zeta_\infty w_\infty}\|_\infty$. If $\|T_{\zeta_\infty w_\infty}\|_\infty \leq \gamma$ stop, else set $i = 1$.
3) For each i, find a suboptimal solution to problem $\mathcal{H}_2/\mathcal{H}_{\infty,\delta_i}$ proceeding as follows:
   a) Obtain $T_i(z)$, $T_i^\infty(z) \in \mathcal{RH}_{\infty,\delta_i}$, with $T_2^\infty(z)$ inner in $\mathcal{RH}_{\infty,\delta_i}$. This can be accomplished by using the change of variable $z = \delta_i \hat{z}$ before performing the factorization (1).
   b) Compute $N(\epsilon, \delta_i)$ from Lemma 2.
   c) Find $Q(z)$ using Theorem 2.
4) Compute $\|T_{\zeta_\infty w_\infty}(z)\|_\infty$. If $\|T_{\zeta_\infty w_\infty}(z)\|_\infty \geq \gamma - \nu$ set $K = \mathcal{F}_l(J, Q)$ and stop, else set $i = i + 1$ and go to 2.

Remark 5: At each stage the algorithm produces a feasible solution to $\mathcal{H}_2/\mathcal{H}_\infty$, with cost μ_i which is an upper bound of the optimal cost μ^o.

IV. A Simple Example

Consider the simple system shown in Fig. 2, consisting of two unity masses coupled by a spring with constant $0.5 \leq k \leq 2$ but otherwise unknown. A control force acts on body 1 and the position of body 2 is measured, resulting in a noncolocated sensor actuator problem. This system has been used as a benchmark during the last few years at the American Control Conference [17] to highlight the issues and trade-offs involved in robust control design. Assume that it is desired to design an internally stabilizing controller subject to the following specifications: i) the closed-loop system must be stable for all possible values of the uncertain parameter $k \in [0.5, 2]$, and ii) the RMS value of the control action u in response to a white noise disturbance acting on m_2 should be minimized.

To fit the problem into the $\mathcal{H}_\infty$ framework, the uncertain spring constant k is modeled as $k = k_o + \Delta$ (with $k_o = 1.25$ and $|\Delta| \leq 0.75$) and, following a standard procedure, Δ is "pulled out" of the system, as shown in Fig. 3. The problem can be stated now as the problem of minimizing $\|T_{yw}\|_2$ over the set of all internally stabilizing controllers, subject to the constraint $\|T_{\zeta v}\|_\infty \leq \frac{4}{3}$.

To fit the problem into our framework, the system was discretized using sample and hold elements at the inputs and outputs, with a sampling time of 0.1 seconds. Finally, to remove the ill-conditioning caused by the poles on the unit circle, a bilinear transformation was used, constraining the poles of the closed-loop system to lie inside the $|z| \leq 0.95$ disk (i.e., $\delta = 0.95$) and the proposed design procedure was used with $\|T_{\zeta v}\|_\delta \leq 1.6$ and $N = 100$, resulting in a controller with 204 states.

Fig. 4 shows the control action in response to an impulse disturbance acting on m_2 for the optimal $\mathcal{H}_\infty$ central controller, the optimal $\mathcal{H}_2$ and the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controllers. These results are summarized in Table I.

Note that the actual value $\|T_{\zeta v}\|_\infty$ obtained with the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller is 1.29. This is due to the fact that $\|T_{\zeta v}\|_\delta$ is an upper bound of $\|T_{\zeta v}\|_\infty$.

Table II show a comparison between the optimal mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller and several reduced order controllers. It is interesting to notice that the controller can be reduced to tenth order with virtually no performance loss. Further reduction to a third-order controller only entails about 10% increase in the $\mathcal{H}_2$ cost. These results seem to support the conjecture of [11] that the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem results in controllers having higher dimension than the plant.

V. Conclusions

In this paper we provide a suboptimal solution to discrete-time mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems. Unlike previous approaches, our method yields a global minimum of the actual $\mathcal{H}_2$ cost rather than of an upper bound, and it is not limited to cases where either the disturbances or the regulated outputs coincide. Although here we considered only the

TABLE II
$\|T_{\zeta v}\|_\infty$ AND $\|T_{u\omega}\|_2$ FOR THE MIXED $\mathcal{H}_2/\mathcal{H}_\infty$ AND REDUCED ORDER CONTROLLERS

	$\|T_{\zeta v}\|_\infty$	$\|T_{u\omega}\|_2$
$\mathcal{H}_2/\mathcal{H}_\infty$	1.292	22.6493
10 ord.	1.281	22.8842
3 ord.	1.292	24.8594

simpler SISO case, we anticipate that the results will extend naturally to multi-input multi-output problems.

Perhaps the most severe limitation of the proposed method is that it may result in very large order controllers (roughly $2N$), necessitating some type of model reduction. Note, however, that this disadvantage is shared by some widely used design methods, such as μ-synthesis or l_1 optimal control theory, that will also produce controllers with no guaranteed complexity bound. Application of some well-established methods in order reduction (noteworthy, weighted balanced truncation) usually succeed in producing controllers of manageable order. The example of Section IV suggests that substantial order reduction can be accomplished without performance degradation. Research is currently under way addressing this issue.

APPENDIX A
PROOF OF LEMMA 1

Proof of Lemma 1: From the maximum modulus theorem, it follows that a controller Q_i that is admissible for $\mathcal{H}_2/\mathcal{H}_{\infty,\delta_i}$ is also admissible for $\mathcal{H}_2/\mathcal{H}_{\infty,\delta_{i+1}}$. Thus, the sequence μ_i is nonincreasing, bounded below by the value of $\|T_{\zeta_2 w_2}\|_2$ obtained when using the optimal $\mathcal{H}_2$ controller. It follows then that it has a limit $\mu \geq \mu^o$. We will show next that $\mu = \mu^o$. Assume by contradiction that $\mu^o < \mu$ and select $\mu^o < \hat{\mu} < \mu$. Since $\inf_{Q\in\mathcal{RH}_\infty}\|R+Q\|_\infty < \gamma$, it follows that there exists $Q_1 \in \mathcal{RH}_\infty$ such that $\|R+Q_1\|_\infty < \gamma$. From the definition of μ^o it follows that, given $\eta > 0$, there exists $Q_o \in \mathcal{RH}_\infty$, $\|R+Q_o\|_\infty \leq \gamma$, such that $\|T_{\zeta_2 w_2}(Q_o)\|_2 \leq \mu^o + \eta$. Let $\hat{Q} \triangleq Q_o + \epsilon(Q_1 - Q_o)$. It follows that

$$\|T_{\zeta_2 w_2}(\hat{Q})\|_2 \leq \mu^o + \eta + \epsilon\|T_2(Q_1 - Q_o)\|_2$$
$$\|R+\hat{Q}\|_\infty \leq \epsilon\|R+Q_1\|_\infty + (1-\epsilon)\|R+Q_o\|_\infty < \gamma.$$

Since $\hat{Q} \in \mathcal{RH}_\infty$ it follows that there exists $\delta_1 < 1$ such that $T_1^\infty + T_2^\infty\hat{Q}$ is analytic in $|z| \geq \delta_1$. Since $\|T_1^\infty + T_2^\infty\hat{Q}\|_\infty < \gamma$, it follows from continuity that there exists $\delta_2 < 1$ such that $\|T_1^\infty + T_2^\infty\hat{Q}\|_{\infty,\delta_2} \leq \gamma$. Therefore, by taking ϵ and η small enough and $\delta \triangleq \max\{\delta_1, \delta_2\} < 1$ we have that $\|T_1^\infty + T_2^\infty\hat{Q}\|_{\infty,\delta} \leq \gamma$ and $\|T_{\zeta_2 w_2}(\hat{Q})\|_2 < \hat{\mu}$. Hence for $\delta_i \geq \delta$, $\mu_i < \hat{\mu}$. This contradicts the fact that the sequence μ_i is nonincreasing and that $\hat{\mu} < \mu = \lim_{\delta_i \to 1}\mu_i$. ◊

REFERENCES

[1] D. S. Bernstein and W. H. Haddad, "LQG control with an $\mathcal{H}_\infty$ performance bound: A Riccati equation approach," *IEEE Trans. Automat. Contr.*, vol. 34, no. 3, pp. 293–305, 1989.

[2] G. Zames, "Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses," *IEEE Trans. Automat. Contr.*, vol. 26, no. 4, pp. 301–320, 1981.

[3] J. Doyle, K. Glover, P. Khargonekar, and B. Francis, "State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems," *IEEE Trans. Automat. Contr.*, vol. 34, no. 8, pp. 831–846, Aug. 1989.

[4] K. Zhou and J. Doyle, "Notes on MIMO control theory," lecture notes, California Instit. Tech., 1990.

[5] J. Doyle, K. Zhou, and B. Bodenheimer, "Optimal control with mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance objectives," in *Proc. 1989 ACC*, Pittsburgh, PA, 1989, pp. 2065–2070.

[6] K. Zhou, J. Doyle, K. Glover and B. Bodenheimer, "Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control," in *Proc. 1990 ACC*, San Diego, CA, 1990, pp. 2502–2507.

[7] H. H. Yeh and S. S. Banda, "Necessary and sufficient conditions for mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control," in *Proc. 29 IEEE CDC*, Hawaii, 1990, pp. 1013–1017.

[8] P. P. Khargonekar and M. A. Rotea, "Mixed $\mathcal{H}_2/H$ control: A convex optimization approach," *IEEE Trans. Automat. Contr.*, vol. 36, no. 7, pp. 824–837, 1991.

[9] I. Kaminer, P. P. Khargonekar, and M. Rotea, "Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control for discrete-time systems via convex optimization," *Automatica*, vol. 29, no. 1, pp. 57–70, 1993.

[10] M. A. Rotea ad P. P. Khargonekar, "$\mathcal{H}_2$ control with an $\mathcal{H}_\infty$ constraint: The state feedback case," *Automatica*, vol. 27, no. 2, pp. 307–316, 1991.

[11] D. B. Ridgely, L. S. Valavani, M. A. Dahleh, and G. Stein, "Solution to the general mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem—Necessary condition for optimality," in *Proc. 1992 ACC*, Chicago, Il., June 24–26, 1992, pp. 1348–1352.

[12] A. Sideris and H. Rotstein, "$\mathcal{H}_\infty$ optimization with time domain constraints over a finite horizon," in *Proc. 29 IEEE CDC*, Hawaii, Dec. 5–7, 1990, pp. 1802–1807.

[13] M. Sznaier, "A mixed $l_\infty/\mathcal{H}_\infty$ optimization approach to robust controller design," in *Proc. 1992 ACC*, Chicago, Il, 1992, pp. 727–732.

[14] S. Boyd *et al.*, "A new CAD method and associated architectures for linear controllers," *IEEE Trans. Automat. Contr.*, vol. 33, no. 3, pp. 268–283, 1988.

[15] A. Sideris and H. Rotstein, "Single input–single output $\mathcal{H}_\infty$ control with time-domain constraints," *Automatica*, vol. 29, no. 4, pp. 969–983, 1993.

[16] B. Francis, *A Course in $\mathcal{H}_\infty$ Optimization Theory*, (Lectures Notes in Control and Information) vol. 88. New York: Springer-Verlag, 1987.

[17] D. Bernstein and B. Wie, "Benchmark problems for robust control design," in *Proc. of the 1991 ACC*, Boston, MA, June 26–28, pp. 1915–1936.

$\mathcal{H}_\infty$ Control of Nonlinear Systems via Output Feedback: Controller Parameterization

Wei-Min Lu and John C. Doyle

***Abstract*— The standard state space solutions to the $\mathcal{H}_\infty$ control problem for linear time invariant systems are generalized to nonlinear time-invariant systems. A class of local nonlinear (output feedback) $\mathcal{H}_\infty$-controllers are parameterized as nonlinear fractional transformations on contractive, stable nonlinear parameters. As in the linear case, the $\mathcal{H}_\infty$ control problem is solved by its reduction to state feedback and output estimation problems, together with a separation argument. Sufficient conditions for $\mathcal{H}_\infty$-control problem to be locally solved are also derived with this machinery.**

I. INTRODUCTION

Linear $\mathcal{H}_\infty$ control theory has a simple state space characterization [3], which has clear connections with traditional methods in optimal control. These facts have stimulated several attempts to generalize the linear $\mathcal{H}_\infty$ results in state space to nonlinear systems [2], [13], [6],

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