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$\mathcal{H}_2$ Control with Time Domain Constraints [1]

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Abstract

In this paper we study the problem of minimizing the $\mathcal{H}_2$ norm of a given transfer function subject to time–domain constraints on the time response of a different transfer function to a given test signal. The main result of the paper shows that this problem admits a minimizing solution in $\mathcal{H}_2$. Moreover, rational solutions with performance arbitrarily close to optimal can be found by constructing a family of approximating problems. Each one of these problems entails solving a finite–dimensional quadratic programming problem whose dimension can be determined before hand.

1 Introduction

In many cases the objective of a control system design can be stated simply as synthesizing an internally stabilizing controller that minimizes the response to some exogenous inputs. When these exogenous inputs are assumed arbitrary but with bounded energy and the outputs are also measured in terms of energy, this problem leads to the minimization of an $\mathcal{H}_\infty$-norm of the closed loop system. The case where the exogenous inputs are bounded persistent signals and the outputs are measured in terms of the peak time–domain magnitude, leads to the minimization of an $\mathcal{L}^1/\ell^1$-norm. $\mathcal{H}_\infty$-optimal control can now be solved by elegant state space formulae [5] while $\mathcal{L}^1/\ell^1$-optimal control can be (approximately) solved by finite linear programming [4]. Finally, the case where the input is a bounded energy signal and performance is measured in terms of the ℓ^∞ norm leads to the generalized $\mathcal{H}_2$ problem [9], also solvable via finite–dimensional convex optimization.

In many cases, following a common practice in engineering, some of the performance requirements are stated in terms of the response of the closed–loop system to a given, fixed test input (such as bounds on the rise time, settling time or maximum error to a step). In this case, if the output is measured in terms of its energy the problems leads to the minimization of the closed–loop $\mathcal{H}_2$-norm, extensively studied in the 1960's and 1970's. On the other hand, if the outputs are measured in terms of the peak time–domain magnitude, it leads to the minimization of $\mathcal{L}^\infty/\ell^\infty$-norm [3, 10, 11, 12, 7, 6, 1].

In general, a realistic control problem is likely to involve specifications on both the energy and peak values of the output. It is well known that, for discrete–time stable systems, the $\mathcal{H}_2$ norm is an upper bound of the ℓ^∞ norm. Thus, in principle one can try to enforce restrictions on the peak value of a (weighted) time–domain response through the minimization of a weighted $\mathcal{H}_2$ norm. However, this approach can be arbitrarily conservative.

In this paper we propose a solution to $\mathcal{H}_2$ problems subject to time domain constraints given in terms of the response to a fixed, given signal. The main result of the paper shows that these problems admit a solution in $\mathcal{H}_2$. Moreover, we show that computing a rational (and thus implementable) controller yielding performance ϵ–away from the optimal can be accomplished by solving a sequence of finite–dimensional quadratic programs.

2 Preliminaries

2.1 Notation

ℓ^1 denotes the space of absolutely summable sequences $h = \{h_i\}$ equipped with the norm $\|h\|_{\ell^1} \doteq \sum_{k=0}^{\infty} |h_k| < \infty$. $\ell^1(Z)$ denotes the Banach space of sequences $\{h_n\} : \sum_{k=-\infty}^{\infty} |h_k| < \infty$. ℓ^∞ denotes the space of bounded sequences $h = \{h_i\}$ equipped with the norm $\|h\|_{\ell^\infty} \doteq$

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$\sup_{k\geq 0}|h_i| < \infty$. We denote by l^p_∞ the space of bounded vector sequences $\{h(k) \in R^p\}$. In this space we define the norm $||h||_{\ell^\infty} \doteq \sup_i ||h_i(k)||_{\ell^\infty}$. Given a sequence $h \in \ell^1$, its z–transform is defined as $H(z) = \sum_{i=0}^{\infty} h_i z^{-i}$. In the sequel, by a slight abuse of notation we will sometimes use the notation $||H||_{\ell^\infty}$ to denote $||h||_{\ell^\infty}$.

By $\mathcal{H}_2(\mathcal{H}_2{}^\perp)$ we denote the Banach space of complex valued matrix functions $G(z)$ with analytic continuation outside (inside) the unit disk, and square integrable there, equipped with the usual $\mathcal{H}_2$ norm $||G||_2^2 \doteq \sup_{\gamma>1} \frac{1}{2\pi} \oint_{|z|=\gamma} |G(z)|_F^2 \frac{dz}{z}$, where $||.||_F$ denotes the Frobenious norm. $\mathcal{RH}_2$ denotes the subspace of $\mathcal{H}_2$ consisting of real rational transfer matrices.

The projection operator $\mathcal{P}_n : \mathcal{H}_2 \rightarrow \mathcal{RH}_2$ is defined by

$$\mathcal{P}_n [G(z)] \doteq \sum_{i=0}^{n-1} G_i z^{-i} \qquad (2-1)$$

2.2 The $\mathcal{H}_2$ with time domain constraints problem

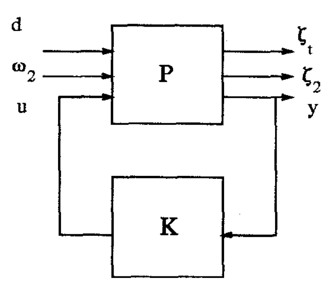

Figure 1: The $\mathcal{H}_2$ with time domain constraints setup

Consider the system shown in Figure 1, where the signal $w_2 \in R^{p_2}$ (white noise) and $d \in R$ represent exogenous disturbances and a known, given test signal respectively, $u \in R^{p_u}$ represents the control action, $\zeta_2 \in R^{m_1}$ and $\zeta_t \in R$ represent regulated outputs, and where $y \in R^{m_y}$ represents the measurements available to the controller.

Assume that the generalized discrete-time plant P is finite-dimensional, linear time invariant. Let $T(z)$ and $S(z)$ denote the closed-loop transfer matrices from w_2 to ζ_2 and from d to ζ_t respectively, obtained when connecting a stabilizing controller from y to u. Using the Youla Parameterization, the set of all such transfer matrices can be parameterized by [13]

$$\begin{aligned} T(z) &= T_{11}(z) + T_{12}(z)Q(z)T_{21}(z) \\ S(z) &= S_{11}(z) + S_{12}(z)Q(z)S_{21}(z) \end{aligned}$$

where T_{ij}, S_{ij} are stable transfer matrices, and $Q(z) \in \mathcal{H}_2$ is the "free parameter" in the parameterization. In order to stress the dependence on Q, the notation $T(Q)$, $S(Q)$ is sometimes used in the sequel. The parameterization allows for precisely stating the $\mathcal{H}_2$ with time–domain constraints problem as:

Problem 1 *Given sequences of upper and lower bounds $\{ub_i\}$ and $\{lb_i\}$, find the optimal value of the performance measure:*

$$\mu \doteq \inf_{Q\in\mathcal{H}_2} ||T_{11} + T_{12}QT_{21}||_2$$

subject to $lb_i \leq (S * d)_i \leq ub_i, i = 0, 1, \dots$ and the corresponding optimal controller Q^*.

Without loss of generality (by using weighting functions and absorbing these weights in the generalized plant (see [11] for details)) this problem can be recast into the following form:

Problem 2 *Find the optimal value of the performance measure:*

$$\mu \doteq \inf_{Q\in\mathcal{H}_2} ||T_{11} + T_{12}QT_{21}||_2$$

subject to $||S_{11} + S_{12}QS_{21}||_{\ell^\infty} \leq 1$

Lemma 1
Let T_{12}, T_{21} have generically full column and row rank respectively, and assume that a solution to Problem 2 exists. Then this solution is unique.

In the sequel we solve Problem 2 by constructing sequences of super and sub–optimal controllers, $\{\underline{Q_i}\}$ and $\{\overline{Q_i}\}$ respectively such that the corresponding $S(Q_i)$ satisfies $||S(Q_i)||_{\ell^\infty} \leq 1$ and such that $||T(Q_i)||_2 \rightarrow \mu$.

3 Problem Solution

3.1 Problem Transformation

It is a standard result that the parameterization of all stabilizing controllers can be selected so that T_{12}, T_{21} are inner and co-inner respectively and such that $R \doteq T_{12}{}^{\sim}T_{11}T_{21}{}^{\sim} \in \mathcal{RH}_2^\perp$

[13]. Since the $\mathcal{H}_2$ norm is invariant under pre(post)–multiplication by inner (outer) matrices, we have that

$$\begin{aligned} \|T_{11} + T_{12}QT_{21}\|_2^2 &= \|R+Q\|_2^2 \\ &= \|G_{sp}\|_2^2 + \|D_G + Q\|_2^2 \end{aligned}$$

where G_{sp} denotes the strictly proper part of $G \doteq R^\sim$ and D_G its feed–through term. It follows that Problem 1 may be reformulated as follows.

Problem 3

$$\inf_{Q \in \mathcal{H}_2} \|Q\|_2$$

subject to $\|S_{11} + S_{12}(Q - D_G)S_{21}\|_{\ell^\infty} \leq 1.$

Problem 3 is a convex infinite-dimensional problem, for which no closed-form solution is known to exist. In this paper, a solution will be computed by taking the limit of the solution to some finite-dimensional minimization problems. For notational simplicity, in the sequel we consider SISO systems, but the proofs generalize to the MIMO case at the price of a more involved notation. Additionally, by redefining S_{11} as $S_{11} - S_{12}D_GS_{21}$ if necessary, we can assume without loss of generality that $D_G = 0$.

3.2 Computation of super–optimal solutions

In this section, a sequence of finite dimensional convex optimization problems is introduced. The n-th problem has $\mathcal{O}(n)$ variables, and its optimal cost $\underline{\mu}^n$ satisfies $\underline{\mu}^n \leq \mu$. The sequence of problems approximates Problem 1 in the sense that $\underline{\mu}^n \to \mu$ and the partial solutions converge to the optimal solution (in the $\mathcal{H}_2$ norm) as $n \to \infty$.

Using the projection operator defined in (2-1), consider the optimization problem

Problem 4

$$\underline{\mu}^n = \inf_{Q \in \mathcal{H}_2} \|Q\|_2^2$$

subject to $\|\mathcal{P}_n(S_1 + S_2Q)\|_{\ell^\infty} \leq 1.$

Problem 4 can be thought of as a finitely–many constraints approximation to the original problem, where the constraints are enforced only over a finite horizon n. In the sequel we show that this problem is equivalent to a finite dimensional quadratic programming problem.

Lemma 2 *Problem 4 is equivalent to:*

$$\underline{\mu}^n = \min_{[Q_0^n \; Q_1^n \; \cdots \; Q_{n-1}^n]} \sum_{i=0}^{n-1} \|Q_i^n\|_F^2$$

subject to

$$\left\| \mathcal{P}_n \left(S_1(z) + S_2(z) \sum_{i=0}^{n-1} Q_i^n z^{-i} \right) \right\|_{\ell^\infty} \leq 1$$

Theorem 1 *Assume that there exists $\hat{Q} \in \mathcal{H}_2$ such that $\|S_1 + S_2Q\|_{\ell^\infty} \leq 1$. Then $\underline{\mu}^n \uparrow \mu$ and $\|Q^n - Q^*\|_2 \to 0$, where $Q^* \in \mathcal{H}_2$ is a solution to Problem 2.*

3.3 Computation of sub–optimal solutions

Theorem 1 shows that a solution to Problem 2 can be obtained by solving a sequence of quadratic programming problems. However, it does not furnish information on how to select n to achieve some desired error bound. To solve this difficulty, in this section we introduce a sequence of suboptimal solutions converging to the optimal from above.

Consider the following finitely many variables approximation to Problem 2:

Problem 5

$$\overline{\mu}^n = \min_{[Q_0^n \; Q_1^n \; \cdots \; Q_{n-1}^n]} \sum_{i=0}^{n-1} \|Q_i^n\|_F^2$$

subject to $\|[S_1(z) + S_2(z)Q^n(z)]\|_{\ell^\infty} \leq 1$ *where* $Q^n(z) = \sum_{i=0}^{n-1} Q_i^n z^{-i}$.

Theorem 2 *$\overline{\mu}^n \downarrow \mu$ and $\|Q^n - Q^*\|_2 \to 0$, where $Q^* \in \mathcal{H}_2$ is the solution to Problem 2.*

In principle, Problem 5 is a semi–infinite dimensional quadratic programming problem, since it has an infinite number of constraints. However, as we show in the sequel, under mild conditions only finitely many of these constraints are active.

Theorem 3 *Assume that S_2 does not have any zeros on $|z| = 1$. Then Problem 5 is equivalent to:*

$$\overline{\mu}^n = \min_{[Q_0^n \; Q_1^n \; \cdots \; Q_{n-1}^n]} \sum_{i=0}^{n-1} \|Q_i^n\|_F^2$$

subject to $|Q_i^n| \leq M_Q$, $i = 0, 1 \ldots, n$ *and*

$$\left\| P_{(n+N)} \left(S_1(z) + S_2(z) \sum_{i=0}^{n-1} Q_i^n z^{-i} \right) \right\|_{\ell\infty} \leq 1$$

where M_Q *and* N *are constants that depend only on the problem data.*

4 The continuous–time case

In this section we consider the continuous–time counterpart of Problem 2, namely:

Problem 6 *Given upper and lower bound functions* $ub(t)$ *and* $lb(t)$, *find the optimal value of the performance measure:*

$$\mu_t \doteq \inf_{Q \in \mathcal{H}_2} \|Q\|_2$$

subject to $lb(t) \leq (S * d)(t) \leq ub(t), t \geq 0$, *and the corresponding optimal controller* Q^*.

The main result of this section shows that this problem is equivalent to a discrete–time problem similar to Problem 2 that can be solved proceeding as in section 3. To establish this result, begin by introducing the *Laguerre* functions, defined as

$$l_i(s) = \frac{\sqrt{(2a)}}{s+a} \left(\frac{s-a}{s+a} \right)^{i-1}, \; i = 1, 2 \ldots$$

where a is a positive real. It is a standard fact (see for instance [8], Chap. 18) that the family $\{l_i\}$ is an orthonormal basis in $\mathcal{H}_2$. Therefore, any function $G(s) \in \mathcal{H}_2$ can be expanded as: $G(s) = \sum_{i=1}^{\infty} \Gamma_i l_i$. Since these functions are orthonormal it follows that $\|G\|_2^2 = \sum_{i=0}^{\infty} \|\Gamma_i\|_F^2$. In the sequel, for simplicity, we will assume that $S_1(s)$ is strictly proper. Since $Q \in \mathcal{H}_2$ this implies that $\phi(Q) = S_1 + S_2 Q$ is also strictly proper. Let S_1, S_2, Q, ub and lb have the following Laguerre expansions:

$$\begin{aligned} S_1(s) &= \sum_{i=1}^{\infty} \sigma_{1,i} l_i(s), \; S_2(s) = \sigma_{2,0} + \sum_{i=1}^{\infty} \sigma_{2,i} l_i(s) \\ ub(s) &= \sum_{i=1}^{\infty} ub_i l_i(s), \; lb(s) = \sum_{i=1}^{\infty} lb_i l_i(s) \\ Q(s) &= \sum_{i=1}^{\infty} \theta_i l_i(s) \end{aligned}$$

Straightforward but tedious computations using the fact that $l_i(s) \cdot l_j(s) = \frac{1}{\sqrt{2a}} (l_{i+j-1} - l_{i+j})$ yield the expansion for $\Phi(s) = \sum_{i=1}^{\infty} \phi_i l_i(s)$ where

$$\begin{aligned} \phi_i &= \sigma_{1,i} + \sum_{j=1}^{i} \hat{s}_{2,i+1-j} q_{j-1} \\ \hat{s}_{2,i} &= \sigma_{2,i} + \sigma_{2,0}\sqrt{2a} \\ q_0 &= \frac{\theta_1}{\sqrt{2a}}, \; q_j = \frac{\theta_{j+1} - \theta_j}{\sqrt{2a}}, \; j \geq 1 \end{aligned}$$

Thus $\|Q\|_2^2 = \sum_0^{\infty} \theta_i^2 = 2a \sum_{i=1}^{\infty} \left(\sum_{j=0}^{i-1} q_j \right)^2 \doteq f(Q)$ and $lb(t) \leq (S * d)(t) \leq ub(t), t \geq 0$ $\iff$ $lb_i \leq \phi_i \leq ub_i, i = 1, \ldots$. Hence Problem 6 is equivalent to the following discrete–time problem:

Problem 7

$$\mu_t = \min_{Q \in \mathcal{H}_2} f(Q)$$

subject to $lb_i \leq |\hat{S}_1(z) + \hat{S}_2(z) Q(z)|_i \leq ub_i, i = 1, \ldots,$ *where* $\hat{S}_1(z) = \sum_{i=1}^{\infty} \sigma_{1,i} z^{-i}$, $\hat{S}_2(z) = \sum_{i=1}^{\infty} \hat{s}_{2,i} z^{-i}$, $Q(z) = \sum_{i=1}^{\infty} q_i z^{-i}$.

Clearly this problem is similar to Problem 2 (the only difference being in the objective quadratic function) and thus can be solved using the techniques proposed in section 3.

5 Illustrative Example

Consider the problem of minimizing the $\mathcal{H}_2$ norm of the sensitivity function for the unstable nonminimum phase system shown in Figure 2, subject to a constraint on the peak of the control due to a unit–impulse disturbance w. Assume that the transfer function of the

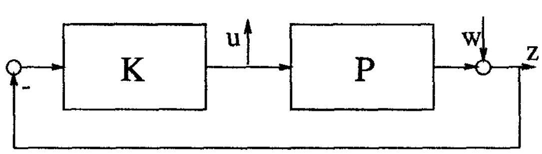

Figure 2: Block diagram for the example

plant is given by $P(z) = \frac{z-2}{z+10}$. In this case the optimal (unconstrained) $\mathcal{H}_2$ controller achieves $\|T_{zw}\|_{H_2}^2 = 1.5155^2$ with $\|T_{uw}\|_{l_\infty} = 9.0600$. Suppose that it is required that the magnitude of the control action must remain below 6.8, i.e. $\|T_{uw}\|_{\ell\infty} \leq 6.8$[1]. An inner–outer factorization

[1]Using the techniques in [3] it can be shown that $\inf_Q \|T_{uw}\|_{\ell\infty} = 6.75$

of the plant is given by

$$\begin{aligned} T_1 &= \frac{(z+10)(z-1.7375)}{(z-0.5)(z+0.1)} \\ T_2 &= -0.05\frac{(z+10)(z-0.2)}{(z-0.5)(z+0.1)} \\ S_1 &= 8.6625\frac{z+10}{(z-0.5)(z+0.1)} \\ S_2 &= -0.05\frac{(z+10)^2}{(z-0.5)(z+0.1)} \end{aligned}$$

In this example the finitely-many variables approximation is unfeasible for $n < 2$. For $n \geq 3$ the sub and super-optimal solutions coincide. Hence in this case the optimal Q is a 2^{nd} order FIR, yielding $||T_{zw}||^2_{\mathcal{H}_2} = 3.22460^2$ and $||T_{uw}||_{l_\infty} = 6.8000$. The corresponding controller has the following state-space realization:

$$K_{opt} = \left[\begin{array}{ccc|c} 1.85 & 0.084 & 0.006 & -0.9 \\ 0.086 & -0.048 & -0.0034 & -0.057 \\ 0 & 1.0 & 0 & 0 \\ \hline 11.14 & 0.48 & 0.034 & -0.43 \end{array}\right]$$

Figure 3 shows a comparison of the impulse responses of T_{uw} for the constrained and unconstrained $\mathcal{H}_2$ problems.

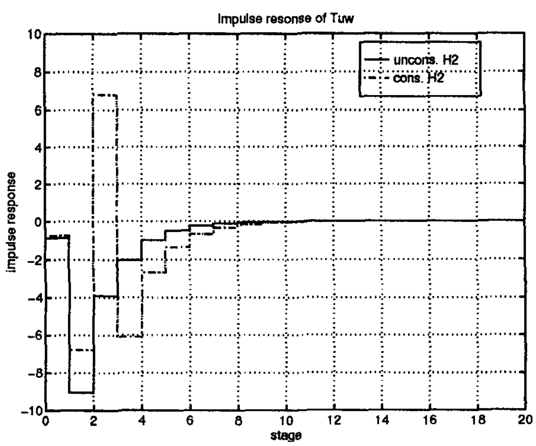

Figure 3: Impulse responses of T_{uw}

6 Conclusions

In this paper we consider the problem of optimizing the $\mathcal{H}_2$ norm of a given system subject to additional specifications given in terms of the response to a given test signal. The main result shows that both in the discrete and continuous time cases this problem admits a solution in $\mathcal{H}_2$. Moreover, suboptimal solutions can be obtained by solving sequences of finite-dimensional quadratic programming problems until the gap between upper and lower bounds of the solution is smaller than a pre-specified tolerance. Additional results show that the sequence of controllers thus obtained converges strongly to the optimal solution.

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