IV. Conclusions

In this paper we have considered optimal routing policies that make use of limited state information, in queueing systems with finite capacities. There are two motivations for studying such policies. First, it is not always realistic to assume that the queue lengths are observed. In this case, the system's manager has to specify a control policy at time zero, i.e., prior to the system's initiation time. Second, it is not always necessary to have complete state information to make optimal routing decisions. Since, in addition, it is typically expensive to obtain this information, it is important to know whether there exist policies that use minimal state information and yet optimize the system's performance.

We have shown that the simple RR policy maximizes departures and minimizes losses in different classes of queueing systems. On one hand, when service times are exponential and capacities are equal, the RR policy outperforms all policies that do not use queue length information. On the other hand, when service times are deterministic and capacities are finite and equal, a simple MRR policy was shown to be optimal, provided that rejections are observed. This is information available in typical communication networks. Moreover, we have shown that the MRR policy coincides with the SUW policy. Note that the SUW policy is not optimal in systems with asymmetric (in terms of the service rate) queues (see for example [3]). In fact, it is very hard to prove optimality of policies under such settings and effort is usually focused on determining policies that perform 'adequately well.'

Interestingly, the SUW policy is not always optimal in systems with deterministic service times and unequal capacities. This makes the buffer allocation problem (i.e., the problem of distributing a fixed number of buffers among the queues) more difficult to attack. It can be shown (see [13]), however, that more balanced allocation schemes provide a better performance in the following sense. Given two allocations C^1 and C^2 such that $C^1 \prec C^2$, then for any policy π^2 employed in C^2 there exists a policy π^1 employed in C^1 that outperforms π^2 (in the sense of optimization of the loss and the departure processes) provided that the two systems start from the same state. The proof (which is omitted because of space limitations) consists of constructing a policy ρ employed in C^1 that copies π^2 and allows for idling in a way that preserves a majorization ordering of queue lengths between ρ and π^2. Since idling does not improve the system's performance the result then follows.

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Feedback Control of Quantized Constrained Systems with Applications to Neuromorphic Controllers Design

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***Abstract*— During the last few years there has been considerable interest in the use of trainable controllers based upon the use of neuron-like elements, with the expectation being that these controllers can be trained, with relatively little effort, to achieve good performance. Good performance, however, hinges on the ability of the neural net to generate a "good" control law even when the input does not belong to the training set, and it has been shown that neural nets do not necessarily generalize well. It has been proposed that this problem can be solved by essentially quantizing the state space and then using a neural net to implement a table lookup procedure. There is little information on the effect of this quantization upon the controllability properties of the system. In this paper we address this problem by extending the theory of control of constrained systems to the case where the controls and measured states are restricted to finite or countably infinite sets. These results provide the theoretical framework for recently suggested neuromorphic controllers, but they are also valuable for analyzing the controllability properties of computer-based control systems.**

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I. Introduction

During the last few years considerable attention has been focused on the use of neural net-based controllers, the expectation being that these controllers can be trained to achieve good performance. In particular, these controllers could be very useful for complex problems that do not admit a closed-form solution. Such is the case of constrained systems. In this case, of considerable practical importance for applications ranging from aerospace to process control, the problem of steering the system from a given initial condition to a desired target set usually does not admit a linear feedback control law as a solution. Therefore, control engineers have to resort to a number of schemas that include (in increasing order of sophistication) switching between several linear controllers, nonlinear controllers and on-line optimization based techniques ([1–3] and references therein). Clearly, a trainable, neural net-based controller could provide a welcome addition to the handful of techniques available for dealing with constrained systems, with the added bonus that such a controller could achieve good performance even in the face of poor or minimal modeling. As an example, we can mention the neuromorphic controller used by Anderson [4] to control an inverted pendulum when the control force is restricted to have bounded magnitude.

The basic idea justifying the use of neural nets as controllers for dynamical systems is that the controller can be trained to generate a desired output for a given input. The underlying assumption is that the neural net has good generalization properties, therefore being capable of generating an appropriate output even when the input is not a member of the training set. It has been shown [5], however, that neural nets do not necessarily generalize well. Therefore, it follows that the asymptotic stability properties of systems utilizing neuromorphic controllers are generally unknown, and this is a major stumbling block preventing their use.

This difficulty can be solved by realizing the fact that the neural net essentially implements a lookup table, and that generalization can be achieved by discretizing the input vectors and mapping them to a fixed number of "cells" in such a way that inputs that are "close" in some sense get mapped to the same cell [6]. This idea is based on the idea of "boxes" [7] and has been used several times in connection with neuromorphic controllers. None of the work available to date addresses the effects of this "quantization" upon the controllability properties of the system and the question of how to select a cell size that would allow the system to reach a "desirable" target set. Clearly, this problem is similar to the problem of investigating the controllability properties of a constrained system when the available state measurements are quantized, i.e., when the only information available at a given instant is that the state of the system belongs to a given "cell." Although the theory of control of constrained systems is well known and the original results due to Lee and Marcus [8] on the controllability of systems under control constraints have been extended in a number of ways to account for different types of constraints (see, for example, [9]), all these extensions always assume that the set of possible control laws is a dense subset of R^n and that the initial condition of the system is perfectly known.

Traditionally, quantization effects have been treated by adding noise sources to the system [10]. This type of analysis provides upper bounds on the errors due to quantization effects, but it is not suitable for extending the theoretical results already known for nonquantized systems to the quantized case. Moreover, recent work by Delchamps [11] showed that quantized systems under linear feedback control exhibit chaotic behavior, which is significantly different from the behavior predicted by the conventional additive noise model.

In [1] we presented a theoretical framework capable of handling the case where the control is restricted to a finite or countably infinite set, for systems under both state and control constraints. In this paper we extend these results to the case where the available measurements are also restricted to a finite or countably infinite set, and we apply the theory to the problem of designing suboptimal neuromorphic controllers for constrained systems. Our approach follows the spirit of [11] in explicitly modeling quantization as a set-membership constraint. The main motivation for this paper is to provide a theoretical framework for some recently suggested neuromorphic controllers [6]. The results presented here, however, also address the need (grown from the increased use of computer-based controllers in recent years) for a general theory of constrained controllability capable of accommodating the quantization effects that may result from the use of a computer in the feedback loop.

The paper is organized as follows: In Section II we introduce the concepts of quantization and quantized controllability. In Section III we use these concepts to characterize regions of state space that can be steered to a desired target set. The main result of the section is a necessary and sufficient condition guaranteeing the reachability of a convex open domain containing the origin and a relationship between the size of this domain (in terms of a Minkowsky functional) and the size of the quantization. In Section IV we use an example to illustrate the application of these theoretical results to the problem of designing suboptimal controllers based upon a partition of state space. Finally, in Section V, we summarize our results, and we indicate directions for future research.

II. Definitions

Definition 1: Consider a closed set $\mathcal{G} \subseteq R^n$. A family $\mathcal{S}$ of closed sets S_i is called a closed cover of $\mathcal{G}$ if $\mathcal{G} = \bigcup_i S_i$.

Definition 2: Consider a closed set $\mathcal{G} \subseteq R^n$ and a closed cover $\mathcal{S} = \{S_i\}$. A quantization $\mathcal{X}$ of $\mathcal{G}$ is a set $\mathcal{X} = \{z_i\}$ containing exactly one element from each set S_i.

Definition 3: Given a quantization $\mathcal{X}$ of a set $\mathcal{G}$, the size of the quantization with respect to some norm $\mathcal{N}$ defined in $\mathcal{G}$ is defined as

$$\frac{1}{s} = \min_i \{r : S_i \subseteq B(z_i, r)\, \forall i\}$$

where $B(z_i, r)$ indicates the $\mathcal{N}$-norm ball centered at z_i and with radius r. A quantization $\mathcal{X}$ with size s will be denoted as $\mathcal{X}_s$.

Consider now the case where the sets of the family S that defines a quantization $\mathcal{X}$ have pairwise disjoint interiors (i.e., $\mathrm{int}(S_i) \cap \mathrm{int}(S_j) = \phi,\ i \neq j$). In this case, S induces an equivalence relation in $\mathcal{G}$ as follows.

Definition 4: Consider a closed cover S of $\mathcal{G}$ with pairwise disjoint interiors, and two points $x_1, x_2 \in \mathcal{G}$. x_1 and x_2 are equivalent modulo S if $\exists i$ such that x_1 and $x_2 \in \mathrm{int}(S_i)$. To complete the partition of $\mathcal{G}$ into equivalence classes, we assign the points that are in $S_i \cap S_j$ (i.e., in the common boundary) arbitrarily to either one of the classes. Two points equivalent modulo S will be denoted as $x_1 \equiv x_2$.

Definition 5: Consider a quantization $\mathcal{X}_s = \{z_i\}$ of a given set $\mathcal{G}$. It follows from Definitions 2 and 4 that for any point $x \in \mathcal{G}$ there exists an element $z \in \mathcal{X}_s$ such that $z \equiv x$. We will define the operator that assigns $x \rightarrow z$ as the quantization operator, and we will denote it as: $z = \mathcal{X}_s(x)$.

In the following definitions we deal with the controllability aspects of the problem and, in particular, with the effects of quantizing the state and control spaces. Consider the constrained linear, time invariant, discrete system modeled by the difference equation

$$x_{k+1} = Ax_k + Bu_k, \qquad k = 0, 1, \cdots$$

$$u_k \in \Omega \subseteq R^m, \qquad x_k \in \mathcal{G} \subset R^n \tag{1}$$

where Ω and $\mathcal{G}$ are closed convex regions containing the origin in their interior and $\mathcal{G}$ is compact.

Definition 6: The system (1) is control quantized null controllable in a region $C \subseteq \mathcal{G}$ if, for any open set $O \subset \mathcal{G}$ containing the origin, there exists a number $s_u(C, O) \in R^+$ such that for all the quantizations Ω_s of Ω with $s \geq s_u$, there exists a sequence of quantized controls $u_k \in \Omega_s$ such that the system can be steered from any initial condition $x_0 \in C$ to O without violating the state constraints.

Consider now the case where state space is quantized. This situation can be modeled by assuming that the only information available to the controller are the quantized state measurements $z_k = \mathcal{X}_s(x_k)$, where $\mathcal{X}_s(\cdot)$ is the quantization operator associated with a given closed-cover $\mathcal{S}$ of $\mathcal{G}$. In the sequel, we will denote a control strategy of the form

$$u_k = f(\mathcal{X}_s(x_k)) \tag{2}$$

where f is a (possibly nonlinear) memoryless map, as an admissible control law.

Definition 7: The system (1) is state quantized null controllable in a region $C \subseteq \mathcal{G}$ if, for any open set $O \subset \mathcal{G}$ containing the origin, there exists a number $s_x(C, O) \in R^+$ such that for all the quantizations $\mathcal{X}_s$ of $\mathcal{G}$ with $s \geq s_x$ and for any initial condition $x_o \in C$, there exist a finite number n and a sequence of admissible controls $u_k = f(\mathcal{X}_s(x_k)) \in \Omega$, $k = 1, 2 \cdots n$, such that $z_k = \mathcal{X}_s(x_k) \in \mathcal{G}$, $k = 0, 1 \cdots n$, and $z_n \in O$.

Finally, we consider the case where both state and control space are quantized.

Definition 8: The system (1) is completely quantized null controllable if there exists a number s_0 such that (1) is state quantized null controllable when the controls are restricted to a quantization Ω_s of Ω with size $s \geq s_0$.

Remark 1: Note that the situation where both the states and the controls are quantized is particularly important for the case of neuromorphic controllers since, in addition to the state-space quantization induced by the "cell" structure, a finite set of control actions is usually required by the controller learning algorithm.

Following previous work in this area [1]–[3], we proceed now to restrict the set $\mathcal{G}$ to be balanced [12] (i.e., $x \in \mathcal{G} \Rightarrow \alpha x \in \mathcal{G}$ for all real α, $|\alpha| \leq 1$). This restriction, while not effecting significantly the number of real world problems that can be handled by our formalism, introduces more structure into the problem. This additional structure will become essential in showing constrained controllability.

Definition 9 [12]: The Minkowsky Functional (or gauge) p of a convex set $\mathcal{G}$ containing the origin in its interior is defined by

$$p(x) = \inf_{r>0}\left\{r : \frac{x}{r} \in \mathcal{G}\right\}. \tag{3}$$

A well-known result in functional analysis (see, for instance, [12]) establishes that if $\mathcal{G}$ is balanced, then p defines a seminorm in R^n. Furthermore, when $\mathcal{G}$ is compact, this seminorm becomes a norm. In the sequel, we will denote this norm as

$$\|x\|_{\mathcal{G}} \triangleq p(x). \tag{4}$$

Remark 2: The set $\mathcal{G}$ can be characterized as the unity ball in $\|\cdot\|_{\mathcal{G}}$. Hence, a point $x \in \mathcal{G}$ iff $\|x\|_{\mathcal{G}} \leq 1$.

III. Theoretical Results

Theorem 1: If

$$\min_{u\in\Omega}\{\|Ax + Bx\|_{\mathcal{G}}\} < 1 \forall x : \|x\|_{\mathcal{G}} = 1 \tag{5}$$

then the system (1) is control quantized null controllable in $\mathcal{G}$.

Proof: The proof of the theorem is a straightforward extension of Corollary 7-1 in [2].

Condition (5) implies that for any initial condition in the boundary of the admissible region, there exists at least one control that brings the system to its interior. It follows that if the problem of controlling the system (1) to the origin without exceeding the constraints is feasible (as it should be in a well posed problem), then the only effects of (5) is to rule out the possibility of the system staying on the boundary for consecutive sampling intervals.

Lemma 1: Let $O \subset \mathcal{G}$ be an open set containing the origin and consider the region $\mathcal{G} - O$. Let

$$\Lambda = \min_{x\in\mathcal{G}-O}\left\{\lambda > 0 : \left(\frac{1}{\lambda}x\right) \in \partial\mathcal{G}\right\} \tag{6}$$

where $\partial\mathcal{G}$ denotes the boundary of the set $\mathcal{G}$. Then

$$\min_{x\in\mathcal{G}-O}\left\{\|x\|_{\mathcal{G}} - \min_{u\in\Omega}\|Ax + Bu\|_{\mathcal{G}}\right\} > \Lambda \min_{\underline{y}\in\partial\mathcal{G}}\left\{1 - \min_{u\in\Omega}\|A\underline{y} + Bu\|_{\mathcal{G}}\right\}. \tag{7}$$

Proof: Given any $x \in \mathcal{G} - O$ it can be expressed as $\lambda_o \underline{y}_o$ with $\underline{y}_o \in \partial\mathcal{G}$ and $0 < \lambda_o \leq 1$. Then

$$\begin{aligned}
&\|x\|_{\mathcal{G}} - \min_{u\in\Omega}\|Ax + Bu\|_{\mathcal{G}} \\
&\quad = \|\lambda_o\underline{y}_o\|_{\mathcal{G}} - \min_{u\in\Omega}\|A\lambda_o\underline{y}_o + Bu\|_{\mathcal{G}} \\
&\quad \geq \|\lambda_o\underline{y}_o\|_{\mathcal{G}} - \min_{u\in\Omega}\|A\lambda_o\underline{y}_o + B\lambda_o u\|_{\mathcal{G}} \\
&\quad = \lambda_o\left\{\|\underline{y}_o\|_{\mathcal{G}} - \min_{u\in\Omega}\|A\underline{y}_o + Bu\|_{\mathcal{G}}\right\} \\
&\quad \geq \min_{\substack{\underline{y}\in\partial\mathcal{G} \\ \lambda\underline{y}\in\mathcal{G}-O}} \lambda\left\{\|\underline{y}\|_{\mathcal{G}} - \min_{u\in\Omega}\|A\underline{y} + Bu\|_{\mathcal{G}}\right\} \\
&\quad \geq \Lambda \min_{\underline{y}\in\partial\mathcal{G}}\left\{1 - \min_{u\in\Omega}\|A\underline{y} + Bu\|_{\mathcal{G}}\right\} \\
&\quad\quad \text{since } \|\underline{y}\|_{\mathcal{G}} = 1,\ \underline{y} \in \partial\mathcal{G}. \quad \Diamond
\end{aligned} \tag{8}$$

Remark: Note that Λ is the minimum of the Minkowsky functional of $\mathcal{G}$ in the region $\mathcal{G} - O$. Since this region is compact, it follows that Λ is well defined. Furthermore, the right-hand side of (7) gives a lower bound on the maximum amount that the norm of the present state of the system ($\|x\|_{\mathcal{G}}$) can be decreased in one stage.

Lemma 2: Let O be an open set containing the origin. If

$$1 - \max_{\|x\|_{\mathcal{G}}=1}\left\{\min_{u\in\Omega}\{\|Ax + Bu\|_{\mathcal{G}}\}\right\} = \delta > 0 \tag{9}$$

then, for any quantization $\mathcal{X}_s = \{z_i\}$ of $\mathcal{G}$ with size $s \geq s_o \triangleq (1 + \|A\|_{\mathcal{G}})/\delta\Lambda$ and for any point $x_o \in \mathcal{G} - O$ such that $z_o = \mathcal{X}_s(x_o) \in \mathcal{G} - O$, there exists an admissible control $u_o = f(\mathcal{X}_s(x_o)) \in \Omega$ such that $\|z_1\|_{\mathcal{G}} < \|z_o\|_{\mathcal{G}}$ where $z_1 = \mathcal{X}_s(Ax_o + Bu_o)$, and where $\|A\|_{\mathcal{G}}$ denotes the induced operator norm (i.e., $\|A\|_{\mathcal{G}} = \max_{\|x\|_{\mathcal{G}}=1}\|Ax\|_{\mathcal{G}}$).

Proof: From the hypothesis and Lemma 1 it follows that

$$\begin{aligned}\max_{u\in\Omega}\{\|z_o\|_{\mathcal{G}} - \|Az_o + Bu\|_{\mathcal{G}}\} &= \|z_o\|_{\mathcal{G}} - \min_{u\in\Omega}\{\|Az_o + Bu\|_{\mathcal{G}}\}\\ &\geq \Lambda\left\{1 - \max_{x\in\mathcal{G}}\left\{\min_{u\in\Omega}\|Ax + Bu\|_{\mathcal{G}}\right\}\right\} = \Lambda\delta. \end{aligned}\tag{10}$$

Define

$$u_0 = \operatorname*{argmin}_{u\in\Omega}\{\|Az_0 + Bu_0\|_{\mathcal{G}}\}$$

$$x_1 = Ax_0 + Bu_0$$

$$z_1 = \mathcal{X}_s(x_1) \triangleq x_1 + \zeta_1,\ \|\zeta_1\|_{\mathcal{G}} \leq \frac{1}{s}. \tag{11}$$

Then

$$\begin{aligned}\|z_0\|_{\mathcal{G}} - \|z_1\|_{\mathcal{G}} &= \|z_0\|_{\mathcal{G}} - \|Ax_0 + Bu_0 + \zeta_1\|_{\mathcal{G}}\\ &= \|z_0\|_{\mathcal{G}} - \|Az_0 + Bu_0 - A\zeta_0 + \zeta_1\|_{\mathcal{G}}\\ &\geq \|z_0\|_{\mathcal{G}} - \|Az_0 + Bu_0\|_{\mathcal{G}} - \|A\zeta_0\|_{\mathcal{G}} - \|\zeta_1\|_{\mathcal{G}}\\ &\geq \|z_0\|_{\mathcal{G}} - \|Az_0 + Bu_0\|_{\mathcal{G}} - \|A\|_{\mathcal{G}}\|\zeta_0\|_{\mathcal{G}} - \|\zeta_1\|_{\mathcal{G}}\\ &\geq \|z_0\|_{\mathcal{G}} - \|Az_0 + Bu_0\|_{\mathcal{G}} - \left(\frac{1 + \|A\|_{\mathcal{G}}}{s}\right)\\ &\geq \Lambda\delta - \left(\frac{1 + \|A\|_{\mathcal{G}}}{s}\right). \end{aligned}\tag{12}$$

Hence, if

$$s > \frac{1 + \|A\|_{\mathcal{G}}}{\Lambda\delta} \tag{13}$$

then $\|z_1\|_{\mathcal{G}} - \|z_0\|_{\mathcal{G}} = \mu < 0$. ◊

In the next theorem we use the results of Lemma 2 to show that condition (9) is a sufficient condition for state quantized null controllability.

Theorem 2: Equation (9) is a sufficient condition for the system (1) to be state quantized null controllable in $\mathcal{G}$.

Proof: To show state quantized null controllability, we have to show that for any open set $O \subset \mathcal{G}$ containing the origin, there exists a number s_0 such that for all the quantizations $\mathcal{X}_s$ of $\mathcal{G}$ with size $s \geq s_0$, and for any initial condition $x_0 \in \mathcal{G}$, there exists a sequence of admissible control laws $\mathcal{U} = \{u_0, u_1 \cdots u_n\}$, where n is a finite number, such that

$$z_k = \mathcal{X}_s(x_k) \in \mathcal{G},\ k = 0, 1 \cdots n$$

$$z_n \in O. \tag{14}$$

Define $s_0 \triangleq (1+\|A\|_{\mathcal{G}})/(\delta\Lambda)$ and consider an arbitrary quantization $\mathcal{X}_r$ with $r \geq s_0$. Let x_o be an arbitrary initial condition in $\mathcal{G}-O$. From the definition of quantization, it follows that there exists $z_0 \in \mathcal{X}_r$ such that $x_0 \equiv z_0$. Obviously, if $z_0 \in O$ the theorem is trivial, so consider the case where $z_0 \notin O$. Then, from Lemma 2 it follows that, as long as $z_k \notin O$, there exists a sequence $\mathcal{U} = \{u_0, u_1 \cdots\}$ such that

$$\begin{aligned}\|z_1\|_{\mathcal{G}} &< \|z_0\|_{\mathcal{G}} - \mu\\ \|z_2\|_{\mathcal{G}} &< \|z_1\|_{\mathcal{G}} - \mu\\ &\vdots\\ \|z_m\|_{\mathcal{G}} &< \|z_{m-1}\|_{\mathcal{G}} - \mu \end{aligned}\tag{15}$$

where $\mu > 0$ and $z_i = \mathcal{X}_r(x_i) = \mathcal{X}_r(Ax_{i-1} + Bu_{i-1})$. It follows then that there exists n_0 such that $z_{n_0} \in O$. ◊

Finally, in the next theorem we show that (9) is a sufficient condition for complete quantized null controllability.

Theorem 3: Equation (9) is a sufficient condition for the system (1) to be completely quantized null controllable in $\mathcal{G}$.

Proof: Since $\|Bu\|_{\mathcal{G}}$ is a continuous function of u, it follows that there exists r such that $\|B\zeta_u\|_{\mathcal{G}} \leq (\Lambda\delta/2)$ for all $\zeta_u \in B(0, r) \subseteq \Omega$ where $B(0, r)$ denotes a ball in some arbitrary norm defined in Ω. The proof follows now from the proof of Theorem 2 by substituting $(\Lambda\delta/2)$ for $\Lambda\delta$. ◊

Corollary: The size of the quantization introduced in Theorems 2 and 3 is inversely proportional to Λ. Hence, as the size of the target set gets smaller, the number of cells increases, while their size decreases. Note, however, that the target set O is achieved through a sequence of intermediate sets O_i, $i = 1, 2 \cdots, n$ with $O_1 \equiv \operatorname{int}(\mathcal{G})$ and $O_n \equiv O$. Since Λ in (6) can be thought of as a lower bound of the ratio of the norm of the next state of the system to the norm of the present state, it follows that to guarantee complete quantized null controllability, it is enough to choose

$$\Lambda = \max_i \Lambda_i; \qquad \Lambda_i = \min_{x\in O_i - O_{i+1}}\left\{\lambda > 0: \left(\frac{1}{\lambda}x\right) \in \partial\overline{O}_i\right\} \tag{16}$$

where $\overline{O}$ denotes the closure of O.

Remark: From (16) it follows that if the sequence of intermediate stages O_i is chosen so that $\Lambda = \Lambda_i\,\forall i$ (i.e., the sets O_i all have the same "shape") then the number of cells in each set roughly decreases as Λ^n. Alternatively, using the same number of cells at each stage results in a "retina" like structure, having coarser resolution far from the target set and increasingly finer resolution closer to the target. Note that this increased resolution could be achieved essentially having only one set of boxes, whose function adaptively changes with the state of the system.

IV. Applications to Suboptimal Controllers Design

As a example of the potential applications of our theory to the optimal control of constrained systems, we will use it to address the problem of determining a "cell size" that guarantees controllability to a given target set. Since in this case the quantization of state space is introduced as an artifact to simplify the search for an optimal trajectory, we will assume that any hardware imposed quantization effects are negligible.

Once a lower bound s_o on the size of the quantizations that guarantee controllability to the desired target set is determined, a suboptimal controller can be implemented as a table lookup schema by essentially finding and storing an optimal control law associated with each cell. Moreover, since neural nets are known to implement table lookup schemas very efficiently [6], it follows that this suboptimal control law could be implemented successfully with a neuromorphic controller, without any assumptions on the generalization properties of the neural net and with guaranteed asymptotic stability of the closed-loop system. This idea is formalized in the following conceptual algorithm.

Algorithm L (Optimal Control Using a lookup Table)

Begin.

1) Determine a lower bound s_o on the size of the quantizations that guarantee controllability to the target set O using (13) and (16). Alternatively, determine the number of boxes to be used with changing resolution, as discussed in the corollary to Theorem 3.

2) Choose a pairwise interior-disjoint closed cover $S = \{S_i\}$ of $\mathcal{G}$ with size $s \geq s_o$. Form a quantization $\mathcal{X}_s = \{z_i\}$ by selecting one representative element from each equivalence class.
3) For each element $z_i \in \mathcal{X}_s$ find the optimal (in some previously defined sense) control law u_i^0, subject to the constraint $\|z_i\|_{\mathcal{G}} - \|A_{z_i} + Bu_i^0\|_{\mathcal{G}} \geq \Lambda\delta$, and store it.
4) While $z_i = \mathcal{X}_s(x_i) \notin O$ use as the next control law, the control law associated with z_i

End.

Next, we show how to apply Algorithm L to a simple example. Since in this paper we are concentrating on the theoretical controllability issue, we will assume that a table lookup procedure is available. The issues concerning the implementation of this procedure by means of a neural net, too extensive to consider here, are left for a future paper on the subject.

A Simple Example

Consider the problem of bringing the angular velocity of a spinning space station with a single axis of symmetry from an initial condition x_o, $\|x_o\|_2 = R_x$ to a final state such that $\|x_f\|_2 \leq R_f$. This situation can model the case where a sophisticated, nonconventional controller is used to bring a system in minimum time to some region (for instance a region where the constraints are not binding) where some relatively easy to design controller can take over. The system can be represented by [3], [12]

$$A = \begin{pmatrix} \cos T & \sin T \\ -\sin T & \cos T \end{pmatrix} B = \begin{pmatrix} \sin T & (1-\cos T) \\ (\cos T - 1) & \sin T \end{pmatrix}$$

$$\mathcal{G} = \{x \in R^2 \colon \|x\|_2 \leq R_x\}, \qquad \Omega = \{u \in R^2 \colon \|u\|_2 \leq 1\} \quad (17)$$

where T is the sampling interval. Let $\alpha^2 \triangleq 2(1 - \cos T)$. Then $B^T B = \alpha^2 I$ and

$$\begin{aligned} \|x_{k+1}\|_2^2 &= \|x_k\|_2^2 + 2x^T A^T Bu + u^T B^T Bu \\ &= \|x_k\|_2^2 + 2x^T B^T u + \alpha^2 \|u\|_2^2. \end{aligned} \quad (18)$$

Hence, by selecting $u = (-Bx_k / \|Bx_k\|_2)$ we have

$$\|x_{k+1}\|_2^2 = \|x_k\|_2^2 - 2\alpha\|x_k\|_2 + \alpha^2 = (\|x_k\|_2 - \alpha)^2. \quad (19)$$

From (19) it follows that

$$\begin{aligned} \delta &= \|x_k\|_{\mathcal{G}} - \|x_{k+1}\|_{\mathcal{G}} \\ &= \frac{1}{R_x}(\|x_k\|_2 - \|x_{k+1}\|_2) = \frac{\alpha}{R_x}. \end{aligned} \quad (20)$$

Since in this case $\|\cdot\|_{\mathcal{G}}$ is simply the euclidian norm scaled by R_x it follows that $\|A\|_{\mathcal{G}} = 1$. Hence, from (13) we have that

$$\frac{1}{s} \leq \frac{1}{2}\frac{\alpha R_f}{R_x^2}. \quad (21)$$

Assume that we want to use a covering formed by square boxes of side l. Then, by choosing the center of each box as the representative element we have

$$\begin{aligned} S_i(l) &\subseteq B\left(z_i, \frac{1}{s}\right)_{\mathcal{G}} = B\left(z_i, \frac{R_x}{s}\right) \\ &\Leftrightarrow l \leq \frac{R_x}{s}\sqrt{2} \leq \frac{\alpha R_f}{\sqrt{2}R_x} = \frac{\alpha\Lambda}{\sqrt{2}}. \end{aligned} \quad (22)$$

Moreover, since the norm of the present state of the system can be decreased at each stage by α (in the region $\|x\|_2 \geq \alpha$) from Theorem 3 and its corollary it follows that l should be selected (see Fig. 1) such that

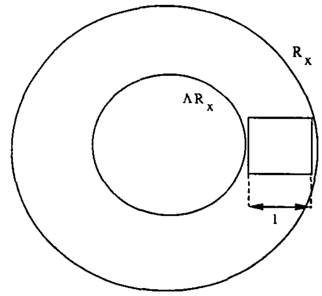

Fig. 1. Selecting the size of the cells.

$$l \leq \alpha$$

$$l \leq R_x - \Lambda R_x = R_x(1 - \Lambda). \quad (23)$$

Hence, the region $\|x\|_2 \leq \alpha$ (which is the region where the constraints are not binding) can be reached, with a degree of stability Λ, by using a quantization such that

$$\frac{\alpha\Lambda}{\sqrt{2}} = R_x(1 - \Lambda) \Leftrightarrow \Lambda = \frac{1}{1 + \frac{\alpha}{R_x\sqrt{2}}}. \quad (24)$$

In our case selecting $T = 2.5$ sec. and $R_x = 20$ yields $\alpha = 1.898\,\Lambda = 0.937$ and $l = 1.258$.

Fig. 2 shows the results of applying the algorithm to the system with initial condition $x_0^T = (20.0\ 0.0)$. Since in this particular case the time-optimal control law has an explicit expression, we simulated the table lookup by computing at each instance the optimal control law associated with the center of the box that contains the present state of the system. Note the proximity between the quantized and true time-optimal trajectories, indicated respectively by "O" and "+." This proximity suggests that the results of Theorem 2 are overly conservative. In fact, experimenting with this problem we have obtained convergence to the region $\|x\|_2 \leq \alpha$ even when $l = \sqrt{2}\alpha$ (the largest l such that at least one square box will fit entirely within the target set).

V. Conclusion

During the last few years, there has been considerable interest in the use of trainable controllers based upon the use of neuron-like elements. These controllers can be trained, for instance by presenting several instances of "desirable" input–output pairs, to achieve good performance, even in the face of poor or minimal modeling. The use of neuromorphic controllers has been hampered, however, by the facts that good performance hinges on the ability of the neural net to generalize the input–output mapping to inputs that are not part of the training set. Through examples [5], it has been shown that neural nets do not necessarily generalize well. Therefore, it follows that the stability properties of the closed-loop system are unknown. Moreover, it is conceivable that poor generalization capabilities may result in limit cycles or even in destabilizing control laws. In this paper we address these problems by proposing a neural net-based controller that results in a schema similar to tabular control and then carefully investigating the properties of such a controller. Perhaps the most valuable contribution of this paper results from the qualitative aspects of (13) that identify the factors that affect any controller based upon the quantization of state space (independently of the specific implementation of the look up schema). Most notably, through the

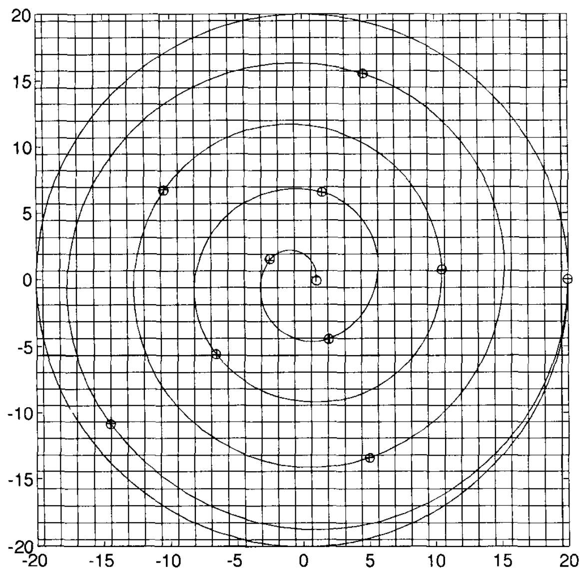

Fig. 2. Time optimal (+) and quantized (o) trajectories for the simple example.

norm of the operator that appears in (13), it is possible to formalize the idea of "poor" modeling and to design a "robust" controller capable of accommodating modeling errors and disturbances.

There are several questions that remain open. Since one of the main reasons for using neural net-based controllers is their ability to yield good performance with imperfect models, the robustness of these controllers to plant perturbations should be investigated. At this point we are working in a neural net implementation of the ideas presented in this paper, and we are investigating their robustness properties. Future articles are planned to report the results of this line of research. Finally, as we noted in the paper, the results of Theorem 2 that guarantee quantized null controllability can be overly restrictive in some cases, since they result from a "worst case" type analysis. A relaxed version of these conditions will be highly desirable.

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Polynomial Solution of the Standard Multivariable $\mathcal{H}_2$-Optimal Control Problem

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***Abstract*—In this work we solve the standard multivariable $\mathcal{H}_2$-optimal control problem using polynomial matrix techniques.**

I. Introduction

There are at least three significant approaches to the design of linear $\mathcal{H}_2$ (or linear quadratic (LQ)-) optimal controllers for multivariable plants. The basic regulator problem has been studied using a time-domain approach in the state space (see, for example, Kwakernaak and Sivan [1]), and a frequency-domain approach using transfer function matrices and Wiener-Hopf theory [2]. As an alternative, a polynomial equation approach has been developed by Kučera [3] which is based on the algebra of polynomial matrices, and this approach provides the distinguishing feature of our presentation. For LQ-type problems, controller synthesis reduces to polynomial spectral factorization and the solution of linear polynomial equations. A deep analysis of the relationship between polynomial (transfer-function) and state-space control synthesis methods may be found in Kučera [4]. An overview of polynomial methods in optimal control and filtering problems is presented in Hunt [5].

Since many different control tasks are encountered in practice, the basic regulator problem solution was later extended to more complex control structures (such as reference tracking [6], [7] and measurement feedforward [8]–[10]) or various types of costing (e.g., including dynamic weights and sensitivity functions [11]).

During practical design work, it may be necessary to perform the synthesis for every particular structure at hand. It is undesirable, however, to have a number of theories with each being specific to only one control structure. Instead, one "general" solution is desirable which can be simplified (adjusted) for every particular practical design. A solution of such generality has been obtained and is presented in this work; it is known as the standard problem. A preliminary solution of this problem was presented in [12], and an alternative polynomial solution can be found in [13].

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