An Exact Solution to Continuous-Time Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Problems

M. Sznaier, H. Rotstein, Juanyu Bu, and A. Sideris

***Abstract*—Multiobjective control problems have been the object of much attention in the past few years, since they allow for handling multiple, perhaps conflicting, performance specifications and model uncertainty. One of the earliest multiobjective problems is the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem, which can be motivated as a nominal LQG optimal control problem subject to robust stability constraints. This problem has proven to be surprisingly difficult to solve, and at this time no closed-form solutions are available. Moreover, it has been shown that except in some trivial cases, the optimal controller is infinite-dimensional.**

In this paper, we propose a solution to general continuous-time mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems, based upon constructing a family of approximating problems, obtained by solving an equivalent discrete-time problem. Each of these approximations can be solved efficiently, and the resulting controllers converge strongly in the $\mathcal{H}_2$ topology to the optimal solution.

***Index Terms*—$\mathcal{H}_2$ control, $\mathcal{H}_\infty$ control, multiobjective control.**

I. Introduction

A large number of practical control problems involve designing a controller that minimizes the worst case response to some exogenous disturbances. The case where the exogenous disturbances w are bounded spectral density signals and the objective is to minimize the power of the output z leads to the well-known $\mathcal{H}_2$ control problem. This problem is appealing since there is a well-established connection between the performance index being optimized and performance requirements encountered in practical situations. Moreover, the resulting controllers are easily found by solving two Riccati equations, and in the state-feedback case exhibit good robustness properties [1]. However, as established in [4], these margins vanish in the ouput feedback case.

Following this paper, several attempts were made to incorporate robustness into the $\mathcal{H}_2$ framework [16], [17], [5], [10]. However, this problem has proven to be surprisingly difficult, and to date no necessary and sufficient robust performance conditions are available [22].

An alternative is to settle for nominal $\mathcal{H}_2$ performance subject to a robust stability constraint. This leads to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem [3], [24], [7], [21], [18], illustrated in Fig. 1, where the objective is to synthesize a controller $u(s) = K(s)y(s)$ such that $\|T_{\zeta_2,w_2}\|_2$ is minimized, subject to the specification $\|T_{\zeta_\infty w_\infty}(s)\|_\infty \leq \gamma$.

A large portion of the work in this field [3], [24], [7], [19] addresses the related problem of minimizing an *upper bound* of the $\mathcal{H}_2$ norm, subject to the $\mathcal{H}_\infty$ constraint. This modified problem has the advantage of leading to a mathematically tractable formulation, but in some cases the resulting controller performs worse than the "central" $\mathcal{H}_\infty$ controller [2].

An alternative approach is to use the Youla parametrization to recast the $\mathcal{H}_2/\mathcal{H}_\infty$ problem as an *infinite* dimensional convex optimization.

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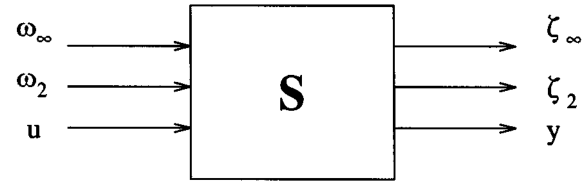

Fig. 1. The generalized plant.

While it is known that the solution to this problem is in general infinite-dimensional [9], recent work has shown that, in the discrete-time version of the problem, performance arbitrarily close to optimal can be achieved using rational controllers. Moreover, ϵ-suboptimal solutions can be found by solving a sequence of truncated problems [21], [18], [6], [13]. The continuous time counterpart of the problem is considerably less developed. An iterative procedure to find the optimal cost and suboptimal controllers was proposed in [18], based also on the solution to a sequence of truncated problems. However, contrary to the discrete time case, results showing convergence of the sequence of controllers and closed-loop systems are not currently available.

In this paper, we propose a solution to continuous-time mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem based upon recasting the problem into a discrete-time equivalent. The transformation, motivated by some of the ideas in [15], preserves both the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, allowing for the use of the results in [21] and [13] to obtain an optimizing sequence that converges to the optimal solution strongly in the $\mathcal{H}_2$ topology. Additionally, by using this technique we establish that, as in the discrete-time case, ϵ-suboptimal performance can be achieved by real rational controllers. Finally, we show that closed-loop systems with a prescribed degree of stability can be obtained by solving a single optimization problem, whose size can be determined *a priori*.

II. Preliminaries

A. Notation

$\mathcal{L}^\infty$ denotes the Lebesgue space of complex valued matrix functions essentially bounded on the $j\omega$ axis, equipped with the norm $\|G(s)\|_\infty \doteq ess\ \sup_\omega \overline{\sigma}(G(j\omega))$, where $\overline{\sigma}$ denotes the largest singular value. By $\mathcal{H}_\infty$ ($\mathcal{H}_\infty^\sim$) we denote the subspace of functions in $\mathcal{L}^\infty$ with a bounded analytic continuation in $Re(s) > 0$ ($\mathrm{Re}(s) < 0$). $\mathcal{RH}_\infty$ denotes the subspace of real rational transfer matrices of $\mathcal{H}_\infty$, and $\mathcal{A}_o$ denotes the subset of $\mathcal{H}_\infty$ functions continuous in the *closed* right-half plane. By $\mathcal{H}_2$ we denote the space of complex valued matrix functions $G(s)$ with analytic continuation in $\mathrm{Re}(s) > 0$ and square integrable on the $j\omega$ axis, equipped with the usual $\mathcal{H}_2$ norm

$$\|G\|_2^2 \doteq \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Trace}[G(j\omega)G^*(j\omega)]\, d\omega.$$

The discrete-time counterparts of $\mathcal{L}^\infty, \mathcal{H}_\infty$, and $\mathcal{H}_2$ will be denoted as $\mathcal{L}^\infty(D), \mathcal{H}_\infty(D)$, and $\mathcal{H}_2(D)$, respectively. Also of interest in this discrete-time setting is the Banach space $\mathcal{H}_{\infty,\delta}$ of transfer functions in $\mathcal{H}_\infty$, which are analytic inside the disk of radius δ, where $\delta > 1$ (usually $\delta \approx 1$), equipped with the norm $\|G(z)\|_{\infty,\delta} \doteq \sup_{|z|<\delta} \overline{\sigma}(G(z))$.

The *Laguerre* functions are defined as

$$l_i(s) = \frac{\sqrt{(2a)}}{s+a}\left(\frac{s-a}{s+a}\right)^{i-1}, \qquad i = 1, 2 \ldots \tag{1}$$

where a is a positive real. It is a standard fact (see for instance [8, Ch. 18]) that the family $\{l_i\}$ is an orthonormal basis in $\mathcal{H}_2$. Therefore, any

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function $G(s) \in \mathcal{H}_2$ can be expanded as $G(s) = \sum_{i=1}^{\infty} \Gamma_i l_i$. Since these functions are orthonormal, it follows that $\|G\|_2^2 = \sum_{i=0}^{\infty} \|\Gamma_i\|_F^2$. The projection operator $\mathcal{P}_n: \mathcal{H}_2 \to \mathcal{RH}_2$ is defined by

$$\mathcal{P}_n[G(s)] \doteq \sum_{i=1}^{n} \Gamma_i l_i. \tag{2}$$

In the sequel, we will assume for simplicity that all the signals involved are scalar, although the results presented here can be generalized to the multi-input/multi-output (MIMO) case (at the cost of more involved notation) proceeding as in [13].

B. The Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Problem

It is well known that by using the Youla parametrization, the set of all closed-loop transfer transfer matrices from w_∞ to ζ_∞ and from w_2 to ζ_2, obtained by connecting an internally stabilizing controller from y to u, can be parameterized as [20]

$$\begin{aligned} T(s) &= T_1(s) - T_2(s)Q(s) \\ S(s) &= S_1(s) - S_2(s)Q(s) \end{aligned} \tag{3}$$

where T_i, S_i are stable transfer matrices and $Q(s) \in \mathcal{H}_\infty$ is the "free parameter" in the parametrization. Hence the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem can be stated as follows.

Problem 1 (Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Problem): Find the optimal value of the performance measure

$$\mu \doteq \inf_{Q \in \mathcal{H}_\infty} \{\|S_1 - S_2 Q\|_2 \text{ such that } \|T_1 - T_2 Q\|_\infty \le 1\} \tag{4}$$

and, given $\epsilon > 0$, a controller Q such that $\|S(s,Q)\|_2 \le \mu + \epsilon$ and $\|T(s,Q)\|_\infty \le 1$.

Note that from the strict convexity of the $\mathcal{H}_2$ norm, if a solution to Problem 1 exists, then it is unique. It can be shown [9] that, except in trivial cases where the solution to the optimal $\mathcal{H}_2$ problem satisfies the $\mathcal{H}_\infty$ constraints, Problem 1 admits a minimizing solution in $\mathcal{H}_\infty$ but not in $\mathcal{A}_o$. Thus, implementability considerations lead to considering the restriction of the problem to rational controllers.

Problem 2 (Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Problem in $\mathcal{A}_o$): Find the optimal value of the performance measure

$$\mu_R \doteq \inf_{Q \in \mathcal{A}_o} \{\|S_1 - S_2 Q\|_2 \text{ such that } \|T_1 - T_2 Q\|_\infty \le 1\} \tag{5}$$

and, given $\epsilon > 0$, find $Q_R \in \mathcal{RH}_\infty$ such that $\|S(Q_R)\|_2 \le \mu_R + \epsilon$ and $\|T(Q_R)\|_\infty \le 1$.

III. Problem Solution

A. Computation of a Solution over $\mathcal{H}_\infty$

In this section, we show that Problem 1 can be solved by considering a sequence of finite-dimensional convex optimization problems, where the nth problem has $\mathcal{O}(n)$ variables. Using the projection operator defined in (2), consider the optimization problem.

Problem 3: Find the optimal value of the performance measure

$$\mu^n \doteq \min_{Q \in \mathcal{H}_\infty} \{\|\mathcal{P}_n[S_1(s) - S_2(s)\mathcal{P}_n(Q(s))]\|_2 \text{ such that } \|T_1 - T_2 Q\|_\infty \le 1\} \tag{6}$$

and the corresponding optimal controller Q^n.

Theorem 1: Assume that Problem 1 is feasible and that $S_2(j\omega) \neq 0$. Then $\mu^n \uparrow \mu$ and the sequence of solutions $\{Q^n(s)\}$ converges normally[1] to a solution of Problem 1 in $\mathrm{Re}(s) > 0$. Moreover, for a fixed n, $Q^n(s)$ can be found by solving a finite-dimensional convex optimization problem and an unconstrained $\mathcal{H}_\infty$ problem.

In order to prove this theorem, we begin by introducing a *discrete-time problem* that is equivalent to Problem 1 (in a sense that will be made clear below). Assume that $S_1(s)$ and $Q(s)$ are strictly proper, and let $S_1(s), S_2(s)$, and $Q(s)$ have the following Laguerre expansions:

$$\begin{aligned} S_1(s) &= \sum_{i=1}^{\infty} \sigma_{1,i} l_i(s) \\ S_2(s) &= \sigma_{2,0} + \sum_{i=1}^{\infty} \sigma_{2,i} l_i(s) \\ Q(s) &= \sum_{i=1}^{\infty} \theta_i l_i(s). \end{aligned} \tag{7}$$

Consider now the following discrete-time mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem.

Problem 4:

$$\mu^d \doteq \min_{Q^d(z) \in \mathcal{H}_\infty(D)} \left\{ \left\| \hat{S}_1(z) - \hat{S}_2(z) Q^d(z) \right\|_2 \text{ such that } \left\| T_1^d(z) - T_2^d(z) Q^d(z) \right\|_\infty \le 1 \right\} \tag{8}$$

where

$$\begin{aligned} \hat{S}_1(z) &= \sum_{i=0}^{\infty} \sigma_{1,i} z^i \\ \hat{S}_2(z) &= \sum_{i=0}^{\infty} \hat{s}_{2,i} z^i, \qquad \hat{s}_{2,i} = \sigma_{2,i} + \sigma_{2,0}\sqrt{2a} \end{aligned} \tag{9}$$

and where $T_1^d(z), T_2^d(z)$ are obtained from $T_1(s), T_2(s)$ via the bilinear transformation

$$z = \frac{s-a}{s+a}. \tag{10}$$

As we show next, Problems 1 and 4 are equivalent in the sense that their solutions are related via a bilinear transformation.

Theorem 2: Problem 1 is feasible if and only if Problem 4 is feasible. Moreover, in that case a controller $Q(s)$ is feasible for Problem 1 and yields a cost $\mu_c = \|S_1 + S_2 Q\|_2$ if and only if the controller $Q^d(z) = Q(s)|_{s=a(1+z/1-z)}$ is feasible for Problem 4 and yields a cost $\mu^d \doteq \|\hat{S}_1(z) - \hat{S}_2(z) Q^d(z)\|_2 = \mu_c$.

Proof: See Appendix A.

Corollary 1: The optimal costs in Problems 1 and 4 coincide.

Proof of Theorem 1: From Theorem 2 and its corollary, it follows that Problem 3 is equivalent (in the sense of Theorem 2) to the following truncated discrete-time problem:

$$\mu_n^d \doteq \min_{Q_n^d(z) \in \mathcal{H}_\infty(D)} \left\| \mathcal{P}_n^d \left[\sum_{i=0}^{\infty} \sigma_{1,i} z^i - \left(\sum_{i=0}^{\infty} \hat{s}_{2,i} z^i \right) \cdot \mathcal{P}_n^d \left(\sum_{i=0}^{\infty} q_i z^i \right) \right] \right\|_2 \text{ subject to } \left\| T_1^d(z) - T_2^d(z) Q_n^d(z) \right\|_\infty \le 1 \tag{11}$$

where $\mathcal{P}_n^d(\sum_{i=0}^{\infty} h_i z^i) \doteq \sum_{i=0}^{n-1} h_i z^i$. The first part of the proof follows now from [13, Theorem 4]. Normal convergence of $Q^n(s)$ in $\mathrm{Re}(s) > 0$ follows from normal convergence of $Q_n^d(z)$ in $|z| < 1$ and the fact that the bilinear transformation (10) maps compact domains $D_c(s) \subset \mathrm{Re}(s) > 0$ into compact domains $D_d(z) \subset |z| < 1$. □

[1] A sequence of complex-valued functions f_n defined in an open set U *converges normally* to f if $\{f_n\}$ is pointwise convergent to f in U and this convergence is uniform on each compact subset of U [11].

B. Computation of a Solution over $\mathcal{RH}_\infty$

Theorem 1 furnishes a procedure to (approximately) compute the optimal $\mathcal{H}_2/\mathcal{H}_\infty$ cost. Moreover, the sequence of controllers (and the corresponding closed-loops) converges, in the normal sense, to the corresponding optima.[2] However, since normal convergence does not imply uniform convergence, one cannot conclude that Q^n will provide an approximate solution to the problem, even if n is taken very large. Indeed, since $S^{\rm opt} \notin \mathcal{A}_o$ [9], the sequence of closed-loop systems $S_1 - S_2 Q^n$ will not approximate $S^{\rm opt}$ in the $\mathcal{H}_\infty$ sense.

In this section, we show that a rational ϵ-suboptimal solution to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem can be found by solving a sequence of truncated problems, each one requiring only a *finite* number of elements of the Laguerre expansion of S. To establish this result, we will consider the restriction of Problem 1 to the subspace $\mathcal{H}_\infty(D_\epsilon) \subset \mathcal{H}_\infty$ formed by functions analytic outside D_ϵ, a closed disk centered at $s_\epsilon = -a(1+\epsilon^2/2\epsilon)$ with radius $r = a(1-\epsilon^2/2\epsilon)$, equipped with the norm

$$\|F\|_{\infty,\epsilon} \doteq \sup_{s \neq \in D_\epsilon} |F(s)|.$$

Note that the impulse response of functions $F \in \mathcal{H}_\infty(D_\epsilon)$ decays faster than $e^{-a\epsilon t}$ and that from the maximum modulus theorem, it follows that $\|F(s)\|_{\infty,\epsilon} \geq \|F(s)\|_\infty$.

For ease of reference, define $G_\delta(z) \doteq T_2^d(\delta z) T_1^d(1/\delta z)$. In the sequel, we will assume that $G_\delta(z)$ has the following (minimal) state-space realization:

$$G = \left(\begin{array}{c|c} A_\delta & b_\delta \\ \hline c_\delta & d_\delta \end{array}\right)$$

with controllability and observability grammians $L_{c,\delta}$ and $L_{o,\delta}$, respectively.

Theorem 3: Consider the restriction of Problem 1 to $\mathcal{H}_\infty(D_\epsilon)$

$$\mu_\epsilon \doteq \min_{Q \in \mathcal{RH}_{\infty,\epsilon}} \{\|S_1 - S_2 Q\|_2 \text{ such that } \|T_1 - T_2 Q\|_{\infty,\epsilon} \leq 1\}. \tag{12}$$

Assume that ϵ is selected small enough so that $T_i, S_j \in \mathcal{H}_\infty(D_\epsilon)$. Then we have the following.

1) Given $\epsilon_1 > 0$, there exists $\epsilon > 0$ (that can be calculated *a priori* in terms of the problem data) such that $\mu_\epsilon \leq \mu + \epsilon_1$.
2) $\mu_\epsilon \geq \mu_R$.
3) Let

$$\mu_\epsilon^n = \min_{[\theta_1\ \theta_2\ \dots\ \theta_n]} \left\| \mathcal{P}_n \left(S_1(s) - S_2(s) \sum_{i=1}^n \theta_i l_i(s) \right) \right\|_2 \quad \text{subject to } \overline{\sigma}[W(\mathbf{\Theta_n})] \leq 1 \tag{13}$$

where $W(.)$ is shown in (14) at the bottom of the page. Then, for every $\epsilon_2 > 0$, there exists $N(\epsilon_2)$ such that $\mu_\epsilon \leq \mu_\epsilon^n + \epsilon_2$ for all $n \geq N(\epsilon_2)$.

Proof: See Appendix A.

Corollary 2: The optimal cost of Problems 1 and 2 are equal, i.e., $\mu = \mu_R$.

Corollary 3: Given $\epsilon_1 > 0$, a suboptimal solution to Problem 1 with cost $\mu_\epsilon \leq \mu + \epsilon_1$ is given by

$$Q_n(s) = \sum_{i=1}^n \theta_i l_i(s) + \left(\frac{s-a}{s+a}\right)^n Q_R(s) \tag{15}$$

where $\mathbf{\Theta_n} = (\theta_1 \quad \theta_2 \quad \dots \quad \theta_n)$ solves (13) for n larger than some precomputable bound N_{ϵ_1} and where $Q_R(s)$ solves the following $\mathcal{H}_\infty$ approximation problem:

$$Q_R(s) = \operatorname*{argmin}_{Q \in \mathcal{H}_\infty(D_\epsilon)} \left\| \left(\frac{s-a}{s+a}\right)^n G(s) - \sum_{i=0}^{n-1} q_i \left(\frac{s-a}{s+q}\right)^{n-i} - Q^\sim(s) \right\|_{\infty,\epsilon.}$$

Remark 1: From Corollary 3, it follows that an approximate solution to Problem 1 can be computed in polynomial time by solving a convex optimization problem with $\mathcal{O}(N_\epsilon)$ variables and an unconstrained $\mathcal{H}_\infty$ problem. Moreover, the resulting closed-loop system has a degree of stability better than $-a\epsilon$ since it is in $\mathcal{H}_\infty(D_\epsilon)$. Note, however, that the value of N_ϵ obtained using the approach outlined in Appendix B is usually very conservative. This difficulty can be circumvented by combining the upper bound introduced in this section with the lower bound introduced in Section III-A to obtain sequences of suboptimal and superoptimal solutions.

Finally, for completeness, we show convergence of the closed-loop systems and of the controllers in the $\mathcal{H}_2$ topology.

Lemma 1: Consider a sequence $0 < \epsilon_i \downarrow 0$. Then, the sequence of corresponding closed loops $S_i \doteq S_1 - S_2 Q_i$ converges in the $\mathcal{H}_2$ topology. Moreover, if S_2 does not have zeros on the $j\omega$-axis, then the sequence of controllers converges in the $\mathcal{H}_2$ topology, i.e., $\|Q_i - Q^*\|_2 \to 0$, where Q^* is a solution to Problem 1.

Proof: The proof, omitted for space reasons, is similar to the proof of [13, Lemma 4].

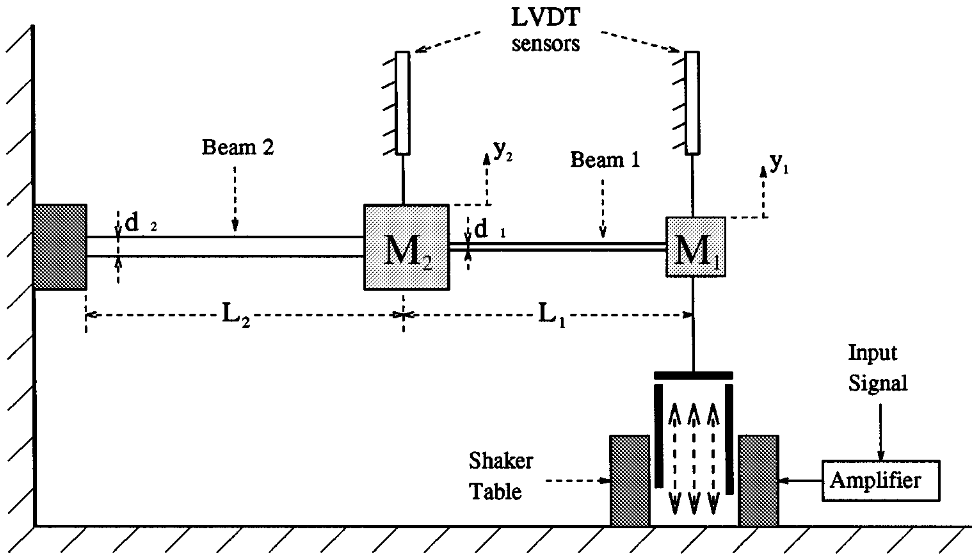
IV. Numerical Example

In this section, we illustrate the proposed framework by applying it to a flexible structure used as a damage mitigation testbed [23]. This structure, illustrated in Fig. 2, consists of two discrete masses supported by cantilever beams, excited by the vibratory motion of a shaker table, and exhibits a very lightly damped resonance at $f = 110$ Hz. The goal is to design a controller so that the mass M_1 tracks low-frequency

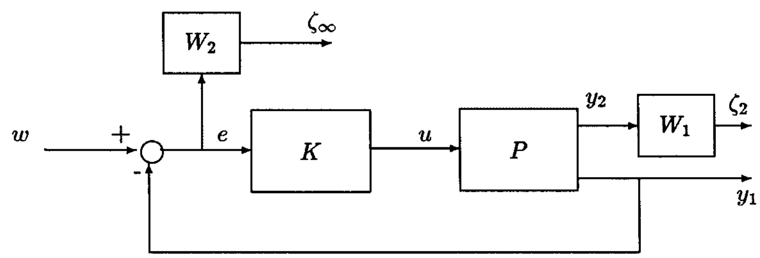
[2]It can be also easily shown that $S_n = S_1 - S_2 Q^n$ is strongly convergent in the $\mathcal{H}_2$ sense by showing that it is a Cauchy sequence.

$$W(\mathbf{\Theta}_n) = \begin{pmatrix} L_{o,\delta}^{1/2} A_\delta^n L_{c,\delta}^{1/2} & L_{o,\delta}^{1/2} b_\delta & L_{o,\delta}^{1/2} A_\delta b_\delta & \dots & L_{o,\delta}^{1/2} A_\delta^{n-1} b_\delta \\ c_\delta A_\delta^{n-1} L_{c,\delta}^{1/2} & d_\delta & c_\delta b_\delta & \dots & c_\delta A_\delta^{n-2} b_\delta \\ \vdots & 0 & d_\delta & \ddots & \vdots \\ c_\delta A_\delta L_{c,\delta}^{1/2} & \vdots & \vdots & \ddots & c_\delta b_\delta \\ c_\delta L_{c,\delta}^{1/2} & 0 & 0 & \dots & d_\delta \end{pmatrix} + \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & q_o & q_1 & \dots & q_{n-1} \\ 0 & 0 & q_o & \dots & q_{n-2} \\ 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & q_o \end{pmatrix}$$

$$\delta = \left(\frac{1+\epsilon}{1-\epsilon}\right), \quad q_o = \frac{\theta_1}{\sqrt{2a}}, \quad q_i = \frac{\theta_{i+1} - \theta_i}{\sqrt{2a}} \delta^i, \qquad i \geq 1. \tag{14}$$



(a)



(b)

Fig. 2. (a) The flexible testbed. (b) Corresponding block diagram.

signals (up to 10 Hz), while minimizing the displacement of the second mass at the resonant frequency.[3]

Control-oriented identification of the structure yields the following seventh-order model [12] as shown in (16) at the bottom of the page.

The problem can be recast into a constrained $\mathcal{H}_2$ minimization where the goal is to minimize $\|W_1 T_{y_2 w}\|_2$[4] subject to $\|W_2 T_{ew}\|_\infty \leq \gamma$, where $T_{y_2 w}$ and T_{ew} denote the closed-loop transfer function from the input to the displacement of the second mass and the tracking error, respectively. The weighting functions W_1 and W_2 are chosen as a notch and low-pass filters, respectively, to reflect the performance specifications

$$W_1(s) = \frac{1.1531s + 5.6566}{0.001s^2 + 0.0091s + 11.384}$$
$$W_2(s) = \left[\frac{10^{-4}s + 1}{0.1s + 1}\right]^2.$$

The optimal $\mathcal{H}_\infty$ controller (found using Matlab's linear matrix inequality (LMI) toolbox hinflmi command) yields $\|T_{y_1\omega}\|_\infty = 0.47$ and $\|T_{y_2\omega}\|_2 = 1534$. Since the plant is open-loop stable, it follows that optimal $\mathcal{H}_2$ performance is achieved in open-loop, resulting in $\|T_{y_1\omega}\|_\infty = 100$. Finally, the proposed design using the values $a = 100, \epsilon = 0.75 \times 10^{-3}, N_\epsilon = 200$, and $\gamma = 0.51$ yields a closed-loop system with $\|T_{y_2\omega}\|_2 = 1254$ and $\|T_{y_1\omega}\|_\infty = 0.51$. The resulting

[3]This displacement is directly related to the stress and causes the structure to fail due to material fatigue. Thus, by minimizing it the controller extends the lifespan of the specimen.

[4]Here we are using the fact that the induced norm from ℓ^2 to ℓ^∞ is precisely the $\mathcal{H}_2$ norm.

$$A = \begin{bmatrix} -0.2185 & 110.2712 & -1.0704 & -0.4213 & -1.1968 & -0.0013 & -0.0777 \\ -110.2801 & -0.9896 & 3.7338 & 2.1488 & 1.8831 & -0.0787 & 0.1621 \\ -1.4114 & -4.5158 & -12.9851 & 42.7061 & -41.2727 & -1.4982 & -2.1739 \\ -0.8933 & -2.5601 & -51.8080 & -5.2078 & -82.6450 & 0.5258 & 3.2744 \\ 1.4331 & 2.2632 & 48.7714 & 82.7986 & -9.4665 & 2.5266 & -1.1314 \\ 0.1010 & 0.2228 & 2.1113 & 0.8446 & -2.8446 & -0.1284 & -145.9587 \\ -0.2674 & -0.6012 & -3.4584 & -4.3014 & 4.8333 & 145.9948 & -0.9545 \end{bmatrix}$$
$$B = [2.7144 \quad 5.7531 \quad 10.4155 \quad 6.1052 \quad -7.5692 \quad -0.6268 \quad 1.7047]^T$$
$$C = \begin{bmatrix} 0.5669 & -0.0827 & 7.2570 & -3.8842 & -3.4930 & 0.6009 & 1.6802 \\ 2.6546 & -5.7525 & 7.4713 & 4.7102 & 6.7150 & -0.1782 & 0.2881 \end{bmatrix}; \qquad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{16}$$

TABLE I
COMPARISON OF RESULTS FOR THE MIXED $\mathcal{H}_2/\mathcal{H}_\infty$ PROBLEM

Type	Controller Order	$\|T_{\zeta_2\omega_2}\|_2$	$\|T_{\zeta_\infty\omega_\infty}\|_\infty$
optimal $\mathcal{H}_\infty$	11	1534	0.47
mixed $\mathcal{H}_2/\mathcal{H}_\infty(LMI)$	11	2168	0.51
mixed $\mathcal{H}_2/\mathcal{H}_\infty$ (29 ord)	29	1254	0.51
mixed $\mathcal{H}_2/\mathcal{H}_\infty$(18 ord)	18	1271	0.52

TABLE II
LOWER BOUND OF $\|T_{y2\omega}\|_2$ AS A FUNCTION OF N

Horizon (N)	50	75	100	125	150	175	200
μ^N	1104	1197	1236	1241	1243	1244	1245

controller can be reduced to order 29 withouth any performance loss and to order 18 with less than 2% performance degradation. For benchmarking purposes, we also designed a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller using the LMI approach proposed in [19], based on the minimization of an upper bound obtained using a single Lyapunov function and implemented in Matlab's hinfmix command. In this case, the eleventh-order controller corresponding to $\gamma = 0.53$ results in $\|T_{y_1\omega}\|_\infty = 0.51$ and $\|T_{y_2\omega}\|_2 = 2168$. Note that due to the potentially conservative nature of the method, in this case performance is worse than that corresponding to the central $\mathcal{H}_\infty$ controller corresponding to $\gamma = 0.47$. These results are summarized in Table I.

Table II shows the lower bound of the cost obtained by solving a sequence of problems of the form of (6) as a function of the horizon N. Note that for $N = 200$ the difference between μ^n and μ_ϵ is less than 1%. Finally, Fig. 3 shows the outputs corresponding to a triangular input wave with frequency $f_s = 5.84$ Hz. As shown in the plot, both the LMI and the proposed controller achieve good tracking of the signal. However, the LMI controller results in larger displacements (and hence larger damage) of the second mass.

V. CONCLUSIONS

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problems arise in the context of both multi-objective control and robust $\mathcal{H}_2$ control. Contrary to both the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ cases, the mixed problem has proved to be surprisingly difficult, and to date no closed-form solutions are available.

In this paper, we propose a solution to continuous-time mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problems, based upon constructing families of super and suboptimal solutions and showing that these families converge to a solution of the original problem. The main idea behind the approach is to construct at each step an equivalent discrete-time problem and then exploit our previous work to find ϵ-suboptimal solutions in polynomial time. This equivalence can be also used to show that, as in the discrete-time case, while the optimal solution is not generically in $\mathcal{A}_o$, optimal performance can be approached arbitrarily close by a real-rational controller.

APPENDIX A
PROOFS OF THEOREMS 2 AND 3

In order to prove these theorems, we need to introduce some additional results. Consider first a function $F \in \mathcal{H}_2$ and its Laguerre expansion $F(s) = \sum_{i=1}^\infty \Gamma_i l_i$. The bilinear transformation (10) maps $F(s) \in \mathcal{H}_\infty$ to $F^d(z) \in \mathcal{H}_\infty(D)$, $F^d(z) = \sum_{i=0}^\infty f_i z^i$, where $f_o \doteq \Gamma_1\sqrt{2a}$ and $f_i \doteq (\Gamma_{i+1} - \Gamma_i)/\sqrt{2a}, i \geq 1$, and where $F^d(1) = 0$. Consider now $\hat{F}(z) \doteq \sum_{i=1}^\infty \Gamma_i z^i$. Straightforward computations show that $\hat{F}(z) = \sqrt{2a}(z/1-z)F^d(z)$. Since $F^d \in \mathcal{H}_\infty(D)$ and $F^d(1) = 0$, it follows that $\hat{F}(z) \in \mathcal{H}_\infty(D)$. From the orthonormality of the Laguerre functions, it follows that $\|F(s)\|_2 = \|\hat{F}(z)\|_2 = \sum_{i=1}^\infty \Gamma_i^2$. Thus $F(s) \in \mathcal{H}_2 \iff \hat{F}(z) \in \mathcal{H}_2(D)$.

Proof of Theorem 2: Given $S_1, S_2, Q \in \mathcal{RH}_\infty$, with the Laguerre expansions (7), define $\Phi(s) \doteq S_1(s) - S_2(s)Q(s)$. Straightforward but tedious computations using the fact that $l_i(s) \cdot l_j(s) = (1/\sqrt{2a})(l_{i+j-1} - l_{i+j})$ yield the following expansion for $\Phi(s)$:

$$\begin{aligned}\Phi(s) &= \sum_{i=1}^\infty \phi_i l_i(s), \quad \phi_i = \sigma_{1,i} + \sum_{j=1}^i \hat{s}_{2,i+1-j} q_{j-1}\\ \hat{s}_{2,i} &= \sigma_{2,i} + \sigma_{2,0}\sqrt{2a}\\ q_0 &= \frac{\theta_1}{\sqrt{2a}}, \quad q_j = \frac{\theta_{j+1} - \theta_j}{\sqrt{2a}}, \qquad j \geq 1.\end{aligned} \tag{17}$$

Note that the $q_i' s$ coincide exactly with the coefficients of the Taylor expansion of $Q(z)$, i.e.,

$$Q(z) = Q(s)|_{s=a(1+z/1-z)} \doteq \sum_{i=0}^\infty q_i z^i.$$

.

Define $\hat{\phi}(z) = \hat{S}_1(z) - \hat{S}_2(z)Q^d(z)$. From the discussion above, it follows that $\phi(s) \in \mathcal{H}_2 \iff \hat{\phi}(z) \in \mathcal{H}_2(D)$ and that $\|\phi(s)\|_2 = \|\hat{\phi}(z)\|_2$. The proof of the theorem follows from the fact that the bilinear transformation preserves the $\mathcal{H}_\infty$ norm and thus $\|T_1(s) - T_2(s)Q(s)\|_\infty = \|T_1^d(z) - T_2^d(z)Q^d(z)\|_\infty$. □

Proof of Theorem 3: Let $\delta \doteq (1 + \epsilon/1 - \epsilon)$. Since the bilinear transformation (10) maps the space $\mathcal{H}_\infty(D_\epsilon)$ to $\mathcal{H}_{\infty,\delta}$, it follows that (12) is equivalent to the following *discrete-time* problem:

$$\mu_\delta \doteq \min_{Q^d \in \mathcal{RH}_{\infty,\delta}} \left\{ \left\| \hat{S}_1(z) - \hat{S}_2(z)Q^d(z) \right\|_2 \text{ such that } \left\| T_1^d(z) - T_2^d(z)Q^d(z) \right\|_{\infty,\delta} \leq 1 \right\}. \tag{18}$$

The proof follows now from [13, Theorem 5].

APPENDIX B
IMPLEMENTATION CONSIDERATIONS

A. Computing a Bound on the Coefficients of the Laguerre Expansion

Let $F(s) = \sum_{i=1}^\infty f_i l_i$ since the Laguerre functions are orthonormal, it follows that

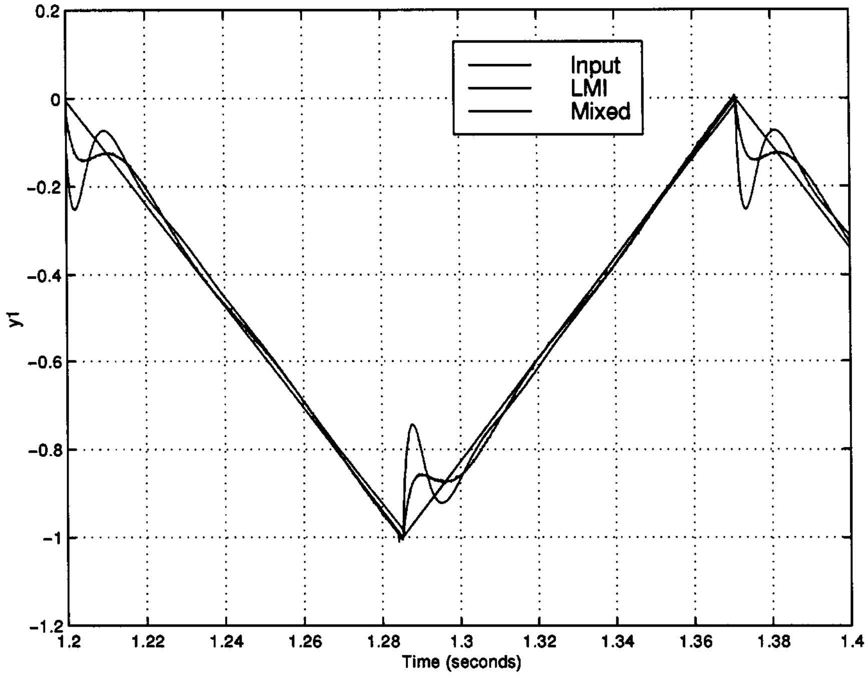
$$f_i = \frac{1}{2\pi j} \oint_C F(s) l_i(-s)\, ds$$

where the integration contour C encloses all the poles of $F(s)$. Since $F(s) \in \mathcal{H}_2(D_\epsilon)$, we can take C as the boundary of the disk D_ϵ. The bilinear transformation $s = a(a\epsilon - s/\epsilon s - a)$ maps C to the $j\omega$ axis, yielding

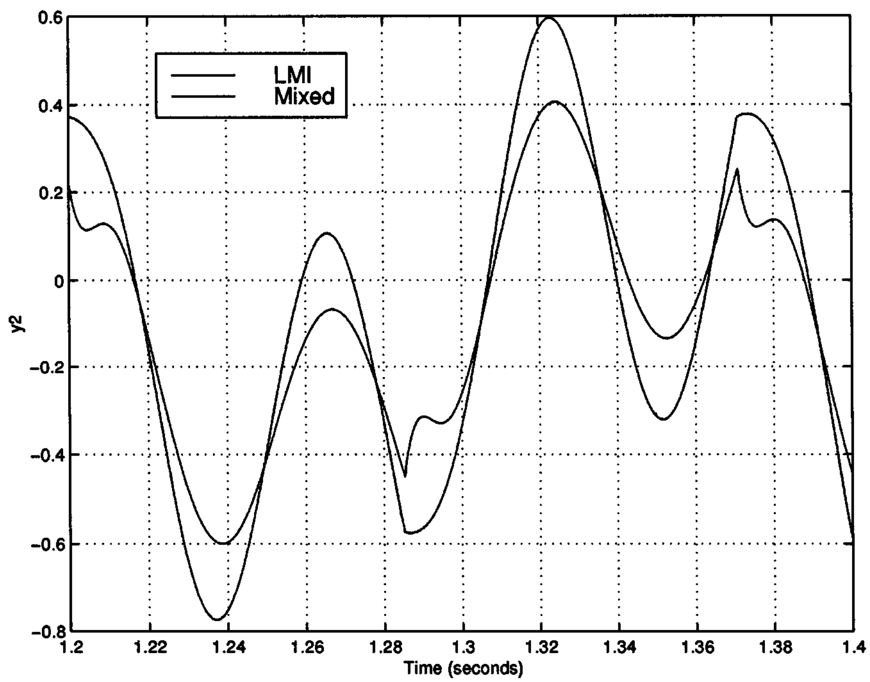
$$f_i = \frac{a\sqrt{2a}}{2\pi}(1+\epsilon)\left(\frac{1-\epsilon}{1+\epsilon}\right)^i \int_{-\infty}^{+\infty} f\left(a\,\frac{a\epsilon - j\omega}{\epsilon j\omega - a}\right) \cdot \frac{1}{(j\omega - a)(\epsilon j\omega - a)}\left(\frac{j\omega + a}{j\omega - a}\right)^{i-1} d\omega.$$

Hence

$$\begin{aligned}|f_i| &\leq \frac{a\sqrt{2a}}{2\pi}(1+\epsilon)\left(\frac{1-\epsilon}{1+\epsilon}\right)^i \int_{-\infty}^{+\infty} \|f\|_\infty \cdot \frac{1}{\sqrt{(\omega^2 + a^2)(\epsilon^2\omega^2 + a^2)}}\, d\omega\\ &\leq \frac{a\sqrt{2a}}{2}(1+\epsilon)\left(\frac{1-\epsilon}{1+\epsilon}\right)^i \|f\|_\infty \frac{1}{\epsilon} \doteq M\left(\frac{1-\epsilon}{1+\epsilon}\right)^i.\end{aligned} \tag{19}$$



(a)



(b)

Fig. 3. Time responses to a triangular input: (a) y_1 and (b) y_2.

B. Additional Considerations

Recall that in Section III-A, we assumed that S_1 was strictly proper. Let D_1, D_2, and D_Q denote the feedthrough terms in S_1, S_2, and Q, respectively. If $D_1 \neq 0$, then the closed-loop system has a finite $\mathcal{H}_2$ norm iff $D_2 \neq 0$ and D_Q is selected as $D_Q \doteq -(D_1/D_2)$. It follows that in this case, the problem is equivalent to solving

$$\mu_R \doteq \inf_{\substack{\tilde{Q} \in \mathcal{H}_\infty \\ \text{strictly proper}}} \{\|\tilde{S}_1 - S_2\tilde{Q}\|_2 \text{ such that } \|\tilde{T}_1 - T_2\tilde{Q}\|_\infty \leq 1\}$$

where

$$\tilde{S}_1 = S_{1_{\rm sp}} - \frac{D_1}{D_2} S_{2_{\rm sp}}, \qquad \tilde{T}_1 = S_{1_{\rm sp}} - \frac{D_1}{D_2} T_2$$

where $S_{i_{\rm sp}}$ denotes the strictly proper portion of S_i. Finally, the constraint $\tilde{Q}(s)$ strictly proper [or, equivalently, $Q^d(1) = 0$ in (4)] can be enforced by simply factoring $Q^d(z)$ as

$$Q^d(z) = \frac{z-1}{z}\hat{Q}(z), \qquad \hat{Q}(z) \in \mathcal{H}_\infty.$$

C. Computational Complexity

Assume that the $\mathcal{H}_\infty$ norm constraint is not "tight," i.e., $\min_{Q_u^d \in \mathcal{H}_\infty} \|\hat{T}_1 - \hat{T}_2 Q_u^d\|_\infty = \gamma^* < 1$. As discussed in [14], it is possible to perturb Q_u^d to Q_1^d so that

$$\left\|\hat{S}_1^d - \hat{S}_2^d Q_1^d\right\|_2 \leq \mu + \epsilon_1$$
$$\left\|\hat{T}_1 - \hat{T}_2 Q_u^d\right\|_\infty \leq 1 - \delta$$

where δ is a function of the problem data times ϵ_1. Introduce the change of variables $z \to \gamma z$, with $\gamma > 1$. Using the fact that $\hat{S}_1^d$, $\hat{S}_2^d$, $\hat{T}_1$, and $\hat{T}_2$ are known and finite-dimensional, it is possible to perturb the problem so that only Q_1^d will include the change of variables. Using elementary manipulations, one can the proceed to estimate the contribution of $(I - \mathcal{P}_n^d)Q_1^d$ on the two and ∞ norms. This approach allows for the computation of a bound on n in terms of the problem data so that $\mu^n \leq \mu + \epsilon$ for any given $\epsilon > 0$ (see [14] for details).

Since Problem 3 can be solved to any desired precision in polynomial time (it is convex), we can then conclude that a controller achieving a performance of at most $\mu + \epsilon$ can be computed in a polynomial number of iterations.

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A Projection Method for Closed-Loop Identification

Urban Forssell and Lennart Ljung

***Abstract*—A new method for closed-loop identification that allows fitting the model to the data with arbitrary frequency weighting is described and analyzed. Just as the direct method, this new method is applicable to systems with arbitrary feedback mechanisms. This is in contrast to other methods, such as the indirect method and the two-stage method, that assume linear feedback. The finite sample behavior of the proposed method is illustrated in a simulation study.**

***Index Terms*—Closed-loop identification, prediction error methods.**

I. INTRODUCTION

In "identification for control," the goal is to construct models that are suitable for control design. It is widely appreciated that small model uncertainty around the crossover frequency is essential for successful control design. Consequently, there has been a substantial interest in identification methods that provide a tunable optimality criterion so that the model can be fit to the data with a suitable frequency weighting. With open-loop experiments this is no problem: it is well known that arbitrary frequency weighting can be obtained by applying a prediction error method to an output error model structure with a suitable fixed noise model/prefilter [recall (see [1]) that the effect of any prefilter may be included in the noise model]. However, open-loop experiments are not always possible since the system might be unstable or has to be controlled for safety or production reasons. In such cases, closed-loop experiments have to be used. The problem is now that the simple approach of using an output error model with a fixed noise model/prefilter will give biased results when applied directly to closed-loop data, unless the fixed noise model correctly models the true noise color (see, e.g., [1, Theorem 8.3]). A way around this would be to use a flexible, parameterized noise model. This would eliminate the bias but the frequency weighting would then not be fixed.

In this paper, we describe and analyze a closed-loop identification method that is consistent and, in the case of undermodeling, allows fitting the model to the data with arbitrary frequency weighting. This

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