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Mixed $\mathcal{L}^1/\mathcal{H}_\infty$ Suboptimal Controllers for Continuous–Time Systems

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Abstract

A successful controller design paradigm must take into account both model uncertainty and design specifications. Model uncertainty can be addressed using either $\mathcal{H}_\infty$ or $\mathcal{L}^1$ robust control theory, depending upon the uncertainty characterization. However, these frameworks cannot accommodate the realistic case where the design specifications include both time and frequency domain constraints. In this paper we present a design procedure for suboptimal $\mathcal{L}^1/\mathcal{H}_\infty$ controllers. These controllers allows for minimizing the worst–case peak output due to bounded persistent disturbances, while, at the same time, satisfying an $\mathcal{H}_\infty$–norm constraint upon some closed–loop transfer function of interest. The main result of the paper shows that rational mixed $\mathcal{L}^1/\mathcal{H}_\infty$ suboptimal controllers can be obtained by solving a sequence of problems, each one consisting of a finite–dimensional convex optimization and a standard, unconstrained $\mathcal{H}_\infty$ problem.

1. Introduction

A large number of control problems involve designing a controller capable of stabilizing a given linear time invariant system while minimizing the worst case response to some exogenous disturbances. This problem is relevant for instance for disturbance rejection, tracking and robustness to model uncertainty (see [1] and references therein). When the exogenous disturbances are modeled as bounded energy signals and performance is measured in terms of the energy of the output, this problem leads to the well known $\mathcal{H}_\infty$ theory [2]. This framework, combined with μ–analysis [3] has been successfully applied to a number of hard practical control problems (see for instance [4]). However, being a frequency–domain based method, $\mathcal{H}_\infty$ can address only a subset of the common performance requirements.

The case where the signals involved are persistent bounded signals leads to the l_1 optimal control theory [1,5–7]. These methods are attractive since they allow for an explicit solution to the robust performance problem [8]. However, they cannot accommodate some common classes of frequency domain specifications (such as $\mathcal{H}_2$ or $\mathcal{H}_\infty$ bounds).

Finally, in [9] a solution was given to a discrete–time mixed $l_1/\mathcal{H}_\infty$ problem, where the l_1 norm of the closed–loop transfer function between an input–output pair of signals is minimized, subject to an $\mathcal{H}_\infty$ constraint upon the transfer function between a different pair of signals. Although this result represents an important step towards developing a metodology capable of handling mixed time–frequency domain specifications, it does not presently have a continuous–time counterpart. Conceivably, this difficulty could be solved by using a discrete–time controller, designed using the theory developed in [9], connected to the continuous–time plant through sample and hold devices [10–13]. However, the use of sample and hold elements usually entails a performance loss, which may be significant, since the control is constrained to remain constant during the sampling period.

In this paper we propose a method to design rational suboptimal $\mathcal{L}^1/\mathcal{H}_\infty$ controllers for continuous–time systems. This method is based upon solving an auxiliary discrete–time $l_1/\mathcal{H}_\infty$ problem [9], obtained using the simple transformation $z = 1 + \tau s$, and then transforming back the resulting controller to the s domain. Thus it only entails solving a *finite dimensional* convex constrained optimization problem and an *unconstrained* $\mathcal{H}_\infty$ problem. The main results of the paper show that: i) the performance of the resulting closed–loop continuous–time system is bounded above (both in the frequency and time domains) by the performance of the auxiliary discrete–time system used in the design; and ii) optimal performance is recovered as the parameter $\tau \to 0$.

The paper is organized as follows: In section II we introduce the notation to be used and we give a formal definition to the mixed $\mathcal{L}^1/\mathcal{H}_\infty$ control problem. Section III contains the bulk of the theoretical results. Here we introduce the discrete time Euler approximating system (EAS) and we show that the peak values of the time and frequency responses of the EAS are upper bounds of the corresponding continuous–time quantities. As an immediate consequence, it follows that suboptimal $\mathcal{L}^1/\mathcal{H}_\infty$ controllers with guaranteed cost can be designed by applying $l_1/\mathcal{H}_\infty$ theory to the EAS. In section IV we present a simple design example and

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we compare our controller to the unconstrained optimal $\mathcal{H}_\infty$ controller. Finally, in section V, we summarize our results and we indicate directions for future research.

2. Problem Formulation

2.1. Notation

By $\mathcal{L}_\infty(j\mathcal{R})$ we denote the Lebesgue space of complex valued transfer functions which are essentially bounded on the imaginary axis with norm $\|T(s)\|_\infty \triangleq \sup_\omega |T(j\omega)|$. $\mathcal{H}_\infty(j\mathcal{R})$ $(\mathcal{H}_\infty(j\mathcal{R})^-)$ denotes the set of stable (antistable) complex functions $G(s) \in \mathcal{L}_\infty(j\mathcal{R})$, i.e analytic in $\Re(s) \geq 0$ $(\Re(s) \leq 0)$. Similarly, $\mathcal{L}_\infty(T)$ denotes the Lebesgue space of complex valued transfer functions which are essentially bounded on the unit circle with norm $\|T(z)\|_\infty \triangleq \sup_\omega |T(e^{j\omega})|$, and $\mathcal{H}_\infty(T)$ $(\mathcal{H}_\infty^-(T))$ denotes the set of stable (antistable) complex functions $G(z) \in \mathcal{L}_\infty(T)$, i.e analytic in $|z| \geq 1$ $(|z| \leq 1)$. $\mathcal{L}^1(R_+)$ denotes the space of measurable functions $f(t)$ equipped with the norm: $\|f\|_1 = \int_0^\infty |f(t)|dt < \infty$; similarly l^1 denotes the space of real sequences, equipped with the norm $\|h\|_1 = \sum_{k=0}^\infty |h_k| < \infty$. The prefix $\mathcal{R}$ will be used to denote subspaces consisting of rational transfer functions. Given a sequence $h \in l_1$ (a function $h(t) \in \mathcal{L}^1$) we will denote its z-transform (Laplace transform) by $H(z)$ $(H(s))$. By a slight abuse of notation given $H(z) \in \mathcal{RH}_\infty(T)$ $(H(s) \in \mathcal{RH}_\infty(j\mathcal{R}))$ we will denote $\|H(z)\|_1 \triangleq \|h\|_1$ $(\|H(s)\|_1 \triangleq \|h(t)\|_1)$.

Given two transfer matrices $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ and Q with appropriate dimensions, the lower *linear fractional transformation* is defined as:

$$\mathcal{F}_l(T,Q) \triangleq T_{11} + T_{12}Q(I - T_{22}Q)^{-1}T_{21}$$

For a discrete-time transfer matrix $G(z)$, we define its conjugate as $G^\sim \triangleq G'(\frac{1}{z})$, where $'$ denotes transpose. Similarly, $G^\sim(s) = G'(-s)$. Finally, throughout the paper we will use packed notation to represent state-space realizations, i.e.

$$G(s) = C(sI - A)^{-1}B + D \triangleq \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

2.2. Statement of the Problem

Consider the system represented by the block diagram 1, where the scalar signals w_∞ (a bounded energy signal), w_1 (a persistent $\mathcal{L}^\infty$ signal) and u represent exogenous disturbances and the control action respectively; and ζ_∞, ζ_1 and y represent the regulated outputs and the measurements respectively. Then, the mixed $\mathcal{L}^1/\mathcal{H}_\infty$ control problem can be stated as: Given the nominal system (S), find an internally stabilizing controller $K(s)$ such that worst case peak amplitude of the performance output $\|\zeta_1\|_\infty$ due to signals inside the $\mathcal{L}^1$-unity ball is minimized, subject to the constraint $\|T_{\zeta_\infty w_\infty}\|_\infty \leq \gamma$.

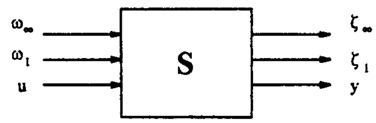

Figure 1: The Generalized Plant

2.3. Problem Transformation

Assume that the system S has the following state-space realization (where without loss of generality we assume that all weighting factors have been absorbed into the plant):

$$\left(\begin{array}{c|ccc} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ C_3 & D_{31} & D_{32} & D_{33} \end{array}\right) \qquad (S)$$

where D_{13} has full column rank, D_{31} has full row rank, and where the pairs (A, B_3) and (C_3, A) are stabilizable and detectable respectively. It is well known (see for instance [14]) that the set of all internally stabilizing controllers can be parametrized in terms of a free parameter $Q \in \mathcal{H}_\infty$ as $K = \mathcal{F}_l(J,Q)$ where J has the following state-space realization:

$$\left(\begin{array}{c|cc} A + B_3F + LC_3 + LD_{33}F & -L & B_3 + LD_{33} \\ \hline F & 0 & I \\ -(C_3 + D_{33}F) & I & -D_{33} \end{array}\right) \qquad (J)$$

and where F and L are selected such that $A + B_3F$ and $A + LC_3$ are stable. By using this parametrization, the scalar closed-loop transfer functions $T_{\zeta_\infty w_\infty}$ and $T_{\zeta_1 w_1}$ can be written as:

$$\begin{array}{rcl} T_{\zeta_\infty w_\infty}(s) & = & T_1^\infty(s) + T_2^\infty(s)Q(s) \\ T_{\zeta_1 w_1}(s) & = & T_1(s) + T_2(s)Q(s) \end{array} \qquad (1)$$

where T_i, T_i^∞, Q are stable transfer functions. Moreover (see [14]), it is possible to select F and L in such a way that $T_2^\infty(s)$ is inner (i.e. $T_2^{\infty\sim}T_2^\infty = I$). By using this parametrization the mixed $\mathcal{L}^1/\mathcal{H}_\infty$ problem can be now precisely stated as solving:

$$\begin{array}{l} \mu^o = \inf_{Q \in \mathcal{RH}_\infty} \|T_{\zeta_1 w_1}\|_1 \\ \text{s. t. } \|T_1^\infty(s) + T_2^\infty(s)Q(s)\|_\infty \leq \gamma \end{array} \qquad (2)$$

3. Problem Solution

In this section we present a method for finding suboptimal rational $\mathcal{L}^1/\mathcal{H}_\infty$ controllers, based upon the use of discrete-time $l_1/\mathcal{H}_\infty$ theory. The main result of this section shows that suboptimal controllers, with cost arbitrarily close to the optimum, can be found by solving a finite-dimensional convex constrained optimization problem and an unconstrained $\mathcal{H}_\infty$ problem.

3.1. Definitions

Definition 1 *Consider the continuous time system (S). Its Euler Approximating System (EAS) is defined as the following discrete time system:*

$$\left(\begin{array}{c|ccc} I+\tau A & \tau B_1 & \tau B_2 & \tau B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ C_3 & D_{31} & D_{32} & D_{33} \end{array}\right) \qquad (EAS)$$

where $\tau > 0$.

3.2. Properties of the Euler Approximating System

In this section we recall some properties of the EAS. The main result of this section shows that the l_1 and $\mathcal{H}_\infty$ norms of the Euler Approximating system are upper bounds of the corresponding continuous-time quantities. Moreover, these upper bounds are non-increasing with τ and converge to the exact value as $\tau \to 0$.

Theorem 1 *Consider the system:*

$$\begin{aligned} \dot{x} &= Ax + B_2 v \\ \zeta &= C_2 x + D_{22} v \end{aligned} \qquad (3)$$

Assume that the corresponding (EAS):

$$\begin{aligned} x_{k+1} &= (I+\tau A)x_k + \tau B_2 v_k \\ \zeta_k &= C_2 x_k + D_{22} v_k \end{aligned} \qquad (4)$$

is asymptotically stable. Then, the system (4) is asymptotically stable and such that:

$$\begin{aligned} \|T_{\zeta v}(s)\|_1 &= \sup_{\substack{v\in\mathcal{L}^\infty,\ \|v\|\le 1 \\ x_o=0}} \|\zeta(t)\|_\infty \\ &\le \|T^{(EAS)}_{\zeta v}(z,\tau)\|_1 = \sup_{\substack{v\in l^\infty,\ \|v\|\le 1 \\ x_o=0}} \|\zeta_k\|_\infty \end{aligned} \qquad (5)$$

Conversely, if (4) is asymptotically stable and $\|T_{\zeta v}\|_1 \triangleq \mu_c$ *then for all* $\mu > \mu_c$ *there exists* $\tau^* > 0$ *such that for all* $0 < \tau \le \tau^*$ *the EAS (4) is asymptotically stable and such that* $\|T^{(EAS)}_{\zeta v}(z,\tau)\|_1 \le \mu$.

Proof: See [15]

Lemma 1 *Consider a strictly decreasing sequence* $\tau_i \to 0$, *and let* $\mu_i = \|T^{EAS}_{\zeta v}(z,\tau_i)\|_1$. *Then the sequence* μ_i *is non-increasing and such that* $\mu_i \to \|T_{\zeta v}\|_1$.

Proof: See [15]

Next we show that the $\|.\|_\infty$ norm of the transfer function of the EAS provides an upper bound of the $\|.\|_\infty$ norm of the transfer function of the continuous-time system.

Lemma 2 *Assume that (4) is asymptotically stable and consider a strictly decreasing sequence geq*$\tau_i \to 0$. *Let* $T_{\zeta v}(s)$ *denote the transfer function of* (4) *and* $T^{EAS}_{\zeta v}(z,\tau_i)$ *the transfer function of the EAS corresponding to* τ_i. *Then:*

$$\begin{aligned} \|T_{\zeta v}(s)\|_\infty &\le \|T^{EAS}_{\zeta v}(z,\tau_i)\|_\infty \ \forall\, i \\ \|T^{EAS}_{\zeta v}(z,\tau_i)\|_\infty &\le \|T^{EAS}_{\zeta v}(z,\tau_j)\|_\infty \ i > j \\ \lim_{\tau_i\to 0} \|T^{EAS}_{\zeta v}(z,\tau_i)\|_\infty &= \|T_{\zeta v}(s)\|_\infty \end{aligned} \qquad (6)$$

Proof: The proof, omitted for space reasons, follows from applying the maximum modulus theorem to the disks C_i, centered at $s = \frac{-1}{\tau_i}$ with radius $\frac{1}{\tau_i}$, and to the closed RHP.

Combining the results of Theorem 1 and Lemmas 1 and 2 yields the main result of this section:

Theorem 2 *Assume that* $\inf_{Q\in\mathcal{RH}_\infty} \|T_{\zeta_\infty w_\infty}(s)\|_\infty = \gamma_o < \gamma$. *Consider a strictly decreasing sequence* $\tau_i \to 0$, *and the corresponding EAS*(τ_i). *Let*

$$\begin{aligned} \mu_i &= \inf_{\substack{Q\in\mathcal{RH}_\infty(T) \\ \|T_{\zeta_\infty w_\infty}\|_\infty \le \gamma}} \|T^{(EAS)}_{\zeta_1 w_1}(z,\tau_i)\|_1 \\ \mu_o &= \inf_{\substack{Q\in\mathcal{RH}_\infty(j\Re) \\ \|T_{\zeta_\infty w_\infty}\|_\infty \le \gamma}} \|T_{\zeta_1 w_1}(s)\|_1 \end{aligned} \qquad (7)$$

Then the sequence μ_i *is non-increasing and such that* $\mu_i \to \mu^o$.

Proof: Given a controller $K(z,\tau_i)$ that internally stabilizes EAS(τ_i), let $S_{cl}(K,z,\tau_i)$ denote the closed-loop system, and $T_{\zeta_1 w_1}(K,z,\tau_i)$ and $T_{\zeta_\infty w_\infty}(K,z,\tau_i)$ the corresponding transfer functions. Assume that $K(z,\tau_i)$ is such that $\|T_{\zeta_\infty w_\infty}(K,z,\tau_i)\|_\infty \le \gamma$. Given any $j > i$, consider the controller $\hat{K}(z)$ obtained from K_i using the change of variable $z \to (1 + \frac{\tau_i(z-1)}{\tau_j})$ and the corresponding closed-loop system $S_{cl}(\hat{K},z,\tau_j)$. Since $j > i$, it follows from Theorem 1 that $S_{cl}(\hat{K},z,\tau_j)$ is internally stable. Moreover, from Lemma 2 we have that:

$$\|T_{\zeta_\infty w_\infty}(\hat{K},z,\tau_j)\|_\infty \le \|T_{\zeta_\infty w_\infty}(K,z,\tau_i)\|_\infty \le \gamma \quad (8)$$

Hence, $\hat{K}$ is a feasible controller for EAS(τ_j). From Theorem 1 we have that:

$$\|T_{\zeta_1 w_1}(\hat{K},z,\tau_j)\|_1 \le \|T_{\zeta_1 w_1}(K,z,\tau_i)\|_1 \qquad (9)$$

It follows then that

$$\begin{aligned} \mu_j = &\inf_{\substack{K \\ \|T_{\zeta_\infty w_\infty}\le\gamma}} \|T_{\zeta_1 w_1}(K,z,\tau_j)\|_1 \le \\ \mu_i = &\inf_{\substack{K \\ \|T_{\zeta_\infty w_\infty}\le\gamma}} \|T_{\zeta_1 w_1}(K,k,\tau_i)\|_1, \text{ for } j > i \end{aligned} \qquad (10)$$

Since μ_i is a non-increasing sequence, bounded below by μ_o, it has a limit $\hat{\mu} \geq \mu_o$. We will show that $\hat{\mu} = \mu_o$ by contradiction. Assume that $\hat{\mu} > \mu_o$ and define $\epsilon \triangleq \hat{\mu} - \mu_o$. Since $\inf_{Q \in \mathcal{RH}_\infty} \|T_1^\infty(s) + T_2^\infty(s)Q(s)\|_\infty < \gamma$, there exists $Q_1 \in \mathcal{RH}_\infty$ such that $\|T_1^\infty(s) + T_2(s)Q_1^\infty(s)\|_\infty = \gamma_1 < \gamma$. From the definition of μ_o it follows that there exists $Q_o \in \mathcal{RH}_\infty$ such that $\|T_1^\infty(s) + T_2^\infty(s)Q_o(s)\|_\infty \leq \gamma$ and $\|T_{\zeta_1 w_1}(s)\|_1 \leq \mu_o + \frac{\epsilon}{8}$. Let $\hat{Q} \triangleq Q_o + \eta(Q_1 - Q_o)$. It follows that:

$$\begin{aligned} \|T_1 + T_2\hat{Q}\|_1 &\leq \mu^o + \tfrac{\epsilon}{8} + \eta\|T_2(Q_1 - Q_o)\|_1 \\ \|T_1^\infty + T_2^\infty\hat{Q}\|_\infty &\leq \eta\|T_1^\infty + T_2^\infty Q_1\|_\infty \\ &+ (1-\eta)\|T_1^\infty + T_2^\infty Q_o\|_\infty < \gamma \end{aligned} \tag{11}$$

Hence, by taking η small enough we have that the controller $K = \mathcal{F}_l(J, \hat{Q})$ yields $\|T_{\zeta_1 w_1}(s)\|_1 \leq \mu_o + \frac{1}{4}\epsilon$ and $\|T_{\zeta_\infty w_\infty}(s)\|_\infty < \gamma$. It follows, from Lemmas 1 and 2, that for τ small enough we have:

$$\begin{aligned} \mu(\tilde{K}) \triangleq \|T_{\zeta_1 w_1}(\tilde{K}, z, \tau)\|_1 &\leq \mu_o + \tfrac{1}{2}\epsilon \\ \|T_{\zeta_\infty w_\infty}(\tilde{K}, z, \tau)\|_\infty &\leq \gamma \end{aligned} \tag{12}$$

Where $\tilde{K}(z) \triangleq K(s)|_{z=1+\tau s}$. Hence $\mu(\tilde{K}) < \hat{\mu}$ which contradicts the definition of $\hat{\mu}$

□

Remark 1 *Theorem 2 shows that the $\mathcal{L}^1/\mathcal{H}_\infty$ problem can be solved by solving a sequence of discrete-time $l_1/\mathcal{H}_\infty$ problems, each one having the form:*

$$\mu^o = \inf_{\substack{Q \in \mathcal{RH}_\infty(T) \\ \|T_1^\infty + T_2^\infty Q\|_\infty \leq \gamma}} \|T_1 + T_2 Q\|_1 \qquad (l_1/\mathcal{H}_\infty)$$

where $T_i, T_i^\infty \in \mathcal{RH}_\infty(T)$.

3.3. A Suboptimal Solution to SISO Mixed $l_1/\mathcal{H}_\infty$ Problems

In [9] it was shown that a rational suboptimal solution to the mixed $l_1/\mathcal{H}_\infty$ problem, with cost arbitrarily close to the optimum, can be found by solving a finite-dimensional convex optimization problem and an unconstrained $\mathcal{H}_\infty$ problem. In this section we briefly review this result:

Lemma 3 (9) *Given $\delta < 1$, let $\mathcal{H}_{\infty,\delta} \triangleq \{Q(z) \in \mathcal{H}_\infty : Q(z)$ analytic in $|z| \geq \delta\}$, and consider the following modified $l_1/\mathcal{H}_\infty$ problem:*

$$\mu_\delta = \inf_{\substack{Q \in \mathcal{RH}_{\infty,\delta} \\ \|T_1^\infty(s) + T_2^\infty(s)Q(s)\|_{\mathcal{H}_{\infty,\delta}} \leq \gamma}} \|T_{\zeta_1 w_1}\|_1 \quad (l_1/\mathcal{H}_{\infty,\delta})$$

where $\|Q\|_{\mathcal{H}_{\infty,\delta}} \triangleq \sup_{|z|=\delta} |Q(z)|$. Then $\mu_\delta \geq \mu_o$ and $\lim_{\delta \to 1} \mu_\delta = \mu_o$.

Next, we recall the main result of [9], showing that if $(l_1/\mathcal{H}_{\infty,\delta})$ is feasible, then a *rational* suboptimal solution, arbitrarily close to the optimum, can be found by solving a truncated problem. Moreover, solving this truncated problem only entails solving a finite-dimensional convex optimization problem and an unconstrained $\mathcal{H}_\infty$ problem.

Theorem 3 *Given $\epsilon > 0$, a suboptimal solution to $(l_1/\mathcal{H}_{\infty,\delta})$, with cost μ_δ^ϵ such that $\mu_\delta \leq \mu_\delta^\epsilon \leq \mu_\delta + \epsilon$ is given by $Q^o = Q_P^o + z^{-N}Q_R^o$ where $Q_P^o = \sum_{i=0}^{N-1} q_i z^{-i}$; $\underline{q}^o = (q_o \dots q_{N-1})'$ solves the following finite dimensional convex optimization problem:*

$$qv^o = \operatorname*{argmin}_{\substack{\underline{q} \in R^N \\ \|Q\|_2 \leq \gamma}} \|\underline{t}_1 + \mathcal{T}\underline{q}\|_1 \tag{13}$$

and Q_R^o solves the unconstrained approximation problem:

$$Q_R^o(z) = \operatorname*{argmin}_{Q_R \in \mathcal{RH}_{\infty,\delta}} \|G(z)^{\sim} + Q_P^o + z^{-N}Q_R(z)\|_{\infty,\delta} \tag{14}$$

where:

$$G \triangleq T_2^\infty T_1^{\infty\sim} \triangleq \left(\begin{array}{c|c} A_G & b_G \\ \hline c_G & d_G \end{array}\right) \tag{15}$$

and $Q =$

$$\begin{pmatrix} yA_G^N x & yA_G^{N-1}b_G & \dots & \dots & yA_G b_G & yb \\ c_G A_G^{N-1}x & c_G A_G^{N-2}b & \dots & \dots & c_G b_G & d_G + q_0 \\ c_G A_G^{N-2}x & c_G A_G^{N-3}b & \dots & \dots & d_G + q_0 & q_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_G A_G x & c_G b_G & d_G + q_0 & \dots & q_{N-3} & q_{N-2} \\ c_G x & d_G + q_0 & q_1 & \dots & q_{N-2} & q_{N-1} \end{pmatrix}$$

where $X > 0$ and $Y > 0$ are the discrete controllability and observability grammians of G; x and y are the positive square roots of X and Y respectively; N is selected such that:

$$\begin{aligned} N &\geq \tfrac{\log_\theta(1-\delta) - \log K}{\log\delta} \\ K &\triangleq \|T_1^\infty\|_{\infty,\delta} + \|T_2^\infty\|_{\infty,\delta}(\gamma + \|G^{\sim}\|_{\infty,\delta}) \end{aligned} \tag{16}$$

and:

$$\begin{aligned} \underline{t}_1 &\triangleq (t_{1_o} \ \dots \ t_{1_{N-1}})' \\ \mathcal{T} &\triangleq \begin{pmatrix} t_{2_o} & 0 & \dots & 0 \\ t_{2_1} & t_{2_0} & \dots & 0 \\ \vdots & & \ddots & \\ t_{2_{N-1}} & \dots & & t_{2_o} \end{pmatrix} \\ \underline{q} &\triangleq (q_o \ \dots \ q_{N-1})' \end{aligned} \tag{17}$$

where t_{i_k} denotes the k^{th} element of the impulse response of $T_i(z)$

Proof: The proof follows from combining Lemma 3 in [9] with the corollary to Theorem 3 in [17].

3.4. Proposed Design Method

From the definition of the EAS it is easily seen that the closed-loop transfer function obtained by applying the rational controller $K(s)$ to (S) is the same as the closed-loop transfer function obtained by applying the controller $K(\frac{z-1}{\tau})$ to the EAS, up to the complex transformation $z = \tau s + 1$. Therefore, if a rational compensator $K(z)$ yielding an $l_1/\mathcal{H}_\infty$ cost μ_d is found for the EAS, then $K(\tau s + 1)$ internally stabilizes (S) and yields an $\mathcal{L}^1/\mathcal{H}_\infty$ cost $\mu_c \leq \mu_d$. It follows that a rational compensator can be synthesized using the EAS with a suitably small τ. These observations are formalized in the following lemma:

Lemma 4 *Consider the mixed $\mathcal{L}^1/\mathcal{H}_\infty$ control problem for SISO continuous time-systems. A suboptimal rational solution can be obtained by solving a discrete-time mixed $l_1/\mathcal{H}_{\infty,\delta}$ control problem for the corresponding EAS, with $\delta = 1 - \tau^2$. Moreover, if $K(z)$ denotes the $l_1/\mathcal{H}_\infty$ controller for the EAS, the suboptimal $\mathcal{L}^1/\mathcal{H}_\infty$ controller is given by $K(\tau s + 1)$.*

Finally, we show that by taking $\tau \to 0$, the proposed design method yields controllers with cost arbitrarily close to the optimal $\mathcal{L}^1/\mathcal{H}_\infty$ cost.

Theorem 4 *Let $\tau_i \to 0$ be a strictly decreasing sequence. Denote by K_i the controller obtained using the design procedure of Lemma 4 with $\tau = \tau_i$ and by $T_{\zeta_1 w_1}(s, K_i)$ the corresponding closed loop transfer function. Then the sequence $\mu_i \triangleq \|T_{\zeta_1 w_1}(s, K_i)\|_1$ is non-increasing and such that $\lim_{i\to\infty} \mu_i = \mu_o$.*

Proof: Using an argument similar to the proof of Theorem 2, it can be easily shown that the sequence μ_i is non-increasing, bounded below by μ_o. Let $\hat{\mu} \triangleq \lim_{i\to\infty} \mu_i$. To complete the proof we need to show that $\hat{\mu} = \mu_o$. Assume to the contrary that $\hat{\mu} > \mu_o$ and let $\epsilon \triangleq \hat{\mu} - \mu_o$. Proceeding as in (11)-(12), we can find an internally stabilizing controller $K(s)$ such that:

$$\begin{array}{l} \|T_{\zeta_1 w_1}(s, K)\|_1 \leq \mu_o + \frac{\epsilon}{4} \\ \|T_{\zeta_\infty w_\infty}(s, K)\|_\infty < \gamma \end{array} \quad (18)$$

From Theorems 1 and 2 it follows that there exists τ_1 such that the closed loop EAS obtained using the controller $K(1 + \tau_1 s)$ satisfies:

$$\begin{array}{l} \mu(K) \triangleq \|T_{\zeta_1 w_1}(K, z, \tau_1)\|_1 \leq \mu_o + \frac{1}{2}\epsilon \\ \|T_{\zeta_\infty w_\infty}(K, z, \tau_1)\|_\infty < \gamma \end{array} \quad (19)$$

Moreover, all the poles of the closed-loop continuous-time system are contained in C_1, a disk with radius $\frac{1}{\tau_1}$, centered at $s = -\frac{1}{\tau_1}$. Let $\overline{C}(\tau)$ denote the disk centered at $\frac{-1}{\tau}$ with radius $\frac{1-\tau^2}{\tau}$. Since the closed-loop system is internally stable, there exists τ such that all the poles of $T_{\zeta_\infty w_\infty}(s)$ are contained in the region $\overline{C}(\tau) \cap C_1$. Since $T_{\zeta_\infty w_\infty}(s)$ is analytic outside this region and $\|T_{\zeta_\infty w_\infty}(z, \tau_1)\|_\infty = \sup_{s \in \partial C_1} |T_{\zeta_\infty w_\infty}(s)| < \gamma$, it follows from continuity, that there exist τ small enough such that $\sup_{s \in \partial(C_1 \cap \overline{C}(\tau))} |T_{\zeta_\infty w_\infty}(s)| \leq \gamma$. Hence, from the maximum modulus theorem we have that:

$$\begin{array}{l} \|T_{\zeta_\infty w_\infty}(z, \tau)\|_{\mathcal{H}_{\infty,\delta}} = \sup_{|z|=\delta} |T_{\zeta_\infty w_\infty}(z, \tau)| \\ = \sup_{s \in \partial\overline{C}(\tau)} |T_{\zeta_\infty w_\infty}(1 + \tau s)| \\ \leq \sup_{s \in \partial(C_1 \cap \overline{C}(\tau))} |T_{\zeta_\infty w_\infty}(1 + \tau s)| \leq \gamma \end{array} \quad (20)$$

It follows then that $K(z)$, $z = 1 + \tau s$, is an admissible controller for problem $l_1/\mathcal{H}_{\infty,\delta}$, yielding a cost $\mu(K) \leq \mu_o + \frac{\epsilon}{2} < \hat{\mu}$ and hence, for $\tau_i \leq \tau$ we have:

$$\begin{array}{rcl} \mu_i \leq \mu(\tau) & = & \inf_{K, \|T_{\zeta_\infty w_\infty}\|_\infty \leq \gamma} \|T_{\zeta_1 w_1}(K, z, \tau)\|_1 \\ & \leq & \mu(K) \leq \mu_o + \frac{\epsilon}{2} \end{array} \quad (21)$$

But, since the sequence μ_i is non-increasing this contradicts the assumption that $\hat{\mu} = \lim_{i\to\infty} \mu_i = \mu_o + \epsilon$.

□

4. A Simple Example

Consider the SISO plant used in [6, 15]

$$P(s) = \frac{s-1}{s-2} \quad (22)$$

The controller that minimises $\|T\|_1 \triangleq \|PC(1+PC)^{-1}\|_1$ is given by:

$$K_{\mathcal{L}^1} = \frac{(s-2)(1.7071 - 4.1213e^{-0.8814s})}{(s-1)(-0.7071 + 4.1213e^{-0.8814s})} \quad (23)$$

and yields $T(s) = 1.7071 - 4.1213e^{-0.8814s}$, with $\|T\|_1 = 5.8284$. It is easily seen that $S(s) \triangleq (1 + PC)^{-1} = 0.7071 + 4.1213e^{-0.8814s}$, with $\|S\|_\infty = 4.8284$. Given the difficulty of physically implementing a non-rational controller, in [15] we developed a method for synthesising rational approximations to the optimal $\mathcal{L}^1$ controller. The rational approximation proposed there yields:

$$\begin{array}{rcl} T(s) & = & 1.8414 - 4.3423\frac{1}{(1+0.1s)^9} \\ S(s) & = & -0.8414 + 4.3423\frac{1}{(1+0.1s)^9} \end{array} \quad (24)$$

with $\|S\|_\infty = 3.9$ and $\|T\|_1 = 6.18$. The $\mathcal{H}_\infty$ controller that minimizes $\|S\|_\infty$ is given by $C(s) = -\frac{4}{3}$ and yields $\|S\|_\infty = 3$ and $\|T\|_1 = 10$. Finally, a mixed $\mathcal{L}^1/\mathcal{H}_\infty$ design yields $\|T\|_1 = 6.41$ and $\|S\|_\infty = 3.45$ The different frequency responses for S and the corresponding impulse responses for T are shown in Figure 2.

5. Discussion and Conclusions

In this paper we address the problem of finding internally stabilising controllers that minimize the peak

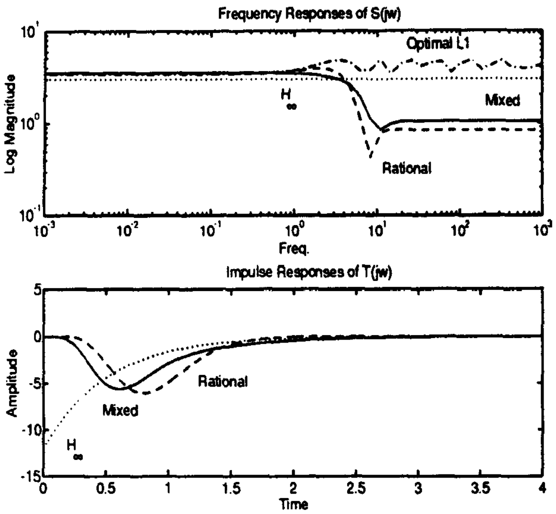

Figure 2: Impulse and Frequency Responses for Different Designs.

amplitude of the worst-case output due to persisten bounded signals, subject to robustness constraints given in the form of an $\mathcal{H}_\infty$ constraint upon the norm of a relevant transfer function. This problem is of importance for example for tracking applications, disturbance rejection, or cases where either the control action or some outputs are subject to hard bounds. It can be thought as the problem of designing a controller capable of guaranteeing an adequate robustness level agains dynamic uncertainty while using the extra available degrees of freedom to optimize a time-domain performance.

The main result of the paper shows that the resulting convex optimization problem can be decoupled into a finite dimensional, albeit non-differentiable, constrained optimization and an unconstrained Nehari approximation problem. This is a notorious departure from previous approaches to solving this types of problems [18-19], where several approximations where required in order to obtain a tractable mathematical problem.

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