Proceedings of the American Control Conference
Albuquerque, New Mexico June 1997
0-7803-3832-4/97/$10.00 © 1997 AACC

An Exact Solution to Continuous–Time Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Problems

M. Sznaier[1]
Electrical Engineering
Penn State University
University Park, PA 16802

H. Rotstein
Dept. of Electrical Engineering
Technion, Haifa 32000, Israel

Juanyu Bu
Electrical Engineering
Penn State University
University Park, PA 16802

A. Sideris
Dept. of Mechanical Eng.
University of California
Irvine, CA 92717

Abstract

The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem can be motivated as a nominal LQG optimal control problem subject to robust stability constraints, expressed in the form of an $\mathcal{H}_\infty$ norm bound. While at the present time there exist efficient methods to solve a modified problem consisting on minimizing an *upper bound* of the $\mathcal{H}_2$ cost subject to the $\mathcal{H}_\infty$ constraint, the original problem remains, to a large extent, still open.

This paper contains a solution to general continuous–time mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems, based upon constructing a family of approximating problems. Each of these approximations consists on a finite–dimensional convex optimization and an unconstrained standard $\mathcal{H}_\infty$ problem. The set of solutions is such that in the limit the performance of the optimal controller is recovered, allowing to establish the existence of an optimal solution. Although the optimal controller is not necessarily finite–dimensional, it is shown that a performance arbitrarily close to the optimal can be achieved with rational (and thus physically implementable) controllers. Moreover, the computation of a controller yielding a performance ϵ-away from optimal requires the solution of a single optimization problem.

1. Introduction

Consider the system illustrated in Fig. 1, where the signals w_∞(an l^2 signal) and w_2(white noise) represent exogenous disturbances, u represents the control action, ζ_∞ and ζ_2 represent regulated outputs, and where y represents the measurements. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem consists of finding an internally stabilizing controller $u(s) = K(s)y(s)$ such that the RMS value of the performance output ζ_2 due to w_2 is minimized, subject to the specification $\|T_{\zeta_\infty w_\infty}(s)\|_\infty \leq \gamma$. This problem was originally introduced in [2] and has received considerable attention since. A large portion of this work (see for instance [2, 5, 20, 18, 8, 7] and references therein) addresses the related problem of minimizing an *upper bound* of the $\mathcal{H}_2$ norm, subject to the $\mathcal{H}_\infty$ constraint. This modified problem, having the advantage of leading to a mathematically tractable formulation, is based upon the intuitively plausible idea that minimizing this upper bound should also reduce the actual objective function. Unfortunately, this may not be the case; numerical results [1] suggest that for some examples the solution to the 'modified" problem may yield an $\mathcal{H}_2$ norm larger than the one achieved by the "central" solution to the pure $\mathcal{H}_\infty$ problem. These examples illustrate the need to develop tools for solving the *exact* $\mathcal{H}_2/\mathcal{H}_\infty$ problem.

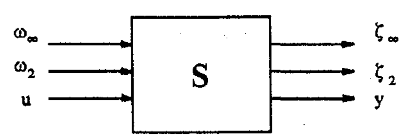


Figure 1: The Generalized Plant

In the state–feedback case, some partial results in this directions were presented in [13]. By fixing instead the order of the controller, [12] used Lagrange multipliers to find necessary conditions for optimality. Unfortunately this approach is prone to numerical difficulties. Moreover, the relationship between controller order and achievable performance is not clear.

An alternative approach is to use the Youla parametrization to recast the $\mathcal{H}_2/\mathcal{H}_\infty$ problem as an *infinite* dimensional convex optimization [3]. Truncation then yields a finite–dimensional problem which is tractable [4]. However, at this moment is not clear how to select the truncation horizon and the effect of this choice upon achievable performance.

[1] Supported in part by NSF under grants ECS–9211169 and ECS–9625920

The approach that we pursue in this paper evolves from the solution to discrete–time mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems presented in [17, 14] combined with some of the ideas in [15, 16]. As in [17, 14] we will show that a *suboptimal* solution to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem, i.e., a solution satisfying the $\mathcal{H}_\infty$ constraint and with performance arbitrarily close to the optimal, can be obtained by solving a finite–dimensional convex optimization problem followed by an unconstrained $\mathcal{H}_\infty$ minimization. Furthermore, sharper results include the existence of an optimal solution, the convergence in the $\mathcal{H}_2$ topology, and the fact that the optimal performance achieved over $\mathcal{H}_\infty$ and the smaller (and physically more meaningful) space $\mathcal{A}_o$ (see bellow for a definition) is the same. Proofs of these results are omitted due to space limitations.

2. Preliminaries

2.1. Notation

$\mathcal{L}^\infty$ denotes the Lebesgue space of complex valued matrix functions which are essentially bounded on the $j\omega$ axis, equipped with the norm:

$$\|G(s)\|_\infty \doteq ess \sup_{\omega} \overline{\sigma}\left(G(j\omega)\right)$$

where $\overline{\sigma}$ denotes the largest singular value. By $\mathcal{H}_\infty(\mathcal{H}_\infty^-)$ we denote the subspace of functions in $\mathcal{L}^\infty$ with a bounded analytic continuation in $Re(s) > 0$ ($Re(s) < 0$)). $\mathcal{RH}_\infty$ denotes the subspace of real rational transfer matrices of $\mathcal{H}_\infty$ and $\mathcal{A}_o$ denotes the subset of $\mathcal{H}_\infty$ functions continuous in the *closed* right–half plane. The norm on $\mathcal{H}_\infty$ is defined by $\|G(z)\|_\infty \doteq ess\sup_{Re(s)>0} \overline{\sigma}(G(s))$. By $\mathcal{H}_2$ we denote the space of complex valued matrix functions $G(s)$ with analytic continuation in $Re(s) > 0$ and square integrable on the $j\omega$ axis, equipped with the usual $\mathcal{H}_2$ norm:

$$\|G\|_2^2 \doteq \sup_{Re(s)>0} \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)|_F^2 d\omega,$$

where $\|.\|_F$ denotes the Frobenious norm.

The discrete time counterparts of $\mathcal{L}^\infty, \mathcal{H}_\infty$ and $\mathcal{H}_2$ will be denoted as $\mathcal{L}^\infty(D), \mathcal{H}_\infty(D)$ and $\mathcal{H}_2(D)$ respectively, i.e. $\mathcal{L}^\infty(D)$ denotes the space of complex–valued functions bounded on the unit circle, equipped with the norm $\|G(z)\|_\infty \doteq ess\sup_{|z|=1} \overline{\sigma}(G(j\omega))$. $\mathcal{H}_\infty(D)$ denotes the subspace of $\mathcal{L}^\infty(D)$ formed by functions analytic inside the unit disk, while $\mathcal{H}_\infty^-(D)$ denotes the subspace of functions analytic outside the unit disk. Note that with these definitions stable functions have all their poles *outside* the unit disk, rather than inside. Also of interest in this discrete–time setting is the space $\mathcal{H}_{\infty,\delta}$ of transfer functions in $\mathcal{H}_\infty$ which are analytic inside the disk of radius δ, where $\delta > 1$ (usually $\delta \approx 1$). When equipped with the norm $\|G(z)\|_{\infty,\delta} \doteq \sup_{|z|<\delta} \overline{\sigma}(G(z))$, $\mathcal{H}_{\infty,\delta}$ becomes a Banach space. Similarly, the space $\mathcal{H}_{2,\delta}$ is defined as the Banach space of transfer matrices having analytic continuation inside $|z| = \delta$ and square integrable there, equipped with the norm:

$$\|G\|_{2,\delta}^2 \doteq \sup_{\gamma<\delta} \frac{1}{2\pi} \oint_{|z|=\gamma} |G(z)|_F^2 \frac{dz}{z},$$

The *Laguerre* functions are defined as

$$l_i(s) = \frac{\sqrt{(2a)}}{s+a}\left(\frac{s-a}{s+a}\right)^{i-1}, \; i = 1, 2 \dots \qquad (1)$$

where a is a positive real. It is a standard fact (see for instance [9], Chap. 18) that the family $\{l_i\}$ is an orthonormal basis in $\mathcal{H}_2$. Therefore, any function $G(s) \in \mathcal{H}_2$ can be expanded as: $G(s) = \sum_{i=1}^{\infty} \Gamma_i l_i$. Since these functions are orthonormal it follows that $\|G\|_2^2 = \sum_{i=0}^{\infty} \|\Gamma_i\|_F^2$. The projection operator $\mathcal{P}_n : \mathcal{H}_2 \to \mathcal{RH}_2$ is defined by

$$\mathcal{P}_n(G(s)) \doteq \sum_{i=1}^{n} \Gamma_i l_i \qquad (2)$$

2.2. The Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Problem

Assume that the generalized plant P is finite-dimensional and linear time invariant. Let $T(s)$ and $S(s)$ denote the closed-loop transfer matrices from w_∞ to ζ_∞ and from w_2 to ζ_2 respectively, obtained when connecting a stabilizing controller from y to u. Using the Youla Parameterization, the set of all such transfer matrices can be parameterized by [19]

$$\begin{aligned} T(s) &= T_{11}(s) - T_{12}(s)Q(s)T_{21}(s) \\ S(s) &= S_{11}(s) - S_{12}(s)Q(s)S_{21}(s), \end{aligned} \qquad (3)$$

where T_{ij}, S_{ij} are stable transfer matrices, and $Q(s) \in \mathcal{H}_\infty$ is the "free parameter" in the parametrization. It is a standard result that the parametrization (3) can be selected so that T_{12}, T_{21} are inner and co-inner respectively, and there exist $T_{12\perp}$, $T_{21\perp}$ such that $\left[\begin{array}{cc} T_{12} & T_{12\perp} \end{array}\right]$ and $\left[\begin{array}{c} T_{21} \\ T_{21\perp} \end{array}\right]$ are unitary. This will be assumed in the sequel. Finally, for simplicity we will assume that all the signals involved are scalar, although the results presented here can be generalized to the MIMO case (at the cost of more involved notation) proceeding as in [14]. In this case (3) reduces to

$$\begin{aligned} T(s) &= T_1(s) - T_2(s)Q(s) \\ S(s) &= S_1(s) - S_2(s)Q(s), \end{aligned}$$

where $T_2(s)$ is an inner function (i.e. $T_2(-s)T_2(s) = 1$). This parametrization allows for precisely stating the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem as:

Problem 1 (Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Problem) *Find the optimal value of the performance measure:*

$$\mu \doteq \inf_{Q\in\mathcal{H}_\infty} \{\|S_1 - S_2 Q\|_2 \; s.\; t. \; \|T_1 - T_2 Q\|_\infty \leq 1\}$$

and, given $\epsilon > 0$, *a controller* Q *such that* $\|S(Q)\|_2 \leq \mu + \epsilon$ *and* $\|T(Q)\|_\infty \leq 1$.

Note that from the strict convexity of the $\mathcal{H}_2$ norm if a solution to problem 1 exists then it is unique. In general, Problem 1 admits a minimizing solution in $\mathcal{H}_\infty$ but not in $\mathcal{A}_o$ [10], implying that the optimal controller cannot be approximated by a rational transfer function. Moreover the optimal closed-loop system is in general not exponentially stable, since exponentially stable functions belong to $\mathcal{A}_o$. From an engineering standpoint, these undesirable properties motivate the following problem:

Problem 2 (Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Problem in $\mathcal{A}_o$)

Find the optimal value of the performance measure:

$$\mu_R \doteq \inf_{Q \in \mathcal{A}_o} \{\|S_1 - S_2 Q\|_2 \ s.\ t.\ \|T_1 - T_2 Q\|_\infty \le 1\}$$

and, given $\epsilon > 0$ find a controller $Q \in \mathcal{A}_o$ such that $\|S(Q_R)\|_2 \le \mu_R + \epsilon$ and $\|T(Q_R)\|_\infty \le 1$.

In the sequel we solve these problems by constructing an optimizing sequence of controllers $\{Q_i\}$ such that the corresponding $T(Q_i)$ satisfies $\|T(Q_i)\|_\infty \le 1$ and such that $\|S(Q_i)\|_2 \rightarrow \mu$.

3. Problem Solution

3.1. Computation of a Solution over $\mathcal{H}_\infty$

In this section, a sequence of finite dimensional convex optimization problems is introduced. The n-th problem has $\mathcal{O}(n)$ variables, and its optimal cost μ^n satisfies $\mu^n \le \mu$. The sequence of problems approximates Problem 1 in the sense that $\mu^n \rightarrow \mu$ and the partial solutions converge to the optimal solution as $n \rightarrow \infty$.

Using the projection operator defined in (2), consider the optimization problem

Problem 3
Find the optimal value of the performance measure:

$$\mu^n \doteq \min_{Q \in \mathcal{H}_\infty} \left\| \mathcal{P}_n \left[S_1(s) - S_2(s) P_n(Q(s)) \right] \right\|_2 \quad s.\ t.\ \|T_1 - T_2 Q\|_\infty \le 1$$

and the corresponding optimal controller Q^n.

The number n is called the "horizon" in the sequel.

Lemma 1 *Problem 3 is convex and $\mu^n \le \mu^{n+1} \le \mu$.*

As a consequence of Lemma 1, $\mu^n \rightarrow \mu^{lim}$. The equality $\mu^{lim} = \mu$ is established next.

Theorem 1
Assume that a feasible solution to Problem 1 exists, and that $S_2 \not\equiv 0$. Then $\mu^n \uparrow \mu$ and the sequence of solutions $\{Q^n(s)\}$ converges normally to a solution of Problem 1. Moreover, for a fixed n the solution to Problem 3 can be found by solving a finite-dimensional convex optimization problem and an unconstrained $\mathcal{H}_\infty$ problem.

Since the sequence $\{Q^n\}$ converges normally, so does the sequence of truncated closed-loop transfer matrices $S_n \doteq \mathcal{P}_n [S_1 - S_2 Q^n]$. Moreover, it can also be easily shown that the sequence S_n is Cauchy sequence in the $\mathcal{H}_2$ topology and hence converges in the $\mathcal{H}_2$-norm. However, since normal convergence does not imply uniform convergence, one cannot de that Q^n will provide an approximate solution to the problem, even if n is taken very large. This difficulty is addressed in the next section.

4. Computation of a Solution over $\mathcal{RH}_\infty$

In this section we show that a rational ϵ-suboptimal solution to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem can be found by solving a sequence of truncated problems, each one requiring only a *finite* number of elements of the Laguerre expansion of S. To establish this result we will show that: *(i)* Problem 1 can be solved by considering a sequence of modified problems; *(ii)* the optimal cost achievable with controllers in $\mathcal{RH}_\infty$ can be made arbitrarily close to the optimum over $\mathcal{H}_\infty$ (i.e. $\mu_R = \mu$); and *(iii)* a suboptimal solution to a modified problem, with cost within ϵ of the optimum, can be found by solving a truncated problem. We begin by considering the following change of variables.

4.1. A Change of Variables

Consider a real rational transfer matrix $F(s)$, and define the mapping

$$F^\epsilon(s) \doteq F\left(\frac{a}{1+\epsilon}\frac{s+a\epsilon}{a+\epsilon s}\right), \quad 0 < \epsilon, \quad a > 0$$

This amounts to considering the bilinear transformation

$$\overline{s} = a\frac{\hat{a}\epsilon - s}{\epsilon s - \hat{a}}, \quad \hat{a} \doteq \frac{a}{1+\epsilon} \tag{4}$$

that maps the closed left half plane of the s-plane to D_ϵ, a closed-disk centered at $s = -a\frac{1+\epsilon^2}{2\epsilon}$, with radius $r = a\frac{1-\epsilon^2}{2\epsilon}$. Thus if $F(s)$ is stable then $F^\epsilon(s)$ has all its poles in D_ϵ. In the sequel we will denote by $\mathcal{H}_\infty(D_\epsilon)$ the subspace of $\mathcal{H}_\infty$ formed by functions analytic outside D_ϵ, equipped with the norm $\|F\|_{\infty,\epsilon} \doteq \sup_{s \notin D_\epsilon} |F(s)|$. Moreover, we will assume that ϵ is selected small enough so that $T_i, S_j \in \mathcal{H}_\infty(D_\epsilon)$. Note that from the Maximum Modulus Theorem it follows that $\|F(s)\|_\infty = \|F^\epsilon(s)\|_{\infty,\epsilon} \ge \|F^\epsilon(s)\|_\infty$.

4.2. A Modified $\mathcal{H}_2/\mathcal{H}_\infty$ Problem

Consider the following modified $\mathcal{H}_2/\mathcal{H}_\infty$ problem:

Problem 4 (Problem $\mathcal{H}_2/\mathcal{H}_{\infty,\epsilon}$) *Find*

$$\mu_\epsilon \doteq \min_{Q \in \overline{\mathcal{RH}}_{\infty,\epsilon}} \{\|S_1 - S_2 Q\|_2 \ s.\ t.\ \|T_1 - T_2 Q\|_{\infty,\epsilon} \le 1\}$$

and the corresponding controller Q^*, where $^-$ denotes closure.

Note that the set $\{Q \in \overline{\mathcal{RH}_\infty} : \|T_1 - T_2Q\|_{\infty,\epsilon} \leq 1\}$ is compact in the $\mathcal{H}_\infty$ topology and thus Q^* is well defined. Comparing the solution to this optimization problem for decreasing ϵ with the solution to Problem 1 gives the following result.

Theorem 2 *Given $\epsilon_1 > 0$ there exists $\epsilon > 0$ such that $\mu_\epsilon \leq \mu + \epsilon_1$.*

Corollary 3 *(i) $\mu_\epsilon \geq \mu_R$. (ii) The optimal cost of Problems 1 and 2 are equal, i.e., $\mu = \mu_R$.*

Thus, although the optimal solution of the $\mathcal{H}_2/\mathcal{H}_\infty$ problem is not generically in $\mathcal{A}_o$, the infimum achievable with controllers in the closure of $\mathcal{RH}_\infty$ is actually equal to the optimal cost over $\mathcal{H}_\infty$.

Finally, we show convergence of the closed-loop systems and of the controllers in the $\mathcal{H}_2$ topology.

Lemma 2 *Consider a sequence $0 < \epsilon_i \downarrow 0$. Then, the sequence of corresponding closed loops $S_i \doteq S_1 - S_2Q_i$ converges in the $\mathcal{H}_2$ topology. Moreover, if S_2 does not have zeros on the $j\omega$-axis then the sequence of controllers converges in the $\mathcal{H}_2$ topology, i.e. $\|Q_i - Q^{\lim}\|_2 \to 0$.*

4.3. Computing an Approximate Solution

From the proof of Theorem 2, if a sub-optimality level $\epsilon_1 > 0$ is given, then for an ϵ which can be computed in terms of the data, the solution Q^* to Problem 4 satisfies $\mu_\epsilon \leq \mu + \epsilon_1$. Moreover, Q^* can be approximated arbitrarily close by

$$Q_n(s) = \sum_{i=1}^{n} \theta_i l_i(s) + \left(\frac{s-a}{s+a}\right)^n Q_R(s)$$

where $\Theta_\mathbf{n} = (\theta_1 \;\; \theta_2 \;\; \ldots \;\; \theta_n)$ solves the following *finite-dimensional* convex optimization problem:

$$\mu_\epsilon^n = \min_{[\theta_1 \;\; \theta_2 \;\; \cdots \;\; \theta_n]} \left\| \mathcal{P}_n \left(S_1(s) - S_2(s) \sum_{i=1}^{n} \theta_i l_i(s) \right) \right\|_2$$
$$s.t.\; \bar{\sigma}(W_\epsilon(\Theta_\mathbf{n})) \leq 1$$

where $W_\epsilon(\Theta_\mathbf{n})$ is obtained from $W(\mathbf{q_n})$ defined in [17, 14] through the change of variables

$$\begin{aligned} q_o &= \frac{\theta_1}{\sqrt{2a}} \\ q_i &= \frac{\theta_{i+1} - \theta_i}{\sqrt{2a}} \left(\frac{1-\epsilon}{1+\epsilon}\right)^i, \; i \geq 1 \end{aligned}$$

and where n is larger than some pre-computable bound N_ϵ. To see this, solve Problem 4 for a fixed $\epsilon > 0$. If $Q_\epsilon^n \doteq \sum_{i=1}^{n} Q_i l_i + \left(\frac{s-a}{s+a}\right)^n Q_{tail}^n(s)$ denotes the solution, then $T^n = T_1 - T_2Q^n$ is such that $\|T^n\|_{\infty,\epsilon} \leq 1$. Moreover, by using the bilinear transformation (4) before performing the Youla parameterization (3) so that T_2 is inner over D_ϵ we have that $\|Q^n\|_{\infty,\epsilon} \leq 1 + \|T_1\|_{\infty,\epsilon}$. Hence,

$$\|S_1 - S_2Q^n\|_{\infty,\epsilon} \leq \|S_1\|_{\infty,\epsilon} + \|S_2\|_{\infty,\epsilon}(1 + \|T_1\|_{\infty,\epsilon}) \doteq K$$

Expanding $S_1 - S_2Q^n = \sum_{i=1}^{\infty} \sigma_i l_i$, it follows that $|\sigma_i| \leq \left(\frac{1-\epsilon}{1+\epsilon}\right)^{i-1} \cdot M$ for some constant M , which yields the following bound for the truncation error:

$$\|(I - \mathcal{P}_n)(S_1 - S_2Q^n)\|_2 \leq \text{constant} \times \frac{M\delta^n}{\sqrt{1-\delta^2}},\; \delta \doteq \frac{1-\epsilon}{1+\epsilon}$$

By taking n sufficiently large, say $n \geq N_\epsilon$, μ_ϵ^n approximates μ_ϵ as closely as desired. Note, though, that N_ϵ is usually very large and hence may not be useful for computations. This difficulty can be circumvented by combining the upper bound introduced in this section with the lower bound introduced in Section 3.1 to obtain sequences of suboptimal and super-optimal solutions.

5. Numerical Example

In this section we present a numerical example to illustrate the results discussed above. Consider the system with the following state space realization:

$$\begin{pmatrix} \zeta_\infty \\ \zeta_2 \\ y \end{pmatrix} = \left[\begin{array}{cc|ccc} \frac{2}{3} & -\frac{8}{3} & 1 & 1 & 3 \\ \frac{4}{3} & \frac{2}{3} & 2 & 2 & 2 \\ \hline 2 & 5 & 0 & 0 & 1 \\ 4 & 3 & 0 & 0 & 1 \\ 1 & -1 & 1 & 1 & 0 \end{array}\right] \begin{pmatrix} \omega_\infty \\ \omega_2 \\ u \end{pmatrix}$$

The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem of interest is to minimize $\|T_{\zeta_2\omega_2}\|_2$ subject to $\|T_{\zeta_\infty\omega_\infty}\|_\infty \leq 4.1$.

The optimal $\mathcal{H}_2$ controller yields $\|T_{\zeta_2\omega_2}\|_2 = 16.64$ and $\|T_{\zeta_\infty\omega_\infty}\|_\infty = 6.338$. The optimal $\mathcal{H}_\infty$ controller yields $\|T_{\zeta_2\omega_2}\|_2 = 89.41$ and $\|T_{\zeta_\infty\omega_\infty}\|_\infty = 2.667$. The proposed method results in a 7th order controller (after model reduction), yielding $\|T_{\zeta_2\omega_2}\|_2 = 17.81$ and $\|T_{\zeta_\infty\omega_\infty}\|_\infty = 4.041$. These results are summarized in Table 1. The frequency responses of $T_{\zeta_2\omega_2}$ and $T_{\zeta_\infty\omega_\infty}$ with different controllers are shown in Figure 2.

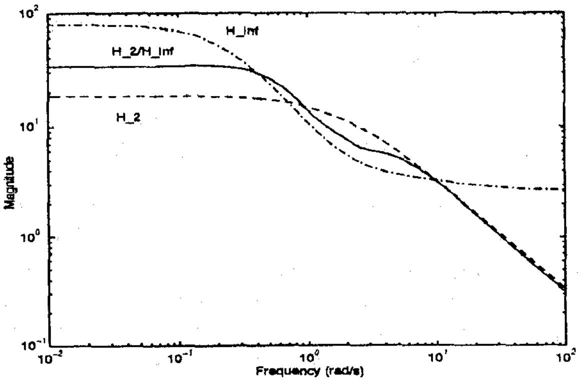
Type	Order	$\|T_{\zeta_2\omega_2}\|_2$	$\|T_{\zeta_\infty\omega_\infty}\|_\infty$
optimal $\mathcal{H}_2$	2	16.64	6.338
optimal $\mathcal{H}_\infty$	2	89.41	2.667
mixed $\mathcal{H}_2/\mathcal{H}_\infty$	7	17.81	4.041

Table 1: Comparison of results: mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem

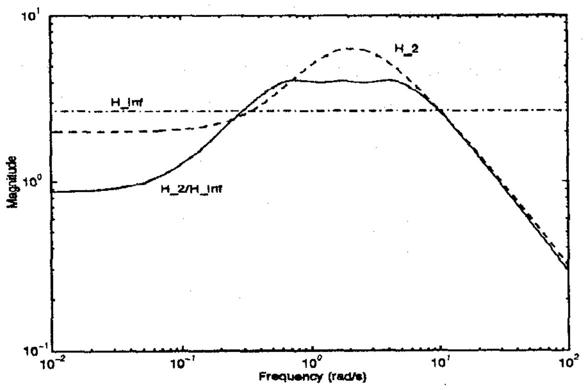
6. Conclusions

In this paper we have proposed a solution to general mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problems. As opposed to most of the literature on the subject (but see the introduction for some exceptions) we deal with the *exact* $\mathcal{H}_2$ object instead of an auxiliary cost function which over-bounds it. The main idea is to construct a family of optimization problems and then show that the set of solutions thus generated converges, in a rather strong sense, to a solution of the original problem. At each step, the optimization problems are convex and have a structure which allows

1990.



(a)



(b)

Figure 2: Frequency responses of closed-loop system: (a) $T_{\zeta_2 w_2}$; (b) $T_{\zeta_\infty w_\infty}$

for efficient computations. Our approach provides additional new insight into some properties of the optimal solutions. This includes the fact that, although an optimal solution is not in general "well-behaved" since it is not continuous on the border of the region of stability (and thus the resulting closed-loop system is not exponentially stable), the optimal performance can be approached arbitrarily close by a real-rational controller. Moreover, from a practical standpoint, our approach allows for finding exponentially stable suboptimal solutions with a prescribed degree of stability, by selecting $\epsilon > 0$ in Problem 4.

References

[1] E. Baeyens and P. Khargonekar. Some examples in mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control. In *Proceedings of the 1994 American Control Conference*, pages 1608–1612, 1994.

[2] D. Bernstein and W. Haddad. LQG control with an $\mathcal{H}_\infty$ performance bound: A Riccati equation approach. *IEEE Transactions on Automatic Control*, 34(2):293–305, 1989.

[3] S. Boyd and C. Barratt. *Linear Controllers Design - Limits of Performance.* Prentice-Hall, Englewood Cliffs, 1990.

[4] X. Chen and J. T. Wen. A Linear Matrix Inequality Approach to the General Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Problem. *Proceedings of the 1995 American Control Conference*, Seattle, WA, June 21–23, 1995, to appear.

[5] J. Doyle, K. Zhou, K. Glover, and B. Bodenheimer. Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance objectives II: Optimal control. *IEEE Transactions on Automatic Control*, 39(8):1575–1587, 1994.

[6] G. Gu, P. Khargonekar, and E. B. Lee. Approximation of infinite-dimensional systems. *IEEE Transactions on Automatic Control*, 34(6):610–618, 1989.

[7] I. Kaminer, P. Khargonekar, and M. Rotea. Mixed $\mathcal{H}_\infty/\mathcal{H}_2$ control for discrete time systems via convex optimization. *Automatica*, 29(1):57–70, 1993.

[8] P. Khargonekar and M. Rotea. Mixed $\mathcal{H}_\infty/\mathcal{H}_2$ control: A convex optimization approach. *IEEE Transactions on Automatic Control*, 36(7):824–837, 1991.

[9] Y. W. Lee. Statistical Theory of Communications. John Wiley, New York, 1960.

[10] A. Megretski. "Pure" mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimal controller is not exponentially stable. Preprint, 1994.

[11] B. Palka. *An Introduction to Complex Function Theory.* Undergraduate Texts in Mathematics. Springer-Verlag, 1992.

[12] D. B. Ridgely, L. S. Valavani and G. Stein. Solution to the general mixed $\mathcal{H}_\infty/\mathcal{H}_2$ control problem–Necessary condition for optimality. In *Proceedings of the 1992 American Control Conference, Chicago, Illinois*, 1992.

[13] M. Rotea and P. Khargonekar. $\mathcal{H}_2$-optimal control with and $\mathcal{H}_\infty$ constraint: The state feedback case. *Automatica*, 27(2):307–316, 1991.

[14] H. Rotstein and M. Sznaier. An Exact Solution to General 4–Blocks Discrete–Time Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Problems via Convex Optimization. *IEEE Transactions on Automatic Control*, to appear.

[15] H. Rotstein and A. Sideris. Continuous time $\mathcal{H}_\infty$ optimization with time domain constraints: the siso case. preprint, 1996.

[16] C. W. Scherer. Multiobjective $\mathcal{H}_2/\mathcal{H}_\infty$ control. *IEEE Transactions on Automatic Control*, 40(6):1054–1062, 1995.

[17] M. Sznaier. An exact solution to general SISO mixed $\mathcal{H}_\infty/\mathcal{H}_2$ problems via convex optimization. *IEEE Transactions on Automatic Control*, 39(12):2511–2517, 1994.

[18] H. Yeh, S. Banda, and B. Chang. Necessary and sufficient conditions for mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control. *IEEE Transactions on Automatic Control*, 37(3):355–358, 1992.

[19] K. Zhou, J. Doyle, and K. Glover. *Robust and Optimal Control.* Prentice-Hall, New Jersey, 1995.

[20] K. Zhou, K. Glover, B. Bodenheimer, and J. Doyle. Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance objectives I: Robust performance analysis. *IEEE Transactions on Automatic Control*, 39(8):1564–1574, 1994.