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Receding Horizon: An easy way to improve performance in LPV systems[1]

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Abstract

During the past few years the problem of stabilizing a Linear Parameter Varying system, while, at the same time, optimizing some measure of performance has been the object of increasing attention. In contrast to the case of linear systems where several optimal synthesis techniques (such as $\mathcal{H}_\infty$, $\mathcal{H}_2$ and ℓ^1) are well established, the counterparts for LPV systems are just starting to emerge. Moreover, at the present time, only sufficient conditions for performance are available, thus leading to potentially conservative designs. In this paper we propose a simple way to improve performance by combining the LMI–based tools currently available with receding horizon techniques.

1 Introduction

A large number of control problems involve designing a controller capable of stabilizing a given system while simultaneously optimizing some performance index. In the case of linear dynamics this problem has been thoroughly explored during the past decade, leading to powerful formalisms such as μ-synthesis and ℓ^1 optimal control theory that have been successfully employed to solve some hard practical problems. More recently, these techniques have been extended to handle multiple, perhaps conflicting, performance specifications [11].

In the case of nonlinear dynamics, a widely used idea among control engineers is to linearize the plant around several operating points and to use *linear* control tools to design a controller for each of these points. The actual controller is implemented using gain scheduling, i.e. the parameters in the linear control law are changed according to the operating condition. While this idea is intuitively appealing, it has several pitfalls [5, 6, 7]. Motivated by these shortcomings, during the past few years considerably attention has been devoted to the problem of synthesizing controllers for Linear Parameter Varying Systems, where the state–space matrices of the plant depend on time–varying parameters whose values are not known a priori, but can be measured by the controller. Assuming that bounds on both the parameter values and their rate of change are known then Affine Matrix Inequalities based conditions are available guaranteeing exponential stability of the system. Moreover, these conditions can be easily used to synthesize stabilizing controllers guaranteeing worst case performance bounds (for instance in an $\mathcal{H}_2$ or $\mathcal{H}_\infty$ sense, see [1, 3, 13] and references therein).

However, a potential drawback of these techniques is that they are based on sufficient conditions, obtained using parameter dependent quadratic Lyapunov functions. Thus, the resulting controllers can be potentially very conservative. Motivated by the approach pursued in [12] and in [8, 9, 10] in this paper we show that performance can be improved by combining these AMI–based tools with receding horizon techniques, and we illustrate these results with a simple example. In the sequel, we consider for simplicity the problem of optimizing the $\mathcal{H}_2$ norm (in a sense that will be precisely defined in the next section), but the results can be easily translated to the $\mathcal{H}_\infty$ (or even multiobjective) case.

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2 Preliminaries

2.1 Notation and Definitions

We consider the following class of LPV systems:

$$\begin{aligned} \dot{x} &= A[\rho(t)]\,x(t) + B_1[\rho(t)]\,w(t) + B_2[\rho(t)]\,u(t) \\ z &= C_1[\rho(t)]\,x(t) + D_{12}[\rho(t)]\,u(t) \\ y &= C_2[\rho(t)]\,x(t) + D_{21}[\rho(t)]\,w(t) \end{aligned} \tag{1}$$

where $x \in R^{n_x}, u \in R^{n_u}, w \in R^{n_w}, z \in R^{m_z}$, and $y \in R^{n_y}$ represent the state, control, exogenous disturbances, regulated variables and outputs available to the controller respectively, $\rho \in R^{n_\rho}$ denotes a vector of time–varying parameters and where the matrix functions $A(.), B(.), C(.)$ and $D(.)$ are continuous. Further, we will assume that at all times $\rho(t) \in \mathcal{P} \subset R^{n_\rho}$, where $\mathcal{P}$ is a given compact set, and that the set of admissible parameter trajectories is given by:

$$\begin{aligned} \mathcal{F}_\nu \doteq \quad & \{\rho \in C^1(R, R^{n_\rho}) : \rho(t) \in \mathcal{P}, \\ & \underline{\nu}_i \leq \dot{\rho}_i(t) \leq \overline{\nu}_i, i = 1, \ldots n_\rho, \forall\, t \in R_+\} \end{aligned} \tag{2}$$

where $\underline{\nu}_i$ and $\overline{\nu}_i$ are given numbers.

Definition 1 ($\mathcal{H}_2$ LPV performance) *Assume that $x = 0$ is an exponentially stable equilibrium point of the system*

$$G[\rho] = \left\{ \begin{aligned} \dot{x} &= A[\rho(t)]\,x(t) + B_1[\rho(t)]\,w(t) \\ z &= C_1[\rho(t)]\,x(t) \end{aligned} \right. \tag{3}$$

for any trajectory $\rho(.) \in \mathcal{F}_\nu$. Then given an initial condition $\rho(0)$ we define the $\mathcal{H}_2$ norm of system (3) as the worst case, over all admissible parameter trajectories, of the $\mathcal{L}_2$ norm of its impulse response, averaged over all possible directions for the input. Equivalently, we can assume an initial condition of the form $x(0) = B_1 w_o$ where w_o is a random variable satisfying $\mathcal{E}(w_o w_o^T) = I$. Thus

$$\|G(\rho_o)\|_{\mathcal{H}_2} \sup_{\rho \in \mathcal{F}_\nu,\ \rho(0)=\rho_o} \mathcal{E} \int_o^\infty z^T(t) z(t) dt \tag{4}$$

where the expectation is taken with respect to w_o.

For simplicity, in the sequel we make the following standard assumptions:

$$D_{12}^T D_{12} = I,\ C_1 D_{12}^T = 0,\ C_2 = I, D_{21} = 0 \tag{5}$$

i.e we consider the state feedback case.

Lemma 1 *[2] Assume that there exist a differentiable function $X(\rho) > 0$, $\forall \rho \in \mathcal{P}$ and such that the following matrix inequality is satisfied:*

$$\begin{aligned} & -\textstyle\sum_{i=1}^{n_\rho} \nu_i \frac{\partial X(\rho)}{\partial \rho_i} + A(\rho)X(\rho) + \\ & + X(\rho)A^T(\rho) + X(\rho)C^T(\rho)C(\rho)X(\rho) \leq 0 \end{aligned} \tag{6}$$

for all $\underline{\nu}_i \leq \nu_i \leq \overline{\nu}_i$ and all $\rho \in \mathcal{P}$. Then the following properties hold:

$$\sup_{\rho \in \mathcal{F}_\nu, \rho(0)=\rho_o} \|z\|_2^2 \leq x^T(0) X^{-1}(\rho_o) x(0) \tag{7}$$

$$\sup_{\rho \in \mathcal{F}_\nu, \rho(0)=\rho_o} \mathcal{E}(\|z\|_2^2) \leq \text{Trace}[B_1^T(\rho_o) X^{-1}(\rho_o) B_1(\rho_o)] \tag{8}$$

Using this lemma it can be shown that, under assumptions (5) a parameter dependent state feedback controller with guaranteed $\|.\|_{\mathcal{H}_2}$ performance can be synthesized by solving the following optimization problem (see [2] for details):

$$\min_{X(\rho)>0, Z} \text{Trace} Z \tag{9}$$

subject to:

$$\begin{bmatrix} -\sum_{i=1}^{n_\rho} \underline{\nu}_i \frac{\partial X(\rho)}{\partial \rho_i} + A(\rho)X(\rho) + X(\rho)A^T(\rho) - B_2(\rho)B_2^T(\rho) & X(\rho)C_1^T(\rho) \\ C_1(\rho)X(\rho) & -I \end{bmatrix} < 0$$

$$\begin{bmatrix} -\sum_{i=1}^{n_\rho} \overline{\nu}_i \frac{\partial X(\rho)}{\partial \rho_i} + A(\rho)X(\rho) + X(\rho)A^T(\rho) - B_2(\rho)B_2^T(\rho) & X(\rho)C_1^T(\rho) \\ C_1(\rho)X(\rho) & -I \end{bmatrix} < 0$$

$$\begin{bmatrix} Z & B_1^T(\rho) \\ B_1(\rho) & X(\rho) \end{bmatrix} > 0 \tag{10}$$

for all $\rho \in \mathcal{P}$. The corresponding control action is given by

$$u = -B_2^T(\rho) X^{-1}(\rho) x \tag{11}$$

and the closed loop $\mathcal{H}_2$ norm bounded by:

$$\sup_{\rho \in \mathcal{F}_\nu} \mathcal{E}(z^T z) \leq \text{TraceZ} \tag{12}$$

While this approach yields a stabilizing controller with guaranteed performance bounds, it is potentially conservative due to the facts (in addition to condition (6) being only sufficient) that (i) it uses a quadratic parameter dependent Lyapunov function ($x^T X^{-1}(\rho) x$), (ii) allows for all possible combinations of the parameters and their derivatives, and (iii) the control law is obtained by minimizing an upper bound ($\sup_{\rho \in \mathcal{P}} \text{Trace}(B_1^T X^{-1} B_1)$) of the cost. In the sequel, we indicate how to improve performance by combining the AMI (6) with receding horizon ideas.

2.2 The Quadratic Regulator Problem for LPV systems

Consider the LPV system (1). In the sequel we consider the following problem. Given an initial condition x_o and an initial value of the parameter ρ, find a parameter dependent state–feedback control law $u[x(t), \rho(t)]$ that minimizes the following performance index

$$J(x_o, \rho_o, u) = \sup_{\substack{\rho \in \mathcal{F}_v \\ \rho(0) = \rho_o}} \int_0^\infty \left[x^T C_1^T C_1 x + u^T u\right] dt, \; x(0) = x_o \tag{13}$$

By using Pontryagin's principle it can be shown that solving this problem is equivalent to solving the following Hamilton–Jacobi–Bellman type partial differential equation:

$$\begin{aligned} 0 = \; & \tfrac{\partial V}{\partial x} A(\rho) x - \tfrac{1}{4} \tfrac{\partial V}{\partial x} B_2(\rho) B_2^T(\rho) \tfrac{\partial V}{\partial x}^T + x^T Q(\rho) x \\ & + \max_{\underline{\nu}_i \le \nu_i \le \overline{\nu}_i} \textstyle\sum_{i=1}^{n_\rho} \tfrac{\partial V}{\partial \rho_i} \nu_i, \; V(0, \rho) = 0 \end{aligned} \tag{14}$$

where $Q(\rho) \doteq C_1^T(\rho) C_1(\rho)$. If this equation admits a C^1 nonnegative solution V, then the optimal control is given by $u(x, \rho) = -\frac{1}{2} B_2^T \frac{\partial V}{\partial x}^T$ and $V(x, \rho)$ is the corresponding optimal cost (or storage function), i.e.

$$V(x, \rho) = \min_u \sup_{\rho \in \mathcal{F}_v} \int_0^\infty \left(x^T Q x + u^T u\right) dt$$

Note that if the solution to (14) is a quadratic function of the form $V(x, \rho) = x^T X(\rho) x$ with $X(\rho) > 0$, then X^{-1} satisfies (6).

3 An Equivalent Finite Horizon Regulation Problem

Unfortunately, the complexity of equation (14) prevents its solution except in some very simple, low dimensional cases. To solve this difficulty, in this section we introduce a finite horizon approximation of the LPV regulation problem, and we analyze its properties. This approximation forms the basis of the proposed method.

Lemma 2 *Consider a compact set $\mathcal{S}$ containing the origin in its interior and assume that the optimal storage function $V(x, \rho)$ is known for all $x \in \mathcal{S}, \rho \in \mathcal{P}$. Let $c = \min_{x \in \partial S} \sup_{\rho \in \mathcal{P}} V(x, \rho)$ where ∂S denotes the boundary of S. Finally, define the set $S_v = \{x : \sup_{\rho \in \mathcal{P}} V(x, \rho) \le c\}$. Consider the following two optimization problems:*

$$\min_u \sup_{\rho \in \mathcal{F}_v, \rho(0) = \rho_o} \left\{ J(x, u, \rho) = \int_0^\infty \left[x^T Q(\rho) x + u^T u\right] dt \right\} \tag{15}$$

$$\min_u \sup_{\substack{\rho \in \mathcal{F}_v \\ \rho(0) = \rho_o}} \left\{ J_T(x, u, \rho) = \int_0^T \left(x^T Q(\rho) x + u^T u\right) dt + V[x(T), \rho(T)] \right\} \tag{16}$$

subject to (1) with $x(0) = x_o$. Then an optimal solution of problem (16) is also optimal for (15) in the interval $[0, T]$ provided that $x(T) \in S_v$.

This lemma shows that if a solution to the HJB equation (14) is known in a neighborhood of the origin, then it can be extended via an explicit finite horizon optimization, well suited for an on–line implementation. This suggest a receding horizon type control combining an an off-line phase to find a local solution to (14) with an on–line phase where a sequence of problems of the form (16) with increasing T is solved, until a solution such that $x(T) \in S_v$ is found. Specifically, let $x(t)$ and $\rho(t)$ denote the current state of system (1) and the value of the parameter respectively. Then the proposed control law is given by:

Algorithm 1

0.- Data: a region S_v, the optimal return function $\Psi(x, \rho)$ for all $x \in S_v$.

1.- If $x(t) \in S_v$, $u = -\frac{1}{2} B_2^T \frac{\partial V(x,\rho)}{\partial x}^T$

2.- If $x(t) \notin S_v$ then solve a sequence of optimization problems of the form (16) until a solution such that $x(T) \in S_v$ is found. Use the corresponding control law $u(t)$ in the interval $[t_o, t_o + \delta t]$.

From the results above it is clear that the resulting control law is globally optimal and thus globally stabilizing. However, the computational complexity associated with finding $V(x, \rho)$ (even only in the region S_v) may preclude the use of this control law in many practical cases. Additionally, the requirement that T should be large enough so that $x(T) \in S_v$ could pose a problem, specially in cases where the system has fast dynamics. Thus, it is of interest to consider a modified control law where an approximation $\Psi(x, \rho)$ (rather than $V(x, \rho)$) and a fixed horizon T are used. To this effect consider a compact set S containing the origin in its interior and let $\Psi: S \times \mathcal{P} \to R_+$, $\Psi \in C^1(R^{n_x} \times R^{n_\rho}, R)$ be a Control Lyapunov Function for system (1) in the sense that the following condition holds:

$$\sup_{\rho \in \mathcal{P}, \underline{\nu}_i \leq \nu_i \leq \overline{\nu}_i} \left\{ \frac{\partial \Psi}{\partial x}[A(\rho)x + B_2(\rho)u] + \sum_i \frac{\partial \Psi}{\partial \rho_i}\nu_i \right\} \leq -\underline{\sigma}\|x\|^2 < 0;\ \forall x \in S_v \tag{17}$$

for some fixed number $\underline{\sigma} > 0$ and some control action u. Let

$$c = \min_{x \in \partial S} \max_{\rho \in \mathcal{P}} \Psi(x, \rho)$$

and define the set

$$S_\Psi \subseteq S = \left\{ x : \max_{\rho \in \mathcal{P}} \Psi(x, \rho) \leq c \right\}$$

Then we propose the following modified law:

Algorithm 2

0.- Data: a CLF $\Psi(x, \rho)$, the region S_Ψ, a horizon T.

1.- If $x(t) \in S_\psi$, $u_\psi(x) =$

$$\underset{|u|}{\operatorname{argmin}} \left\{ u: \sup_{\rho \in \mathcal{P}, \underline{\nu}_i \leq \nu_i \leq \overline{\nu}_i} \{ \frac{\partial \Psi}{\partial x}[A(\rho)x + B_2(\rho)u] + \sum_i \frac{\partial \Psi}{\partial \rho_i}\nu_i \leq -\underline{\sigma}\|x\|^2 < 0 \right\}$$

2.- If $x(t) \notin S_\Psi$ then

$$u_\Psi = \underset{u}{\operatorname{argmin}} \left\{ \sup_{\rho \in \mathcal{F}_v} \left[\int_t^{T+t} \left(x^T Q x + u^T u \right) dt + \Psi[x(T+t), \rho(T+t)] \right] \right\} \tag{18}$$

subject to:

$$\sup_{\rho \in \mathcal{P}} (u_\Psi^T u_\psi + x^T Q x + \dot{\Psi})|_{T+t} \leq 0 \tag{19}$$

Theorem 1 *The control law u_Ψ generated by Algorithm 2 has the following properties:*

1. *It renders the origin a globally asymptotically stable equilibrium point of (1)*
2. *Coincides with the globally optimal control law when $\Psi(x, \rho) = V(x, \rho)$.*

Remark 1 *A suitable choice for Ψ is given by $\Psi(x, \rho) = x^T X^{-1}(\rho) x$, where X denotes a solution to the set of AMIs (10) introduced in section 2. Moreover, from the Euler Lagrange conditions for optimality, it can be easily shown that in this case the constraint (19) is redundant, since it is satisfied by the control law that minimizes (18). As we show next, for this choice of Ψ, the control law u_ψ performs no worse than the AMI based control law (11), in the sense that both have the same worse-case upper bound.*

Lemma 3 *For any feasible parameter trajectory $\tilde{\rho}$ the following holds:*

$$\begin{aligned} J_\psi(x_o, \tilde{\rho}) &\doteq \int_0^\infty \left(\tilde{x}_\psi^T \tilde{Q} \tilde{x}_\psi + \tilde{u}_\psi^T \tilde{u}_\psi \right) dt \\ &\leq x(0)^T X^{-1}(\rho_o) x(0) \end{aligned} \tag{20}$$

where $\tilde{x}_\psi$ and $\tilde{u}_\psi$ denote the trajectory and control corresponding to the parameter trajectory $\tilde{\rho}$, obtained when using the control law (18).

4 Illustrative Example

Consider an LPV system with the following state space realization:

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ 1 & 0.5\rho - 1.5 \end{bmatrix},\ B_2 = [0 \quad 1]^T, \\ C_1 &= \sqrt{2} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},\ D_{12} = [0 \quad 1]^T \\ \mathcal{P} &= \{\rho: 0 \leq \rho \leq 1\};\ \underline{\nu} = -2, \overline{\nu} = 2 \end{aligned} \tag{21}$$

It can be easily verified that the following matrix function satisfies the AMIs (10):

$$\begin{aligned} X(\rho) &= X_o + X_1\rho + X_2\rho^2 \\ X_o &= \begin{bmatrix} 0.2210 & -0.3505 \\ -0.3505 & 1.1272 \end{bmatrix} \\ X_1 &= \begin{bmatrix} -0.0239 & 0.0924 \\ 0.0924 & -0.3577 \end{bmatrix} \\ X_2 &= \begin{bmatrix} 0.0243 & -0.0683 \\ -0.0683 & 0.2180 \end{bmatrix} \end{aligned} \tag{22}$$

for all $\rho \in \mathcal{F}_\nu$. Figure 1 trajectories starting from the initial condition $[0\ 2]^T$ for the AMI–based (x_{1ami}, x_{2ami}) and the proposed controller respectively. The latter was implemented using $T = 2$ as horizon and $\Psi = x^T X^{-1}(\rho)x$. For the specific parameter history shown there, the receding horizon controller yields $J = 6.91$ versus $J = 8.30$ for the control law (11), a performance improvement of roughly 20%. Similar results were obtained for other initial conditions and parameter trajectories.

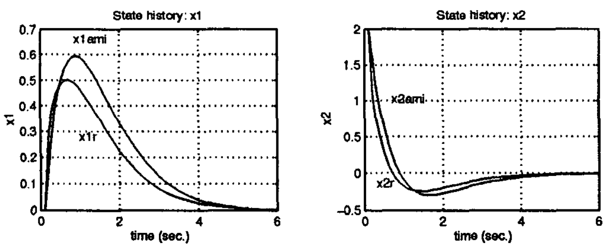

Figure 1: state, control and (normalized) parameter trajectories for the Example

5 Conclusions

In contrast with the case of linear plants, tools for simultaneously addressing performance and stability of linear parameter varying systems have emerged relatively recently. Moreover, these tools are based on sufficient conditions and thus they can be arbitrarily conservative.

In this paper, motivated by some earlier results on regulation of LTV systems [12] we propose a new suboptimal $\mathcal{H}_2$ regulator for LPV systems. This regulator is based upon recasting the infinite horizon regulation problem into a (approximately) equivalent finite horizon form, using an idea originally proposed in [9] and latter extended to the nonlinear case [10]. The main result of the paper shows that this control law is guaranteed to stabilize the system and yield optimal performance, provided that the optimal return function $V(x,\rho)$ is known in a neighborhood of the origin and that there is enough time to solve, on line, certain optimization problem. In the second part of the paper we propose a simplified control law that requires knowing only an approximation Ψ (rather that V). Such an approximation is readily available from techniques currently used to deal with LPV systems. The main result of this section shows that the proposed control law is stabilizing *for any* choice of the horizon T, and, in the worst case, is guaranteed to perform no worse than an AMI based controller. These results were illustrated with a simple example where the proposed controller resulted in about 20% performance improvement.

An additional advantage of the proposed framework is that it can be easily extended to incorporate constraints or nonlinear dynamics, proceeding as in [10]. Research in this direction is currently being pursued, as well as into incorporating robust stability and performance.

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