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An Exact Solution to General 4-Blocks Discrete–Time Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Problems via Convex Optimization

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Abstract

The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem can be motivated as a nominal LQG optimal control problem, subject to robust stability constraints, expressed in the form of an $\mathcal{H}_\infty$ norm bound. A related modified problem consisting on minimizing an *upper bound* of the $\mathcal{H}_2$ cost subject to $\mathcal{H}_\infty$ constraints was introduced in [2]. Although there presently exist efficient methods to solve this modified problem, the original problem remains, to a large extent, still open. In [8] we developed a method to solve exactly the simpler SISO case. In this paper we extend this method to general MIMO systems. As in [8], the main result of this paper shows that the proposed method involves solving a sequence of problems, each one consisting of a finite–dimensional convex optimization and an unconstrained $\mathcal{H}_\infty$ problem

1. Introduction

During the last decade, a large research effort has been devoted to the problem of designing robust controllers, capable of guaranteeing stability in the face of plant uncertainty. As a result, a powerful $\mathcal{H}_\infty$ framework has been developed, addressing the issue of robust stability in the presence of norm–bounded plant perturbations. Since suboptimal $\mathcal{H}_\infty$ controllers are not unique, the extra degrees of freedom available can then be used to optimize some performance measure. This leads naturally to a robust performance problem: design a controller guaranteeing a desired level of performance in the face of plant uncertainty. However, in spite of a large research effort [10], this problem has not completely been solved.

Alternatively, the extra degrees of freedom can be used to solve a problem of the form *nominal performance with robust stability.* In this case the controller yields a desired performance level for the nominal system while guaranteeing stability for all possible plant perturbations. A problem of this form that has been the object of much attention lately is the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem: Given the system represented by the block diagram 1, where the signals $w_\infty \in R^{p_1}$ (an l^2 signal) and $w_2 \in R^{p_2}$ (white noise) represent exogenous disturbances, $u \in R^{p_u}$ represents the control action, $\zeta_\infty \in R^{m_1}$ and $\zeta_2 \in R^{m_2}$ represent regulated outputs, and where $y \in R^{m_y}$ represents the measurements; find an internally stabilizing controller $u(z) = K(z)y(z)$ such that the RMS value of the performance output ζ_2 due to w_2 is minimized, subject to the specification $\|T_{\zeta_\infty w_\infty}(z)\|_\infty \leq \gamma$.

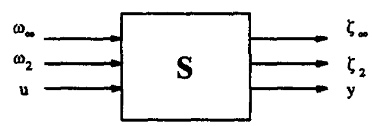

Figure 1: The Generalized Plant

Different versions of this problem have been studied recently. Bernstein and Haddad [2] considered the case where $w_2 = w_\infty$ and obtained necessary conditions for solving the *modified* problem of minimizing an *upper bound* of $\|T_{w_2\zeta_2}\|_2$, subject to the $\mathcal{H}_\infty$ constraint. In [11] the dual problem of minimizing this upper bound for the case $w_2 \neq w_\infty$, $\zeta_2 = \zeta_\infty$ was considered and sufficient conditions for optimality where given. Finally, in [9] these conditions where shown to be necessary and sufficient. In [4] Khargonekar and Rotea showed that the modified problem can be cast into the format of a constrained convex optimization problem over a bounded set of matrices and solved using non–differentiable optimization techniques.

The approaches mentioned above provide a solution to the *modified* problem. However, recent

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numerical results [1] suggest that the gap between the upper bound and the true $\mathcal{H}_2$ cost may be significant. Since little is known about the quality of this approximation, it is interesting to seek exact solutions, even if they are computationally more involved. Recently, mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control using fixed–order controllers was analyzed using a Lagrange multipliers based approach and necessary conditions for optimality were obtained [5]. However, these conditions involve solving coupled non–linear matrix equations and finding the neutrally stable solution to a Lyapunov equation, which leads to numerical difficulties. Moreover, in [6] it was shown that even in the state–feedback case, the optimal controller must be dynamic, and it is conjectured that in the general case it may have higher order than the plant. This makes a fixed order approach less attractive, since there is little a priori information on the order of the optimal controller.

Recently, an exact solution method was developed for the simpler case of SISO systems [8]. In this paper we extend this approach to MIMO systems. As in [8], the main result of the paper shows that a suboptimal solution to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem, with performance arbitrarily close to the optimal, can be obtained by solving a finite–dimensional convex optimization problema and an unconstrained $\mathcal{H}_\infty$ problem.

2. Preliminaries

2.1. Notation

$\mathcal{L}_\infty$ denotes the Lebesgue space of complex valued matrix functions which are essentially bounded on the unit circle. By $\mathcal{H}_\infty(\mathcal{H}_\infty^-)$ we denote the space of transfer matrices $G(z) \in \mathcal{L}_\infty$ which are analytic outside (inside) the unit disk. If $G(z) \in \mathcal{L}_\infty$ then its norm is defined in the standard way as $\|G(z)\|_\infty \stackrel{\Delta}{=} \sup_{0\leq\theta\leq\pi} \bar{\sigma}\left(G(e^{j\theta})\right)$ where $\bar{\sigma}$ denotes the largest singular value. By $\mathcal{RH}_\infty$ we denote the subspace of real rational transfer matrices of $\mathcal{H}_\infty$. Similarly, $\mathcal{RH}_{\infty,\delta}$ denotes the subspace of transfer matrices in $\mathcal{RH}_\infty$ which are analytic outside the disk of radius δ, $0 < \delta < 1$, equipped with the norm $\|G(z)\|_{\infty,\delta} \stackrel{\Delta}{=} \sup_{0\leq\theta\leq\pi} \bar{\sigma}\left(G(\delta e^{j\theta})\right)$. $\|G(z)\|_2$ is defined in the usual way as $\|G\|_2^2 \stackrel{\Delta}{=} \frac{1}{2\pi}\oint_{|z|=1} \frac{|G(z)|_F^2}{z} dz$ where $\|.\|_F$ denotes the Frobenious norm. For a transfer matrix $G(z)$, $G^{\sim} \stackrel{\Delta}{=} G^T(\frac{1}{z})$. Throughout the paper we will use packed notation to represent state–space realizations, i.e. $G(z) \in \mathcal{RH}_\infty$ will be written as:

$$\begin{aligned} G(z) &= C(zI-A)^{-1}B + D = D + \sum_{i=0}^{\infty} CA^iBz^{-(i+1)} \\ &\stackrel{\Delta}{=} \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \end{aligned}$$

For notational convenience, we will sometimes write $G_0 = D$ and $G_i = CA^{i-1}B$, $i = 1, 2, \cdots$, and define $\mathbf{G_n} \stackrel{\Delta}{=} [G_0 \ \cdots \ G_{n-1}]$. Finally, given two transfer matrices $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ and Q with appropriate dimensions, the lower *linear fractional transformation* is defined as:

$$\mathcal{F}_l(T,Q) \stackrel{\Delta}{=} T_{11} + T_{12}Q(I - T_{22}Q)^{-1}T_{21}$$

2.2. Problem Transformation

Assume that the system S has the following state–space realization (where without loss of generality we assume that all weighting factors have been absorbed into the plant):

$$\left(\begin{array}{c|ccc} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ C_3 & D_{31} & D_{32} & D_{33} \end{array}\right) \qquad (S)$$

where D_{13} has full column rank, D_{31} has full row rank, and where the pairs (A, B_3) and (C_3, A) are stabilizable and detectable respectively. It is well known (see for instance [10]) that the set of all internally stabilizing controllers can be parametrized in terms of a free parameter $Q \in \mathcal{H}_\infty$ as:

$$K = \mathcal{F}_l(J, Q) \qquad (1)$$

where a state–space realization of (J) can be found for instance in [10] By using this parametrization, the closed–loop transfer matrices $T_{\zeta_\infty w_\infty}$ and $T_{\zeta_2 w_2}$ can be written as:

$$\begin{aligned} T_{\zeta_\infty w_\infty}(z) &= T_{11}(z) + T_{12}(z)Q(z)T_{21}(z) \\ T_{\zeta_2 w_2}(z) &= V_{11}(z) + V_{12}(z)Q(z)V_{21} \end{aligned} \qquad (2)$$

where T_{ij}, V_{ij} are stable transfer matrices. By using this parametrization the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem can be now precisely stated as:

Problem 1 *(Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem:)* Find the optimal value of the performance measure:

$$\mu^o = \inf_{Q\in\mathcal{RH}_\infty} \|T_{\zeta_2 w_2}\|_2 \qquad (\mathcal{H}_2/\mathcal{H}_\infty)$$

$$\text{s. t. } \|T_{11}(z) + T_{12}(z)Q(z)T_{21}(z)\|_\infty \leq \gamma$$

where $\{V_i\}$ and $\{Q_i\}$ are the coefficients of the impulse responses of $T_{\zeta_2 w_2}$ and Q respectively.

Remark 1 *It is well known (see for instance [10]), that it is possible to select (J) in such a way that* $T_{12}(z)$ *is inner and* T_{21} *is co-inner. If* T_{12} (T_{21}) *is not square, it is possible to choose* $T_{12\perp}$ ($T_{21\perp}$) *such that* $T_{12a} \stackrel{\Delta}{=} [T_{12} \quad T_{12\perp}]$ ($T_{21a}^{\sim} \stackrel{\Delta}{=} [T_{21}^{\sim} \quad T_{21\perp}^{\sim}]$) *is a unitary matrix. This fact can be used to reduce* $\|T_{\zeta_\infty w_\infty}\|_\infty$ *to the form:*

$$\begin{aligned} \|T_{\zeta_\infty w_\infty}\|_\infty &= \left\| T_{11} + T_{12a} \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} T_{21a} \right\|_\infty \\ &= \left\| G + \begin{bmatrix} Q^{\sim} & 0 \\ 0 & 0 \end{bmatrix} \right\|_\infty \end{aligned} \tag{3}$$

where $G \stackrel{\Delta}{=} T_{21a} T_{11}^{\sim} T_{12a} \in \mathcal{RH}_\infty$ *has a state-space realization:*

$$G = \left(\begin{array}{c|cc} \hat{A} & B_a & B_b \\ \hline C_a & D_{aa} & D_{ab} \\ C_b & D_{ba} & D_{bb} \end{array} \right) \tag{4}$$

Remark 2 *In the sequel, for notational simplicity we will call:*

$$\begin{aligned} B_e &= [B_a \quad B_b] \\ C_e &= \begin{bmatrix} C_a \\ C_b \end{bmatrix} \\ D_{er} &= [D_{aa} \quad D_{ab}] \\ D_{ec} &= \begin{bmatrix} D_{aa} \\ D_{ba} \end{bmatrix} \end{aligned} \tag{5}$$

We will also assume, without loss of generality, that $\gamma = 1$ *and that* $\inf_{Q \in \mathcal{RH}_\infty} \|T_{11} + T_{12} Q T_{21}\|_\infty \stackrel{\Delta}{=} \gamma^* < 1$. *This last assumption guarantees both the existence of suboptimal* $\mathcal{H}_\infty$ *controllers and non-trivial solutions to the mixed* $\mathcal{H}_2/\mathcal{H}_\infty$ *problem.*

3. Problem Solution

Problem ($\mathcal{H}_2/\mathcal{H}_\infty$) is an *infinite-dimensional* optimization problem. In principle, one can attempt to solve this problem following an approach similar to the one in [3]. This entails a double approximation, since the free parameter Q is approximated by a finite impulse response while the constraint is approximated by computing its value at a finite number of frequency points. Thus, there is neither guarantee that the solution obtained be feasible, nor that the actual cost be bounded above by the objective function. Moreover, the computational cost associated with such a scheme may be prohibitively expensive. In this paper we will pursue a different route. Using some results from [8][7], we will show that, as in the simpler SISO case, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem can be solved by considering a sequence of problems, each one requiring the solution of a finite dimensional convex optimization problem. To establish this result we will proceed as follows: i) introduce a modified $\mathcal{H}_2/\mathcal{H}_\infty$ problem, ii) show that the original problem can be solved by solving a sequence of modified problems (Lemma 1); iii) show that an approximate solution (arbitrarily close to the optimum) to each modified problem can be found by solving a truncated problem (Lemma 3); and finally iv) show that solving the truncated problem entails solving a finite dimensional convex optimization problem and a standard $\mathcal{H}_\infty$ problem (Theorem 2).

3.1. A Modified $\mathcal{H}_2/\mathcal{H}_\infty$ Problem

In this section we show that a rational suboptimal solution to $\mathcal{H}_2/\mathcal{H}_\infty$, with cost arbitrarily close to the optimum, can be found by solving a sequence of truncated problems, each one requiring consideration of only a *finite* number of elements of the impulse response of $T_{\zeta_2 w_2}$.

Problem 2 *(Modified* $\mathcal{H}_2/\mathcal{H}_\infty$ *problem:)* Given $V_{ij}(z), T_{ij}(z) \in \mathcal{RH}_{\infty,\delta}$, find

$$\mu_\delta^o = \inf_{Q \in \mathcal{RH}_{\infty,\delta}} \|V_{11} + V_{12} Q V_{21}\|_2 \qquad (\mathcal{H}_2/\mathcal{H}_{\infty,\delta})$$

subject to:

$$\|R + \begin{bmatrix} Q(z) & 0 \\ 0 & 0 \end{bmatrix}\|_{\infty,\delta} \leq 1$$

where $\delta < 1$ and $R^{\sim} \stackrel{\Delta}{=} G \in \mathcal{RH}_{\infty,\delta}$.

Lemma 1 *Consider an increasing sequence* $\delta_i \to 1$. *Let* μ^o *and* μ_i *denote the solution to problems* $\mathcal{H}_2/\mathcal{H}_\infty$ *and* $\mathcal{H}_2/\mathcal{H}_{\infty,\delta_i}$ *respectively. Then the sequence* $\mu_i \to \mu^o$.

Proof: The proof, ommited for space reasons, is similar to the proof of Lemma 1 in [8].

Lemma 2 *For every* $\epsilon > 0$, *there exists* $N(\epsilon, \delta)$ *such that if* $Q \in \mathcal{H}_{\infty,\delta}$ *satisfies the constraint* $\|R(z) + \begin{bmatrix} Q(z) & 0 \\ 0 & 0 \end{bmatrix}\|_{\infty,\delta} \leq \gamma$, *it also satisfies*

$\sum_{i=N}^{\infty} ||V_k||_F^2 \leq \epsilon^2$, *where* V_k *denote the coefficients of the impulse response of* $T_{\zeta_2 w_2} = V_{11} + V_{12}QV_{21}$.

Proof: Since $Q \in \mathcal{H}_{\infty,\delta}$, $T_{\zeta_2 w_2}$ is analytic in $|z| \geq \delta$ and:

$$V_k = \frac{1}{2\pi j}\oint_{|z|=\delta} T_{\zeta_2 w_2}(z) z^{k-1} dz \qquad (6)$$

Hence:

$$\begin{aligned} ||V_k||_F &\leq \sqrt{m_2}\bar{\sigma}(V_k) \\ &\leq \sqrt{m_2}||T_{\zeta_2 w_2}||_{\infty,\delta}\delta^k \\ \textstyle\sum_{i=N}^{\infty} |V_k|_F^2 &\leq m_2 \frac{||T_{\zeta_2 w_2}||_{\infty,\delta}^2 \delta^{2N}}{1-\delta^2} \end{aligned} \qquad (7)$$

Since $||.||_{\infty,\delta}$ is submultiplicative, we have:

$$\begin{aligned} &||T_{\zeta_2 w_2}(z)||_{\infty,\delta} \\ &\leq ||V_{11}||_{\infty,\delta} + ||V_{12}||_{\infty,\delta}||Q||_{\infty,\delta}||V_{21}||_{\infty,\delta} \\ &\leq ||V_{11}||_{\infty,\delta} + ||V_{12}||_{\infty,\delta}||V_{21}||_{\infty,\delta}(1 + ||R||_{\infty,\delta}) \\ &\stackrel{\Delta}{=} K \end{aligned} \qquad (8)$$

The desired result follows by selecting $N \geq N_o = \frac{1}{2}\frac{\log\epsilon^2(1-\delta^2) - m_2 \log K^2}{\log\delta}$ □

Lemma 3 *Consider the following optimization problem:*

$$\min_{Q_i \in R^{m_u \cdot m_y}} ||\underline{v}_1 + \mathcal{V}_{12}\underline{Q}\mathcal{V}_{21}||_F \qquad (\mathcal{H}_2/\mathcal{H}_{\infty,\delta}^{\epsilon})$$

$$\text{s. t. } \left\| G(z) + \begin{bmatrix} Q(z)^{\sim} & 0 \\ 0 & 0 \end{bmatrix} \right\|_{\infty,\delta} \leq 1$$

where:

$$\begin{aligned} \underline{v}_1 &= [V_{11_o}^T \; \dots \; V_{12_{N-1}}^T]^T \\ \mathcal{V}_{12} &= \begin{bmatrix} V_{12_o} & 0 & \dots & 0 \\ V_{12_1} & V_{12_o} & \dots & 0 \\ \vdots & & \ddots & \\ V_{12_{N-1}} & \dots & & V_{12_o} \end{bmatrix} \\ \underline{Q} &= \begin{bmatrix} Q_o & 0 & \dots & 0 \\ Q_1 & Q_o & \dots & 0 \\ \vdots & & \ddots & \\ Q_{N-1} & \dots & & Q_o \end{bmatrix} \\ \mathcal{V}_{21}^T &= \left[V_{21_o}^T \; V_{21_1}^T \; \dots \; V_{21_{N-1}}^T \right]^T \end{aligned} \qquad (9)$$

and where Q_k, V_{ij_k} *denote the* k^{th} *element of the impulse response of* $Q(z), V_{ij}(z)$ *respectively. Let* Q^{ϵ} *and* $T^{\epsilon}_{\zeta_2 w_2}$ *denote the optimal solution and define* $\mu_\delta^\epsilon = ||T^{\epsilon}_{\zeta_2 w_2}||_2$. *Then* $\mu_\delta^o \leq \mu_\delta^\epsilon \leq \mu_\delta^o + \epsilon$.

Proof: The proof, ommited for space reasons follows from Lemma 2 and the definitions of μ_δ^o and μ_δ^ϵ.

By combining the results of Lemmas 1, 2 and 3, the following result is now apparent:

Lemma 4 *Consider an increasing sequence* $\delta_i \to 1$. *Let* μ^o *and* $\mu^\epsilon_{\delta_i}$ *denote the solution to problems* $\mathcal{H}_2/\mathcal{H}_\infty$ *and* $\mathcal{H}_2/\mathcal{H}^\epsilon_{\infty,\delta_i}$ *respectively. Then the sequence* $\mu^\epsilon_{\delta_i}$ *has an accumulation point* $\hat{\mu}_\epsilon$ *such that* $\mu^o \leq \hat{\mu}_\epsilon \leq \mu^o + \epsilon$.

3.2. Handling the $\mathcal{H}_\infty$ Constraint

In the previous section we showed that the $\mathcal{H}_2/\mathcal{H}_\infty$ problem can be solved by solving a sequence of truncated problems. In this section we show that each problem $\mathcal{H}_2/\mathcal{H}^\epsilon_{\infty,\delta}$ can be *exactly* solved by solving a finite dimensional convex optimization problem and an unconstrained $\mathcal{H}_\infty$ problem. To establish this result we recall first a result from [7] establishing a necessary and sufficient condition for the feasibility of the $\mathcal{H}_\infty$ constraint when the first N parameters in the expansion $Q(z) = Q_0 + Q_1 z^{-1} + \dots + Q_{n-1}z^{-(n-1)} + \dots$ are fixed. We begin by considering the following Riccati equations:

$$\begin{aligned} \hat{X} &= \hat{A}\hat{X}\hat{A}^T + B_e B_e^T + \left(\hat{A}\hat{X}C_a^T + B_e D_{er}^T\right) \\ &\quad \left(I - D_{er}D_{er}^T - C_1\hat{X}C_1^T\right)^{-1}\left(C_a\hat{X}\hat{A}^T + D_{er}B_e^T\right) \\ \hat{Y} &= \hat{A}^T\hat{Y}\hat{A} + C_e^T C_e + \left(\hat{A}^T\hat{Y}B_a + C_e^T D_{ec}\right) \\ &\quad \left(I - D_{ec}^T Dec - B_1^T\hat{Y}B_1\right)^{-1}\left(B_a^T\hat{Y}\hat{A} + D_{ec}^T C_a\right) \end{aligned} \qquad (10)$$

From [7], there exist a Q satisfying the *strict* $\mathcal{H}_\infty$ constraint if and only if there exist positive-definite solutions $\hat{X}$ and $\hat{Y}$ to these Riccati equations such that $\rho(\hat{X}\hat{Y}) < 1$. For ease of notation, let $x \stackrel{\Delta}{=} \hat{X}^{1/2}$, $y \stackrel{\Delta}{=} \hat{Y}^{1/2}$.

Theorem 1 *Let* G *have a state-space realization as in (4), and let* $Q^n(z) = \sum_{i=0}^{n-1} Q_i z^{-i}$. *Then there exists a* $Q^n_{tail}(z) \in \mathcal{H}_\infty$ *such that* $\left\| \begin{bmatrix} G_{11} - \sum_{i=0}^{n-1} Q_i^T z^i - z^n Q^n_{tail}(z) & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \right\|_\infty \leq 1$ *if* *and only if* $\overline{\sigma}(W(\mathbf{Q_n})) \leq 1$, *where:*

$$W(\mathbf{Q_n}) = \begin{bmatrix} y\hat{A}^n x & y\hat{A}^{n-1}B_a & \cdots & y\hat{A}B_a & yB_a & y\hat{A}^{n-1}B_b & y\hat{A}^{n-2}B_b & \cdots & y\hat{A}B_b & yB_b \\ C_a\hat{A}^{n-1}x & C_a\hat{A}^{n-2}B_a & \cdots & C_aB_a & D_{aa} & C_a\hat{A}^{n-2}B_b & C_a\hat{A}^{n-3}B_b & \cdots & C_aB_b & D_{ab} \\ C_a\hat{A}^{n-2}x & C_a\hat{A}^{n-3}B_a & \cdots & D_{aa} & 0 & C_a\hat{A}^{n-3}B_b & C_a\hat{A}^{n-4}B_b & \cdots & D_{ab} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_a x & D_{aa} & 0 & \cdots & 0 & D_{ab} & 0 & 0 & \cdots & 0 \\ C_b\hat{A}^{n-1}x & C_b\hat{A}^{n-2}B_a & \cdots & C_bB_a & D_{ba} & C_b\hat{A}^{n-2}B_b & C_b\hat{A}^{n-3}B_b & \cdots & C_bB_b & -Q_0^t \\ C_b\hat{A}^{n-2}x & C_b\hat{A}^{n-3}B_a & \cdots & D_{ba} & 0 & C_b\hat{A}^{n-3}B_b & C_b\hat{A}^{n-4}B_b & \cdots & -Q_0^t & -Q_1^t \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_b x & D_{ba} & 0 & \cdots & 0 & -Q_0^t & -Q_1^t & -Q_2^t & \cdots & -Q_{(n-1)}^t \end{bmatrix}$$

Proof: see [7].

3.3. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Controller Design

Combining Lemma 4 and Theorem 1 yields the main result of the paper:

Theorem 2 *A suboptimal solution to the mixed* $\mathcal{H}_2/\mathcal{H}_{\infty,\delta}$ *control problem, with cost* $\mu_\delta \leq \mu_\delta^\epsilon \leq \mu_\delta + \epsilon$ *is given by* $Q^o = Q_F^o + z^{-N}Q_R^o$ *where* $Q_F^o = \sum_{i=0}^{N-1} Q_i z^{-i}$, $\underline{Q}^o = [Q_o \dots Q_{N-1}]$ *solves the following finite dimensional convex optimization problem:*

$$\underline{Q}^o = \operatorname*{argmin}_{\substack{\underline{Q} \in R^{m_u \cdot N m_y} \\ \|W_1\|_2 \leq 1}} \|\underline{v}_1 + \mathcal{V}_{12}\underline{Q}\mathcal{V}_{21}\|_2$$

and Q_R *solves the approximation problem*

$$Q_R^o(z) = \operatorname*{argmin}_{Q_R \in \mathcal{RH}_{\infty,\delta}} \|T_{11}(z) + T_{12}Q_F^oT_{21}(z) + z^{-N}T_{12}Q_R(z)T_{21}(z)\|_{\infty,\delta} \tag{11}$$

where $N(\epsilon,\delta)$ *is selected according to Lemma 2,* $\underline{v}_1$ *and* $\mathcal{V}_{ij}$ *are defined in (9), and* W *is defined in Theorem 1.*

From Theorem 2 it follows that a suboptimal solution to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem (with cost arbitrarily close to the optimum) can be found using the following iterative algorithm:

1. *Data:* An increasing sequence $\delta_i \rightarrow 1, \epsilon > 0, \nu > 0$.
2. Solve the unconstrained $\mathcal{H}_2$ control problem (using the standard $\mathcal{H}_2$ (LQG) theory). Compute $\|T_{\zeta_\infty w_\infty}\|_\infty$. If $\|T_{\zeta_\infty w_\infty}\|_\infty \leq 1$ stop, else set $i = 1$.
3. For each i, find a suboptimal solution to problem $\mathcal{H}_2/\mathcal{H}_{\infty,\delta}$ proceeding as follows:
   (a) Obtain $T_{ij}(z), V_{ij}(z) \in \mathcal{RH}_{\infty,\delta_i}$, with $T_{12}(z)$ and $T_{21}(z)$ inner and co–inner in $\mathcal{RH}_{\infty,\delta_i}$, respectively. This can be accomplished by using the change of variable $z = \delta_i\tilde{z}$ before performing the factorization (1).
   (b) Compute $N(\epsilon,\delta_i)$ from Lemma 2.
   (c) Find $Q(z)$ using Theorem 2.
4. Compute $\|T_{\zeta_\infty w_\infty}(z)\|_\infty$. If $\|T_{\zeta_\infty w_\infty}(z)\|_\infty \geq \gamma - \nu$ stop, else set $i = i+1$ and go to 3.1.

Remark 3 *From the maximum modulus theorem, it follows that at each stage the algorithm produces a feasible solution to the mixed* $\mathcal{H}_2/\mathcal{H}_\infty$ *control problem, with cost* μ_i *which is an upper bound of the optimal cost* μ^o.

4. A Simple Example

Consider the following plant:

$$A = \begin{bmatrix} 1.1314 & 1.1815 & -.1791 \\ -.9064 & .2005 & .1689 \\ -.5154 & -.3643 & .7966 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -.0715 \\ -.1253 \\ .0104 \end{bmatrix} \quad B_2 = \begin{bmatrix} .0142 & .1967 \\ -.0043 & .0906 \\ .0519 & -.0999 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} -.0631 \\ -.2842 \\ -.1383 \end{bmatrix}$$

$$C_1 = [.1173 \ \ .0853 \ \ -.0379]$$

$$C_2 = \begin{bmatrix} .1612 & -.0574 & -.2380 \\ .2318 & .1363 & -.0082 \end{bmatrix}$$

$$C_3 = [-.0815 \ \ .1149 \ \ -.1224]$$

$$D = \begin{bmatrix} -.0834 & -.0197 & .0897 & -.1230 \\ -.0339 & -.0621 & -.0507 & -.0369 \\ -.1171 & -.0060 & .0297 & .0050 \\ .1279 & -.1227 & .0144 & .0687 \end{bmatrix}. \tag{13}$$

The pure $\mathcal{H}_\infty$ problem (i.e., the minimization of $\|T\|_\infty$) yields $\|T^*\|_\infty = 0.912$ and $\|S\|_2 = 1.037$. On the other hand, the minimization of the $\mathcal{H}_2$ part of the problem gives $\|T\|_\infty = 2.243$ and

$||S^*||_2 = 0.3722$. Finally, minimizing $||S||_2$ subject to $||T||_\infty \leq 1$ using the proposed synthesis method yields $||S||_2 = 0.5$. Fig. 4 shows the frequency response plots of the controllers obtained for increasing value of N. Note that the low frequency behavior of the controller is achieved with relatively short horizon lengths, and subsequently meaningful changes occur only in the high frequency range. For $N = 50$, the resulting controller would be of

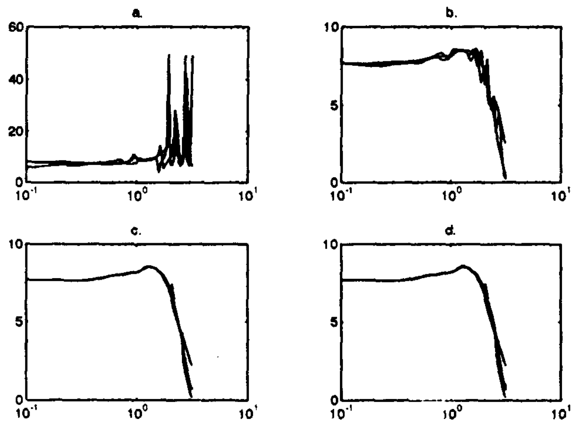

Figure 2: Controller frequency responses: a. $n = 5$, 10, 15. b. $n = 20$, 25 30. c. $n = 35$, 40 50. d. $n = 50$ and reduced order.

order 52 to control a plant with two states, however, by using balanced truncations we computed an eleventh order controller which satisfied the $\mathcal{H}_\infty$ constraint and achieved virtually the same $\mathcal{H}_2$-norm. This controller was further model reduced to third order again using balanced truncation, at the expense of a slight violation of the $\mathcal{H}_\infty$ constraint and achieving essentially the same $\mathcal{H}_2$ norm. This controller has a transfer function:

$$k_r(z) = \frac{5.0473z^3 + 9.1395z^2 + 4.6276z + 1.2564}{z^3 + .9548z^2 + .4584z + .1811}. \tag{14}$$

5. Conclusions

In this paper we provide a sub-optimal solution to discrete-time MIMO mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems. Unlike previous approaches, our method yields a global minimum of the actual $\mathcal{H}_2$ cost rather than of an upper bound and it is not limited to cases where either the disturbances or the regulated outputs coincide.

Perhaps the most severe limitation of the proposed method is that may result in very large order controllers (roughly N), necessitating some type of model reduction. Note however that this disadvantage is shared by some widely used design methods, such as μ-synthesis or l_1 optimal control theory, that will also produce controllers with no guaranteed complexity bound. Application of some well established methods in order reduction (noteworthy, weighted balanced truncation) usually succeed in producing controllers of manageable order. Research is currently under way addressing the issue of model reduction in the presence of mixed performance objectives.

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