ii) It follows from Lemma 1 that $\delta(F_{s^o(\alpha)}, F(\alpha, s^o(\alpha))) = \delta(F_{s^o(\alpha)}, (1-\alpha)^{-1}F_{s^o(\alpha)}) \leq \alpha(1-\alpha)^{-1}M(s^o(\alpha))$, where $M(s^o(\alpha)) \leq M_\infty < \infty$ for all $\alpha \in (0,1)$, hence $\delta(F_{s^o(\alpha)}, F(\alpha, s^o(\alpha))) \to 0$ as $\alpha \searrow 0$. Note that $s^o(\alpha) \to \infty$ as $\alpha \searrow 0$. Since $F(\alpha, s^o(\alpha)) \supseteq F_\infty \supseteq F_{s^o(\alpha)}$ for all $\alpha \in (0,1)$ and $F_{s^o(\alpha)} \to F_\infty$ as $\alpha \searrow 0$, we conclude that $F(\alpha, s^o(\alpha)) \to F_\infty$ as $\alpha \searrow 0$.

## Appendix III
## Proof of Theorem 3

We refer to the proof of Theorem 2 for the definition of $M_\infty$. Let $\varepsilon > 0$ and recall that $0 < M_\infty < \infty$ and $F_s \subseteq F_\infty$ for all $s \in \mathbb{N}$. Since $F_s$ and $F_\infty$ are convex and contain the origin, it follows that $\alpha(1-\alpha)^{-1}F_s \subseteq \alpha(1-\alpha)^{-1}F_\infty$ for any $s \in \mathbb{N}$ and $\alpha \in [0,1)$. Note that the inclusion $\alpha(1-\alpha)^{-1}F_\infty \subseteq \mathbb{B}_p^n(\varepsilon)$ is true if $\alpha(1-\alpha)^{-1}M_\infty \leq \varepsilon$ or, equivalently, if $\alpha \leq \varepsilon(\varepsilon + M_\infty)^{-1}$. Hence, (8) is true for any $s \in \mathbb{N}$ and $\alpha \in [0, \bar{\alpha}]$, where $\bar{\alpha} \triangleq \varepsilon(\varepsilon + M_\infty)^{-1} \in (0,1)$. Clearly, (4) is also true if we choose $\alpha \in (0, \bar{\alpha}]$ and $s = s^o(\alpha)$. This establishes the existence of a suitable couple $(\alpha, s)$ such that (4) and (8) hold simultaneously.

Let $(\alpha, s)$ be such that (4) and (8) are true. Since $F(\alpha, s) = (1-\alpha)^{-1}F_s$ is a convex and compact set that contains the origin, $F(\alpha, s) = (1-\alpha)^{-1}F_s = (1 + \alpha(1-\alpha)^{-1})F_s = F_s \oplus \alpha(1-\alpha)^{-1}F_s$. Since $F_s \subseteq F_\infty \subseteq F(\alpha, s) \subseteq F_s \oplus \mathbb{B}_p^n(\varepsilon) \subseteq F_\infty \oplus \mathbb{B}_p^n(\varepsilon)$, it follows that $F(\alpha, s)$ is an RPI, outer $\varepsilon$-approximation of the mRPI set $F_\infty$.

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# An Algorithm for Sampling Subsets of $\mathcal{H}_\infty$ With Applications to Risk-Adjusted Performance Analysis and Model (In)Validation

Mario Sznaier, Constantino M. Lagoa, and Maria Cecilia Mazzaro

***Abstract*—In spite of their potential to reduce computational complexity, the use of probabilistic methods in robust control has been mostly limited to parametric uncertainty, since the problem of sampling causal bounded operators is largely open. In this note, we take steps toward removing this limitation by proposing a computationally efficient algorithm aimed at uniformly sampling suitably chosen subsets of $\mathcal{H}_\infty$. As we show in the note, samples taken from these sets can be used to carry out model (in)validation and robust performance analysis in the presence of structured dynamic linear time-invariant uncertainty, problems known to be NP-hard in the number of uncertainty blocks.**

***Index Terms*—Model (in)validation, risk-adjusted control, robust performance, sampling, structured uncertainty.**

## I. Introduction

Many problems arising in robust control have poor computational properties. Examples are validating a system model and assessing its robust performance properties in the presence of structured linear time invariant dynamic uncertainty, both NP-hard in the number of uncertainty blocks [4], [19]. Tractable relaxations are available, but can be arbitrarily conservative [18].

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Given these difficulties, during the past few years considerable attention has been devoted to the use of probabilistic methods [2], [3], [5], [8], [14], [16], where the computational burden grows moderately with the size of the problem [12].

However, at the present time, the domain of applicability of probabilistic techniques has been largely restricted to the finite-dimensional parametric uncertainty case.[1] The main reason for this limitation is that up to now the problem of sampling causal bounded operators has been largely open. In the first part of this note, we propose an algorithm to remove this limitation when the set to be sampled, $\mathcal{D}$, consists of operators that can be completed to belong to balls in $\mathcal{H}_\infty$. By combining matrix dilation and Carathéodory–Fejér interpolation results, we show that the problem of generating operators uniformly distributed over these sets can be reduced to that of generating finite-dimensional vectors uniformly distributed over a convex set. In general, this is also a hard problem. Most of the solution methods available require designing a random walk whose stationary distribution is the required one [9]. However, as we show here, for the class of subsets of $\mathcal{H}_\infty$ considered in this note, use of Parrot's Theorem allows for solving the problem by simply sampling a sequence of intervals, leading to a computationally efficient algorithm.

In the second part of this note, we exploit these results to develop a *risk-adjusted* framework that allows for validating a given system model from experimental data, and to assess its *finite-horizon* robustness properties in the presence of structured uncertainty.

## II. Notation

In the sequel, $\mathcal{H}_\infty$ denotes the subspace of transfer matrices analytic in $|z| < 1$ and essentially bounded on $|z| = 1$, equipped with the norm: $\|G\|_\infty \doteq ess\ \sup_{|z|<1} \overline{\sigma}(G(z))$, where $\overline{\sigma}(.)$ denotes maximum singular value. $\mathcal{BH}_\infty$ and $\mathcal{BH}_\infty^n$ denote the unit ball in $\mathcal{H}_\infty$ and the set of $(n-1)^{\text{th}}$ order FIR transfer matrices that can be completed to belong to $\mathcal{BH}_\infty$, i.e. $\mathcal{BH}_\infty^n \doteq \{H(z) = \mathbf{H}_0 + \mathbf{H}_1 z + \ldots + \mathbf{H}_{n-1} z^{n-1} : H(z) + z^n G(z) \in \mathcal{BH}_\infty, \text{ for some } G(z) \in \mathcal{H}_\infty\}$, respectively.

## III. Sampling the Class $\mathcal{BH}_\infty^n$

Using a risk-adjusted approach to perform model (in)validation and to assess robust performance requires solving the following problem.

*Problem 1:* Given $n$, generate suitably distributed samples from a finite dimensional representation of the convex set $\mathcal{BH}_\infty^n$.

In the aforementioned problem, $n$ is given by the specific application under consideration: for model invalidation problems, $n$ is given by the number of experimental data points; for performance analysis, $n$ corresponds to the horizon length of interest.

In principle, sampling general convex sets is a hard problem. However, as we will show in the sequel, in the case under consideration here, the special structure of the problem can be exploited to obtain a computationally efficient algorithm.

### A. Reducing the Problem to Sampling Finite Dimensional Sets

We begin by showing how Problem 1 can be reduced to the problem of sampling a *finite-dimensional* convex set. From the Carathéodory–Fejér Theorem (see, for instance, [1]) it follows that given the first $n$ Markov parameters $\mathbf{H}_i \in \mathbf{R}^{s\times m}$, $i = 0, 1, \ldots, n-1$ of a matrix operator $H(z) \in \mathcal{H}_\infty$, the corresponding $H(z) \in \mathcal{BH}_\infty^n$ if and only if $\overline{\sigma}(\mathbf{T}_H^n) \le 1$, where

$$\mathbf{T}_H^n(\mathbf{H}_0, \mathbf{H}_1, \ldots, \mathbf{H}_{n-1}) \doteq \begin{bmatrix} \mathbf{H}_{n-1} & \cdots & \mathbf{H}_1 & \mathbf{H}_0 \\ \mathbf{H}_{n-2} & \cdots & \mathbf{H}_0 & \mathbf{0} \\ \vdots & & & \vdots \\ \mathbf{H}_0 & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}.$$

Thus, a natural representation for $\mathcal{BH}_\infty^n$ in Problem 1 is the set $\mathcal{C}_{\mathcal{H}_n} \doteq \{\{\mathbf{H}_i\}_{i=0}^{n-1} : \overline{\sigma}(\mathbf{T}_H^n) \le 1\}$. This leads to the following problem.

*Problem 2:* Given $n > 0$, generate uniform samples over the convex set $\mathcal{C}_{\mathcal{H}_n}$.

*Remark 1:* Is worth emphasizing that uniformly sampling $\mathcal{C}_{\mathcal{H}_n}$ is not equivalent to uniformly sampling either $\mathcal{BH}_\infty^n$ or $\mathcal{BH}_\infty$. Rather, it assigns the same probability to each equivalence class in these sets formed by transfer matrices having the same first $n$ Markov Parameters. This is natural in the context of the applications addressed in this note, model (in)validation and finite horizon performance assessment, where the property in question depends only on these parameters.

In the sequel, we present an algorithm for generating uniform samples over arbitrary finite dimensional convex sets and we solve Problem 2 as a special case.

### B. Generating Uniform Samples Over Convex Sets

Given an arbitrary convex set $\mathcal{C} \subset \mathbf{R}^n$ define its projection and section, respectively, by

$$\begin{aligned} \mathrm{Proj}^l(\mathcal{C}) &\doteq \left\{\mathbf{x} \in \mathbf{R}^l : \left[\mathbf{x}^T \mathbf{y}^T\right]^T \in \mathcal{C}, \text{ for some } \mathbf{y} \in \mathbf{R}^{k-l}\right\} \\ \mathcal{S}_{\mathcal{C}}^k(\mathbf{y}) &\doteq \left\{x \in \mathbf{R}^k : \left[\mathbf{y}^T \mathbf{x}^T\right]^T \in \mathcal{C}\right\}. \end{aligned} \tag{1}$$

Finally, given $\mathbf{x} \in \mathcal{C}$, partition the vector conformably to some given structure in the following form $\mathbf{x} = [\mathbf{x}_1^T\ \mathbf{x}_2^T\ \cdots\ \mathbf{x}_m^T]^T$, where $\mathbf{x}_i \in \mathbf{R}^{n_i}$ and $\sum_{i=1}^m n_i = n$. Consider now the following algorithm.

*Algorithm 1*

1) Let $k = 0$. Generate $N_1$ samples, $\mathbf{x}_1^l$, $l = 1, 2, \ldots, N_1$, uniformly distributed over the set $\mathcal{I}_o \doteq \mathbf{Proj}^{n_1}(\mathcal{C})$.
2) Let $k := k + 1$. For every generated sample $(\mathbf{x}_1^l, \mathbf{x}_2^l, \ldots, \mathbf{x}_{k-1}^l)$, let

$$\mathcal{C}_k\left(\mathbf{x}_1^l, \mathbf{x}_2^l, \ldots, \mathbf{x}_{k-1}^l\right) \doteq \mathcal{S}_{\mathcal{C}}^{n^*}\left(\left[\left(\mathbf{x}_1^l\right)^T \left(\mathbf{x}_2^l\right)^T \cdots \left(\mathbf{x}_{k-1}^l\right)^T\right]^T\right)$$
$$\mathcal{I}_k\left(\mathbf{x}_1^l, \mathbf{x}_2^l, \ldots, \mathbf{x}_{k-1}^l\right) \doteq \mathrm{Proj}^{n_k}(\mathcal{C}_k)$$

with $n^* \doteq \sum_{i=k}^m n_i$. Generate $N_k \doteq \lfloor \alpha_k N_1 \mathbf{vol}(\mathcal{I}_k) \rfloor$ samples uniformly over the set $\mathcal{I}_k$, where $\alpha_k$ is an arbitrary positive constant.
3) If $k < m$, go to step 2). Else, stop.

*Remark 2:* Note that each set generated by this algorithm contains $N_1$ independent samples. As we will show in Section IV, this allows for exploiting the results in [16] to obtain bound on the number of samples required to guarantee that a property holds with a given probability. In

[1]An exception is the work of Zhou [21], [22] using boundary N–P interpolation theory to estimate the probability that a given model is not (in)validated by a set of frequency-domain experiments.

contrast, these results cannot be used for samples generated using for instance Markov Chain based methods, since they are not independent.

Next we show that the probability distribution of the samples generated by this algorithm converges, with probability one, to a uniform distribution as $N_1 \to \infty$.

*Theorem 1:* Consider any set $\mathcal{A} \subseteq \mathcal{C}$. For a given $N_1$, denote by $ns_t(N_1)$ and $ns_{\mathcal{A}}(N_1)$[2] the total number of samples generated by Algorithm 1 and the number of those samples that belong to $\mathcal{A}$, respectively. Then

$$\frac{ns_{\mathcal{A}}(N_1)}{ns_t(N_1)} \xrightarrow{w.p.1} \frac{\mathrm{vol}(\mathcal{A})}{\mathrm{vol}(\mathcal{C})}. \tag{2}$$

*Proof:* See Appendix A. ∎

*Remark 3:* The main reason that prevents the estimate of probability produced by the samples generated by Algorithm 1 from being unbiased is that, in general, at step $s$

$$\frac{\lfloor N_1 \alpha_s v_s\left(\mathbf{X}_1^k, \mathbf{X}_2^m, \ldots, \mathbf{X}_{s-1}^n\right)\rfloor}{N_1} \neq \alpha_s v_s\left(\mathbf{X}_1^k, \mathbf{X}_2^m, \ldots, \mathbf{X}_{s-1}^n\right)$$

where $v_s(.) \doteq \mathrm{vol}[I_s(\mathbf{X}_1^{\mathbf{k}}, \ldots, \mathbf{X}_{s-1}^n)]$, due to the rounding. Indeed, for any union of hyper-rectangles $\mathcal{A} \subseteq \mathcal{C}$ satisfying

$$\frac{\lfloor N_1 \alpha_s v_s\left(\mathbf{X}_1^k, \mathbf{X}_2^m, \ldots, \mathbf{X}_{s-1}^n\right)\rfloor}{N_1} = \alpha_s v_s\left(\mathbf{X}_1^k, \mathbf{X}_2^m, \ldots, \mathbf{X}_{s-1}^n\right)$$

it can be shown that, for any value of $N_1$

$$\frac{E[N_{\mathcal{A}}]}{E[N_t]} = \frac{\mathrm{vol}(\mathcal{A})}{\mathrm{vol}(\mathcal{C})}.$$

Unfortunately, this equality is not true in general. However, as shown next, the difference between these values can be made very small even for relatively small values of $N_1$.

*Theorem 2:* Consider a set $\mathcal{A} \subseteq \mathcal{C}$. Then, there exist constants $k_1$, $k_2$ and $k_3$ such that, for any $N_1$

$$\left|\frac{E[N_{\mathcal{A}}]}{E[N_t]} - \frac{\mathrm{vol}(\mathcal{A})}{\mathrm{vol}(\mathcal{C})}\right| \leq \frac{1}{N_1}\frac{k_1}{k_2 + \frac{k_3}{N_1}}.$$

*Proof:* See Appendix B ∎

### C. $\mathcal{BH}_\infty^n$ as a Simpler Case

In the case of general convex sets $\mathcal{C}$, Algorithm 1 requires knowledge of the volume of the projection sets up to a multiplying constant. However, as we show in the sequel, for sets of the form $\mathcal{C}_{\mathcal{H}_n} \doteq \{\{\mathbf{H}_i\}_{i=0}^{n-1} : \overline{\sigma}(\mathbf{T}_H^n) \leq 1\}$ it is possible to *analytically* find these quantities. Since these are precisely the sets arising in the context of Problem 1, and since the linear spaces $\mathbf{R}^{s\times m}$ and $\mathbf{R}^{sm}$ are isomorphic, it follows that this problem can be efficiently solved by applying Algorithm 1.

Given $\{\mathbf{H}_0, \mathbf{H}_1, \ldots, \mathbf{H}_{k-1}\}, 1 \leq k \leq n$, consider the problem of determining the set

$$\begin{aligned}&\mathrm{Proj}^{n_k}\left(\mathcal{C}_k(\mathbf{H}_0, \mathbf{H}_1, \ldots, \mathbf{H}_{k-1})\right)\\ &\quad \doteq \{\mathbf{H}_k : (\mathbf{H}_0, \ldots, \mathbf{H}_{k-1}, \mathbf{H}_k, \mathbf{H}_{k+1}, \ldots, \mathbf{H}_{n-1}) \in \mathcal{C}_{\mathcal{H}_n},\\ &\qquad \text{for some}(\mathbf{H}_{k+1}, \ldots, \mathbf{H}_{n-1})\}.\end{aligned} \tag{3}$$

From Parrott's Theorem [20, p. 40], it follows that the set (3) is given by

$$\left\{\mathbf{H}_k : \overline{\sigma}\left(\mathbf{T}_H^{k+1}(\mathbf{H}_0, \mathbf{H}_1, \ldots, \mathbf{H}_k)\right) \leq 1\right\}.$$

[2]Note that both $ns_t$ and $ns_{\mathcal{A}}$ are random variables.

An explicit parameterization of this set can be obtained as follows. Partition

$$\mathbf{T}_H^{k+1}(\mathbf{H}_0, \mathbf{H}_1, \ldots, \mathbf{H}_k) = \begin{bmatrix}\mathbf{H}_k & \mathbf{B}\\ \mathbf{C} & \mathbf{A}\end{bmatrix} \tag{4}$$

and let the matrices $\mathbf{Y}$ and $\mathbf{Z}$ be a solution of the linear equations

$$\mathbf{B} = \mathbf{Y}(\mathbf{I} - \mathbf{A}^T\mathbf{A})^{\frac{1}{2}} \tag{5}$$

$$\mathbf{C} = (\mathbf{I} - \mathbf{A}\mathbf{A}^T)^{\frac{1}{2}}\mathbf{Z} \tag{6}$$

$$\overline{\sigma}(\mathbf{Y}) \leq 1 \quad \overline{\sigma}(\mathbf{Z}) \leq 1. \tag{7}$$

Then

$$\begin{aligned}\left\{\mathbf{H}_k : \overline{\sigma}(\mathbf{T})_H^{k+1} \leq 1\right\} = \Big\{\mathbf{H}_k : \mathbf{H}_k &= -\mathbf{Y}\mathbf{A}^T\mathbf{Z}\\ &+ (\mathbf{I} - \mathbf{Y}\mathbf{Y}^T)^{\frac{1}{2}}\mathbf{W}(\mathbf{I} - \mathbf{Z}^T\mathbf{Z})^{\frac{1}{2}}, \overline{\sigma}(\mathbf{W}) \leq 1\Big\}.\end{aligned}$$

Hence, generating uniform samples over the set (3) reduces to the problem of uniformly sampling the set $\{\mathbf{W} : \overline{\sigma}(\mathbf{W}) \leq 1\}$. Algorithms to do sampling over such sets are readily available (see, for instance, [5]). In addition, this parameterization allows for easily computing, up to a multiplying constant, the volume of the set $\mathrm{Proj}^{n_k}(\mathcal{C}_k)$, required in step 2) of Algorithm 1. This follows from the fact that $\mathrm{Proj}^{n_k}(\mathcal{C}_k(\mathbf{H}_0, \mathbf{H}_1, \ldots, \mathbf{H}_{k-1}))$ is a linear transformation of the set $\mathcal{M}(\{\mathbf{W} : \overline{\sigma}(\mathbf{W}) \leq 1\})$ and, thus

$$\mathbf{J}(\mathbf{H}_0, \mathbf{H}_1, \ldots, \mathbf{H}_{k-1}) = \frac{\mathrm{vol}\left(\mathrm{Proj}^{n_k}\left(\mathcal{C}_k(\mathbf{H}_0, \mathbf{H}_1, \ldots, \mathbf{H}_{k-1})\right)\right)}{\mathrm{vol}\left(\mathcal{M}\left(\{\mathbf{W} : \overline{\sigma}(\mathbf{W}) \leq 1\}\right)\right)}$$

where

$$\mathbf{J}(\mathbf{H}_0, \mathbf{H}_1, \ldots, \mathbf{H}_{k-1}) = \left|(\mathbf{I} - \mathbf{Y}\mathbf{Y}^T)^{\frac{1}{2}}\right|^m \left|(\mathbf{I} - \mathbf{Z}^T\mathbf{Z})^{\frac{1}{2}}\right|^s \tag{8}$$

is the Jacobian of the previous transformation (see [5], Appendix F). Combining these observations leads to the following algorithm for solving Problem 1.

*Algorithm 2*

1) Let $k = 0$. Generate $N_1$ samples uniformly distributed over the set

$$\{\mathbf{H}_0 : \overline{\sigma}(\mathbf{H}_0) \leq 1\}.$$

2) Let $k := k + 1$. For every generated sample $(\mathbf{H}_0^l, \mathbf{H}_1^l, \ldots, \mathbf{H}_{k-1}^l)$, partition $\mathbf{T}_H^{k+1}$ as in (4), and find a solution $\mathbf{Y}$ and $\mathbf{Z}$ to (5). Generate $\lfloor N_1 \mathbf{J}(\mathbf{H}_0, \mathbf{H}_1, \ldots, \mathbf{H}_{k-1})\rfloor$ samples uniformly over the set $\{\mathbf{W} : \overline{\sigma}(\mathbf{W}) \leq 1\}$ and for each of those samples $\mathbf{W}^i$, take

$$\mathbf{H}_k^i = -\mathbf{Y}\mathbf{A}^T\mathbf{Z} + (\mathbf{I} - \mathbf{Y}\mathbf{Y}^T)^{\frac{1}{2}}\mathbf{W}^i(\mathbf{I} - \mathbf{Z}^T\mathbf{Z})^{\frac{1}{2}}.$$

3) If $k < m$, go to step 2). Else, stop.

## IV. Applications

In this section, we apply our theoretical framework to the problems of model (in)validation and finite horizon performance analysis.

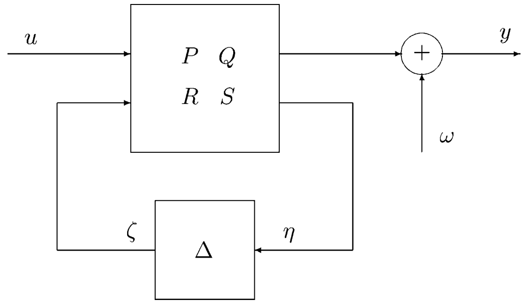


Fig. 1. Model (in)validation setup.

### A. Application 1: Risk-Adjusted Model (In)Validation

Consider the lower LFT interconnection, shown in Fig. 1, of a known model $M$ and structured dynamic LTI uncertainty $\Delta$. The block $M$

$$M \doteq \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \tag{9}$$

consists of a nominal model $P$ of the actual system and a description, given by the blocks $Q$, $R$, and $S$[3] of how uncertainty enters the model. The block $\Delta$ is known to belong to a given set $\mathbf{\Delta}_{st}(\gamma) \doteq \{\Delta : \Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_l),\ \|\Delta_i\|_\infty \leq \gamma, \forall i = 1, \ldots, l\}$. Finally, the signals $u$ and $y$ represent a known test input and its corresponding output, respectively, corrupted by measurement noise $\omega \in \mathcal{N} = \{\omega \in \ell_p^m : \|\omega\|_{\ell^\infty} \leq \epsilon_t\}$. Given the time-domain measurements $u \doteq \{\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n\}$, $y \doteq \{\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n\}$, the goal is to determine if they are consistent with the assumed *a priori* information $(M, \mathcal{N}, \mathbf{\Delta}_{st})$, i.e., whether the consistency set

$$\mathcal{T}(y) = \{(\Delta, \omega) : \Delta \in \mathbf{\Delta}_{st}, \omega \in \mathcal{N} \text{ and } \mathbf{y}_k = (\mathcal{F}_l(M, \Delta) * u + \omega)_k,\ k = 0, \ldots, n\} \tag{10}$$

is nonempty.

It is well know that this problem is NP-hard in the number of uncertainty blocks [19]. In the sequel we propose to avoid this difficulty by pursuing a *risk-adjusted* approach. The basic idea is to sample the set $\mathbf{\Delta}_{st}$ in an attempt to find an element that, together with an admissible noise, explains the observed experimental data. If no such uncertainty can be found, then we can conclude that, with a certain probability, the model is invalid. Note that, given a *finite* set of $n$ input–output measurements, since $\Delta$ is causal, *only the first* $n$ *Markov parameters* affect the output $y$. Thus, we only need to generate uniform samples of the first $n$ Markov parameters of elements of the set $\mathbf{\Delta}_{st}$. Combining this observation with Algorithm 2, leads to the following model (in)validation algorithm.

*Algorithm 3*

Given $\gamma_{st}$, select $N_1$ and take $ns(N_1)$ samples of $\mathbf{\Delta}_{st}(\gamma_{st})$, $\{\Delta^i(z)\}_{i=1}^{ns}$, according to the procedure described in Section III-C.

1) At step $s$, let $\omega^s \doteq \{(y - \mathcal{F}_l(M, \Delta^s) * u)_k\}_{k=0}^{n}$.
2) If $\omega^s \in \mathcal{N}$, stop. Otherwise, consider next sample $\Delta^{s+1}(z)$ and go back to step 2).

Clearly, the existence of at least one $\omega^s \in \mathcal{N}$ is equivalent to $\mathcal{T}(y) \neq \emptyset$. The algorithm finishes, either by finding one admissible uncertainty $\Delta^s(z)$ that makes the model set not invalidated by the data or after $ns$ steps, in which case the model is deemed to be invalid. Straightforward application of the results in [16], shows that if $N_1$ is chosen such that

$$N_1 \geq \frac{\ln\left(\frac{1}{\delta}\right)}{\ln\left(\frac{1}{(1-\epsilon)}\right)} \tag{11}$$

where $(\epsilon, \delta)$ are two positive constants in (0, 1), then with probability greater than $1 - \delta$, the probability of rejecting a model which is not invalidated by the data is smaller that $\epsilon$.[4]

Thus, by introducing an (arbitrarily small) risk of rejecting a possibly *good* candidate model, we can substantially alleviate the computational complexity entailed in validating models subject to structured uncertainty. In addition, as pointed out in [21] the deterministic approach to model invalidation is optimistic since a candidate model will be accepted even if there exists only a very small set of pairs (uncertainty, noise) that validate the experimental record. On the other hand, both the approach in [21] and the one proposed here will reject (with probability close to 1) such models. The main difference between these approaches is related to the experimental data and the *a priori* assumptions. The approach in [21] uses frequency domain data and relies heavily on the whiteness of the noise process and independence between samples at different frequencies. On the other hand, the approach pursued in this note is based on time-domain data and the risk estimates are independent of the specific probability density function of $\Delta$. [16].

In order to illustrate the proposed method, assume that

$$\begin{aligned}
P(z) &= \frac{0.2(z+1)^2}{18.6z^2 - 48.8z + 32.6} \\
Q(z) &= [1 \quad 0 \quad -1] \quad R(z) = [0 \quad 1 \quad 1]^T \\
S(z) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\hat{\Delta}(z) &= \begin{bmatrix} \frac{0.125(5.1-4.9z)}{(6.375-3.6250z)} & 0 & 0 \\ 0 & \frac{0.1(5.001-4.9990z)}{(6.15-3.85z)} & 0 \\ 0 & 0 & \frac{0.05(5.15-4.85z)}{(6.95-3.05z)} \end{bmatrix}.
\end{aligned} \tag{12}$$

Our experimental data consists of a set of $n = 20$ samples of the impulse response of $\hat{G}(z) = \mathcal{F}_\ell(P, \hat{\Delta})$, corrupted by noise $\|\omega\|_{\ell^\infty} \leq 0.0041$. The noise bound $\epsilon_t$ represents a 10% of the peak value of the impulse response. Our goal is to find the minimum size of the uncertainty, $\gamma_{st}$, so that the model is not invalidated by the data. A coarse lower bound $\gamma_{st} \geq 0.0158$ can be obtained by performing an LMI-based invalidation test using *unstructured* uncertainty [7]. Starting from this value of $\gamma$, we generated three sets with $N_1 = 300$,[5] one for each of the scalar blocks, with $\|\Delta_i(z)\|_\infty \leq \gamma_{st}$, and, at each given value of $\gamma_{st}$, we evaluated the function

$$f(\Delta^s) = \epsilon_t - \left\| \{\mathcal{F}_l(M, \Delta^s) * u - y\}_{k=0}^{n} \right\|_{\infty[0,n]}$$

for all $\Delta^s \in \mathbf{\Delta}_{st}(\gamma_{st})$. If $\forall \Delta^s$, $f(\Delta^s) < 0$, then the model is invalidated by the data with high probability. It is then necessary to increase the value of $\gamma_{st}$ and continue the (in)validation test. In this particular example, the test was repeated over a grid of 1000 points of the interval $\mathcal{I}$ until we obtained the value $\gamma_{st}$ of 0.0775, the minimum value of $\gamma_{st}$ for which the model was not invalidated by the given experimental evidence.

The proposed approach differs from the one in [7] in that here the invalidation test is performed by searching over $\mathbf{\Delta}_{st}$ with the hope of finding one admissible $\Delta \in \mathbf{\Delta}_{st}$ that makes the model not invalid;

[3]We will assume that the structured singular value, $\mu_\Delta(S) < \gamma^{-1}$ so that the interconnection $\mathcal{F}_l(M, \Delta)$ is well-posed.

[4]These probabilities are measured with respect to the pdf of the samples generated by Algorithm 1.

[5]This guarantees probability of at least 0.985 that $\mathrm{prob}\{f(\Delta) > 0\} \leq 0.015$.

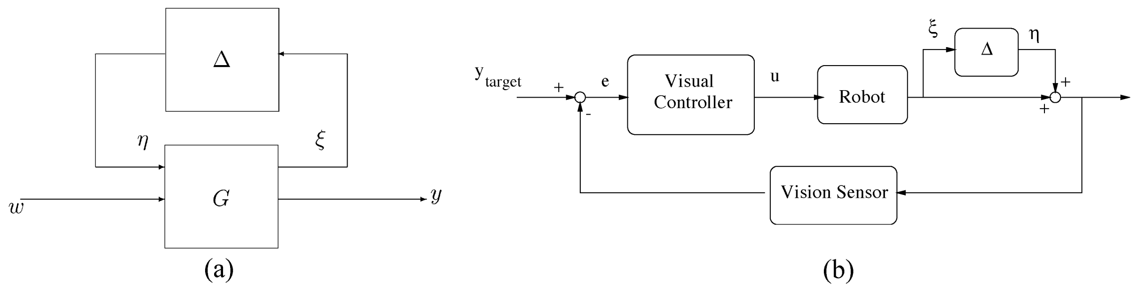


Fig. 2. (a) Robust performance analysis problem. (b) Application to active vision.

TABLE I
RISK-ADJUSTED ASSESSMENT OF ROBUST PERFORMANCE

| $\gamma$ | $\max_{k\in[0,8],\Delta^i} \frac{\lvert y^i(k)\rvert-50}{50}100$ | $\max_{k\in[9,39],\Delta^i} \lvert y^i(k)\rvert - 10(0.95)^{k-9}$ | **RP** |
|---|---|---|---|
| 0.0111 | 1.5819% | −1.0190 | Pass |
| 0.0203 | 2.8938% | −0.0706 | Pass |
| 0.0221 | 3.1590% | 0.1212 | **Fail** |

while there it is done by searching over the class of unstructured uncertainties $\mathbf{\Delta}_u$ and by introducing, at each step, diagonal similarity scaling matrices with the aim of invalidating the model. More precisely, if at step $k$ the model subject to unstructured uncertainty remains *not* invalidated (which is equivalent to the existence of at least one feasible pair $(\zeta, \mathbf{D}_k)$ so that a given matrix $\mathbf{M}(\zeta, \mathbf{D}_k) \leq 0$), one possible strategy is to select the scaling $\mathbf{D}_{k+1}$ so as to maximize the trace of $\mathbf{M}$. See [6, Ch. 9, pp. 301–306] for details. However, for this particular example $\mathbf{D}_k = \mathrm{diag}(d_{1k}, d_{2k}, d_{3k})$ and this last condition becomes

$$\sup_{d_{1k},d_{2k},d_{3k}} -n\left(1+\frac{1}{\gamma^2}\right)(d_{2k}+d_{3k}) + n\left(1-\frac{1}{\gamma^2}\right)d_{1k}$$
$$d_{1k}, d_{2k}, d_{3k} \geq 0.$$

For $0 < \gamma < 1$, clearly the supremum is achieved at $d_{1k} = 0$, $d_{2k} = 0$ and $d_{3k} = 0$. As an alternative searching strategy, one may attempt to randomly check condition $\mathbf{M}(\zeta, \mathbf{D}_k) \leq 0$ by sampling appropriately the scaling matrices, following [19]. Using 6000 samples led to a value of $\gamma_{st}$ of 0.031 05 for which the model was invalidated by the data. For values of $\gamma_{st}$ in [0.031 05,0.125] nothing can be concluded regarding the validity of the model.

Combination of these bounds with the risk-adjusted ones obtained earlier shows that the model is definitely invalid for $\gamma_{st} \leq 0.031\,05$, invalid with probability 0.999 in $0.031\,05 < \gamma_{st} < 0.0755$ and it is not invalidated by the experimental data available thus far for $0.0755 \leq \gamma_{st} \leq 0.125$. Thus, these approaches, rather than competing, can be combined to obtain sharper conditions for rejecting candidate models.

### B. Application 2: Finite Horizon Robust Performance Analysis

Consider the interconnection shown in Fig. 2(a), where $w \in R^{n_w}$ and $y \in R^{n_y}$ represent a fixed, known test signal and the corresponding regulated output. We will assume that a robustly stabilizing controller has been already found so that the interconnection is stable for all $\Delta \in \mathcal{D}(\gamma) \doteq \{\Delta \in \mathcal{H}_\infty : \Delta = \mathrm{diag}\{\Delta_i\}, \|\Delta\|_\infty \leq \gamma\}$. The goal is to establish if time domain constraints of the form

$$\|y(k)\|_\infty \leq \phi_i(k), \qquad k = 0, 1, \ldots n \tag{13}$$

hold for all $\Delta \in \mathcal{D}(\gamma)$ and all test signals $w$ in a given set $\mathcal{W}_t$. Typical choices are $\mathcal{W}_t = \{\text{unit step}\}$ or $\mathcal{W}_t = \{\text{unit impulse}\}$ and $\phi_i(.)$:

$$\begin{aligned} \phi(k) &= M, \qquad k = 0, 1, \ldots, k_1 \\ \phi(k) &= Ma^{k-k_1}, \qquad k_1 \leq k \leq n, \qquad 0 < a < 1. \end{aligned} \tag{14}$$

The aforementioned problem is a generalization of the time-domain constrained $\mathcal{H}_\infty$ [15] and $\mathcal{H}_2$ [17] control problems, that considers robust, rather than nominal, performance. As we briefly outline next, it can be solved by sampling the set $\mathcal{D}(\gamma)$ in an attempt to find an element $\Delta_{\mathrm{worst}}$ such that the corresponding interconnection violates the constraints. If no such uncertainty can be found, then we can conclude that, with a certain probability, the constraints are robustly satisfied. As before, since only the first $n+1$ Markov parameters of $\Delta$ affect the output $y$ in $[0, n]$, the problem also reduces to sampling $\mathcal{BH}_\infty^n$.

Next, we illustrate this approach using the problem illustrated in Fig. 2(b), where the goal is to internally stabilize the plant and to track target motions, $w_{\mathrm{target}}$, using as measurements images possibly corrupted by noise. Assuming that the velocity of the target and the measurement noise are $\ell^2$ signals and that the tracking error is measured in the $\ell^\infty$ sense leads to an optimal $\mathcal{H}_2$ control problem. Moreover, in order to avoid saturation and target walk off problems, the control action in response to a 25 units step input should satisfy (see [17] for details)

$$|y(k)| \leq f_b(k) \quad f_b(k) \doteq \begin{cases} 50(1+0.1), & 0 \leq k < 9 \\ 10(0.95)^{k-9}, & 9 \leq k \leq 39. \end{cases} \tag{15}$$

A constrained $\mathcal{H}_2$ controller satisfying these specifications for the nominal plant was designed in [17]. Our goal here is to assess its robustness, by determining how much uncertainty can be tolerated before the specifications are violated.

Table I shows the results obtained by applying the proposed risk-adjusted approach with $ns = 500$ samples, corresponding to a probability $\delta = 0.0095$ of passing the tests while in fact the risk of exceeding the constraints is above $\epsilon = 0.0095$. As shown there, the control constraints are violated if $\|\Delta\|_\infty \geq 0.0221$.

## V. CONCLUSION

Probabilistic methods have the potential to address both the issue of the conservatism of worst-case bounds and the computational complexity entailed in their computation. However, up to the present time application of these methods has been limited to the case of finite-dimensional parametric uncertainty, largely due to the unavailability of methods for generating samples from sets of bounded causal operators.

In this note, we take steps toward removing this limitation by proposing a computationally efficient algorithm for sampling sets of the form $\mathcal{BH}_\infty^n$. As we show in the note, samples generated with the proposed algorithm can be used to perform model (in)validation in the presence of structured uncertainty and robust performance over finite horizons. In addition, these sets can be also used to assess, up to an arbitrary precision, infinite horizon $\epsilon$ robust performance against structured dynamic uncertainty, by resorting to an iterative procedure. Salient features of the proposed approach are that: i) the computational complexity scales linearly, rather than exponentially, with the number of uncertainty blocks, and ii) the results are independent of the actual probability distribution of both the uncertainty and the noise. Finally, these results can be directly applied to the problem of synthesizing robust controllers by using the samples generated in combination with a stochastic gradient algorithm [13].

## APPENDIX

### A. Proof of Theorem 1

For the sake of notational simplicity we will prove the result for the case where the number of partitions of the vector $x$ is $m = 4$, but the same reasoning applies to arbitrary dimensions.

Consider a rectangle $R \doteq R_1 \times R_2 \times R_3 \times R_4 \subseteq \mathcal{C}$, where $R_i \subset \mathbf{R}^{k_i}$, $i = 1, 2, 3, 4$. Let $N_t(N_1)$ and $N_R(N_1)$ be the total number of samples generated and the number of hits of $R$, respectively. We will show that

$$\frac{N_R}{N_t} \xrightarrow{w.p.1} \frac{\text{vol}(R_1)\text{vol}(R_2)\text{vol}(R_3)\text{vol}(R_4)}{\text{vol}(\mathcal{C})}.$$

In other words, the ratio converges with probability one to a constant which is equal to the probability of the rectangle $R$ under a uniform distribution over the set $\mathcal{C}$. Henceforth, the symbol $\rightarrow$ denotes convergence with probability one.

Let $\mathbf{X}_1^k$ be the $k$th sample of the first component of the vector. Similarly, denote by $\mathbf{X}_2^{mk}$ and $\mathbf{X}_3^{nmk}$ the $m$th sample of the second component of the vector when the first component is $\mathbf{X}_1^k$, and the $n$th sample of the third component of the vector when the first two components are $\mathbf{X}_1^k$ and $\mathbf{X}_2^{mk}$, respectively. Finally, denote by $v_k \doteq \text{vol}[I_k(\mathbf{X}_1, \ldots, \mathbf{X}_{k-1})]$. Consider

$$\frac{N_t}{N_1^4} = \frac{1}{N_1}\sum_{k=1}^{N_1}\frac{1}{N_1}\sum_{m=1}^{\lfloor \alpha_2 N_1 v_2(\mathbf{X}_1^k)\rfloor}\frac{1}{N_1}\sum_{n=1}^{\lfloor N_1\alpha_3 v_3(\mathbf{X}_1^k,\mathbf{X}_2^{mk})\rfloor} \times\frac{1}{N_1}\left\lfloor N_1\alpha_4 v_4\left(\mathbf{X}_1^k,\mathbf{X}_2^{mk},\mathbf{X}_3^{nmk}\right)\right\rfloor.$$

Using the following equalities:

$$E\left[v_4\left(\mathbf{X}_1^k,\mathbf{X}_2^{mk},\mathbf{X}_3^{nmk}\right)|\mathbf{X}_1^k,\mathbf{X}_2^{mk}\right] = \frac{\text{vol}\left[\mathcal{S}_2\left(\mathbf{X}_1^k,\mathbf{X}_2^{mk}\right)\right]}{v_3\left(\mathbf{X}_1^k,\mathbf{X}_2^{mk}\right)}$$
$$E\left[\mathcal{S}_2\left(\mathbf{X}_1^k,\mathbf{X}_2^{mk}\right)|\mathbf{X}_1^k\right] = \frac{\text{vol}\left[\mathcal{S}_3\left(\mathbf{X}_1^k\right)\right]}{v_2\left(\mathbf{X}_1^k\right)}$$
$$E\left[\text{vol}\left[\mathcal{S}_3\left(\mathbf{X}_1^k\right)\right]\right] = \frac{\text{vol}(\mathcal{C})}{v_1}$$

the fact that $\lim_{N_1\to\infty}\lfloor N_1 a\rfloor/N_1 = a$ for any $a$ and applying the Strong Law of Large Numbers (e.g. see [11]), one obtains

$$\frac{N_t}{N_1^4} \rightarrow \frac{\alpha_2\alpha_3\alpha_4}{v_1}\text{vol}(\mathcal{C}) \text{ as } N_1 \rightarrow \infty.$$

Next, consider the number of hits of the rectangle $R$, which we denote by $N_R$. The Strong Law of Large Numbers implies that

$$\left.\frac{N_R}{N_1}\right| \mathbf{X}_1^k \in R_1, \mathbf{X}_2^{mk} \in R_2, \mathbf{X}_3^{nmk} \in R_3 \rightarrow \alpha_4 v_4 \times \left(\mathbf{X}_1^k,\mathbf{X}_2^{mk},\mathbf{X}_3^{nmk}\right)\frac{\text{vol}(R_4)}{v_4\left(\mathbf{X}_1^k,\mathbf{X}_2^{mk},\mathbf{X}_3^{nmk}\right)} = \alpha_4\text{vol}(R_4)$$

which is independent of the values of $\mathbf{X}_1^k$, $\mathbf{X}_2^{mk}$ and $\mathbf{X}_3^{nmk}$. Repeating the same reasoning, we obtain

$$\frac{N_R}{N_1^4} \rightarrow \frac{1}{v_1}\alpha_2\alpha_3\alpha_4\text{vol}(R_1)\text{vol}(R_2)\text{vol}(R_3)\text{vol}(R_4).$$

Hence, as $N_1 \rightarrow \infty$

$$\frac{N_R}{N_t} \rightarrow \frac{\text{vol}(R_1)\text{vol}(R_2)\text{vol}(R_3)\text{vol}(R_4)}{\text{vol}(\mathcal{C})}.$$

### B. Proof of Theorem 2

As in the proof of Theorem 1, only $m = 4$ is considered and it is assumed that

$$\mathcal{A} \doteq R_1 \times R_2 \times R_3 \times R_4 \subseteq \mathcal{C}$$

where $R_i \subset \mathbf{R}^{k_i}$, $i = 1, 2, 3, 4$, satisfy $\text{vol}(R_i) = dx_i$. The proof can be easily generalized for other values of $m$ and other sets $\mathcal{A}$.

Using the notation in the proof of Theorem 1, we first consider $E[N_t]$. The reasoning to follow relies on the fact that, given two random variables, $X$ and $Y$, $E[Y] = E[E[Y|X]]$. Indeed

$$E\left[\frac{N_t}{N_1^4}\right] = E\left[\frac{1}{N_1}\sum_{k=1}^{N_1}\frac{1}{N_1}\sum_{m=1}^{\lfloor \alpha_2 N_1 v_2(\mathbf{X}_1^k)\rfloor}\frac{1}{N_1} \times\sum_{n=1}^{\lfloor N_1\alpha_3 v_3(\mathbf{X}_1^k,\mathbf{X}_2^{mk})\rfloor}\frac{1}{N_1}\left\lfloor N_1\alpha_4 v_4\left(\mathbf{X}_1^k,\mathbf{X}_2^{mk},\mathbf{X}_3^{nmk}\right)\right\rfloor\right].$$

Moreover

$$E\left[\frac{1}{N_1}\sum_{n=1}^{\lfloor N_1\alpha_3 v_3(\mathbf{X}_1^k,\mathbf{X}_2^{mk})\rfloor}\frac{1}{N_1} \times\left\lfloor N_1\alpha_4 v_4\left(\mathbf{X}_1^k,\mathbf{X}_2^{mk},\mathbf{X}_3^{nmk}\right)\right\rfloor|\mathbf{X}_1^k,\mathbf{X}_2^{mk}\right] = \frac{\left\lfloor N_1\alpha_3 v_3\left(\mathbf{X}_1^k,\mathbf{X}_2^{mk}\right)\right\rfloor}{N_1}\left(\alpha_4\frac{\text{vol}\left(\mathcal{S}_2\left(\mathbf{X}_1^k,\mathbf{X}_2^{mk}\right)\right)}{v_3\left(\mathbf{X}_1^k,\mathbf{X}_2^{mk}\right)} - \frac{\varepsilon_4}{N_1}\right)$$

where $\varepsilon_4 \in [0, 1]$. Repeating the previous reasoning, one obtains

$$E\left[\frac{N_t}{N_1^4}\right] = \frac{\alpha_2\alpha_3\alpha_4}{v1}\text{vol}(\mathcal{C}) - \frac{\beta_1+\beta_2+\beta_3}{N_1}$$

where $\beta_1$, $\beta_2$ and $\beta_3$ above are bounded functions of $N_1$. Hence, there exists a constant $\beta$ such that

$$-\frac{\beta}{N_1} \le E\left[\frac{N_t}{N_1^4}\right] - \frac{\alpha_2\alpha_3\alpha_4}{v_1}\text{vol}(\mathcal{C}) \le 0.$$

Next consider $E[N_{\mathcal{A}}/N_1^4]$ which equals

$$E\left[\frac{1}{N_1^4}\sum_{k=1}^{N_1}\sum_{m=1}^{\lfloor \alpha_2 N_1 v_2(\mathbf{X}_1^k)\rfloor}\sum_{n=1}^{\lfloor N_1\alpha_3 v_3(\mathbf{X}_1^k,\mathbf{X}_2^{mk})\rfloor} \times\sum_{l=1}^{\lfloor N_1\alpha_4 v_4(\mathbf{X}_1^k,\mathbf{X}_2^{mk},\mathbf{X}_3^{nmk})\rfloor} I_{\mathbf{X}_1^k\in R_1,\mathbf{X}_2^{mk}\in R_2,\mathbf{X}_3^{nmk}\in R_3,\mathbf{X}_4^{nmkl}\in R_4}\right]$$

and where $I$ denotes the indicator function. Repeating the previous reasoning, one obtains

$$E\left[\frac{N_{\mathcal{A}}}{N_1^4}\right] = dx_1dx_2dx_3dx_4\frac{\alpha_2\alpha_3\alpha_4}{v_1} - \frac{\gamma_1+\gamma_2+\gamma_3}{N_1}$$

where $\gamma_1$, $\gamma_2$ and $\gamma_3$ above are bounded functions of $N_1$. Hence, there exists a constant $\gamma$ such that

$$-\frac{\gamma}{N_1} \le E\left[\frac{N_{\mathcal{A}}}{N_1^4}\right] - dx_1 dx_2 dx_3 dx_4 \frac{\alpha_2\alpha_3\alpha_4}{v_1} \le 0.$$

The proof is completed by noting that given the aforementioned results, one can determine constants $k_1$, $k_2$ and $k_3$ such that

$$\left|\frac{E[N_{\mathcal{A}}]}{E[N_t]} - \frac{\mathrm{vol}(\mathcal{A})}{\mathrm{vol}(\mathcal{C})}\right| = \left|\frac{E\left[\frac{N_{\mathcal{A}}}{N_1^4}\right]}{E\left[\frac{N_t}{N_1^4}\right]} - \frac{\mathrm{vol}(\mathcal{A})}{\mathrm{vol}(\mathcal{C})}\right| \le \frac{1}{N_1}\frac{k_1}{k_2 + \frac{k_3}{N_1}}.$$

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# Comments on "Explicit Criterion for the Positive Definiteness of a General Quartic Form"

Fei Wang and Liqun Qi

***Abstract*—The purpose of this note is to point out that the result in the above paper is incomplete, and to give a complete and improved result.**

***Index Terms*—Positive definiteness, quartic polynomial, roots.**

The above paper [2] gives a necessary and sufficient condition for the positive definiteness of the following quartic form of two variables:

$$V(x_1, x_2) = k_0x_1^4 + k_1x_1^3x_2 + k_2x_1^2x_2^2 + k_3x_1x_2^3 + k_4x_2^4 \quad (1)$$

for all $x_1$ and $x_2$, where $k_0$, $k_1$, $k_2$, $k_3$ and $k_4$ are real numbers. The purpose of this paper is to point out that the result in [2] is incomplete, and to give a complete and improved result.

Let $k_0 = a_0$, $k_1 = 4a_1$, $k_2 = 6a_2$, $k_3 = 4a_3$, and $k_4 = a_4$.

i) If $x_2 = 0$, (1) reduces to

$$V(x_1, 0) = a_0x_1^4.$$

Then, $V(x_1, x_2) > 0$ holds for all $x_1 \ne 0$ and $x_2 = 0$ if and only if $a_0 > 0$.

ii) If $x_2 \ne 0$, then

$$V(x_1, x_2) = x_2^4\left[a_0\left(\frac{x_1}{x_2}\right)^4 + 4a_1\left(\frac{x_1}{x_2}\right)^3 + 6a_2\left(\frac{x_1}{x_2}\right)^2 + 4a_3\left(\frac{x_1}{x_2}\right) + a_4\right].$$

Let $x = x_1/x_2$. Then, $V(x_1, x_2) > 0$ holds for all $x_2 \ne 0$ if and only if

$$f(x) = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0 \quad (2)$$

has no real roots for all values of $x$ and $a_0 > 0$.

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