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An (Almost) Exact Solution to General SISO Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Problems via Convex Optimization

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Abstract

The mixed $(\mathcal{H}_2/\mathcal{H}_\infty)$ control problem can be motivated as a nominal *LQG* optimal control problem, subject to robust stability constraints, expressed in the form of an $\mathcal{H}_\infty$ norm bound. A related modified problem consisting on minimizing an *upper bound* of the $\mathcal{H}_2$ cost subject to $\mathcal{H}_\infty$ constraints was introduced in [1]. Although there presently exist efficient methods to solve this modified problem, the original problem remains, to a large extent, still open. In this paper we propose a method for solving general discrete–time SISO $(\mathcal{H}_2/\mathcal{H}_\infty)$ problems. This method involves solving a sequence of problems, each one consisting of a finite–dimensional convex optimization and an unconstrained Nehari approximation problem

1. Introduction

During the last decade, a large research effort has been devoted to the problem of designing robust controllers, capable of guaranteeing stability in the face of plant uncertainty. As a result, a powerful $\mathcal{H}_\infty$ framework has been developed, addressing the issue of robust stability in the presence of norm–bounded plant perturbations. Since its introduction, the original formulation of Zames [2] has been substantially simplified, resulting in efficient computational schemes for finding solutions. Of particular importance is [3] where a state–space approach is developed and an efficient procedure is given to compute suboptimal $\mathcal{H}_\infty$ controllers. In general, these controllers are preferred, since optimal $\mathcal{H}_\infty$ controllers may exhibit some undesirable properties. Since suboptimal controllers are seldom unique, the extra degrees of freedom available can then be used to optimize some performance measure. This leads naturally to a robust performance problem: design a controller guaranteeing a desired level of performance in the face of plant uncertainty. However, in spite of a large research effort [4], this problem has not completely been solved.

Alternatively, the extra degrees of freedom can be used to solve a problem of the form *nominal performance with robust stability*. In this case the controller yields a desired performance level for the nominal system while guaranteeing stability for all possible plant perturbations. A problem of this form that has been the object of much attention lately is the mixed $(\mathcal{H}_2/\mathcal{H}_\infty)$ control problem.

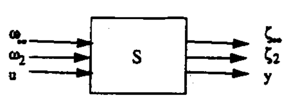

Figure 1: The Generalized Plant

Consider the system represented by the block diagram 1, where S represents the system to be controlled; the scalar signals w_∞ (a bounded power signal), w_2 (white noise) and u represent exogenous disturbances and the control action respectively; and ζ_∞, ζ_2 and y represent the regulated outputs and the measurements respectively. Then, the mixed $(\mathcal{H}_2/\mathcal{H}_\infty)$ control problem can be stated as: Given the nominal system (S), find an internally stabilizing controller

$$u(z) = K(z)y(z) \qquad (C)$$

such that the power semi–norm of the performance output $\|\zeta_2\|_{\mathcal{P}}$ due to w_2 is minimized subject to the specification:

$$\sup_{w_\infty \in \mathcal{P},\, \|w_\infty\| \leq 1} \|\zeta_\infty\|_{\mathcal{P}} = \|T_{\zeta_\infty w_\infty}(z)\|_\infty \leq \gamma \qquad (P)$$

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Different versions of this problem have been studied recently. Bernstein and Haddad [1] considered the case where $w_2 = w_\infty$ and obtained necessary conditions for solving the *modified* problem of minimizing an *upper bound* of $\|T_{w_2\zeta_2}\|_2$, subject to the $\mathcal{H}_\infty$ constraint. In [5] and [6] the dual problem of minimizing this upper bound for the case $w_2 \neq w_\infty$, $\zeta_2 = \zeta_\infty$ was considered and sufficient conditions for optimality where given. Finally, in [7] these conditions where shown to be necessary and sufficient. However these conditions involve solving several coupled Riccati equations, and at this point there are no effective procedures for achieving this. In [8] Khargonekar and Rotea (see also [9] for the discrete–time version) showed that the modified problem can be cast into the format of a constrained convex optimization problem over a bounded set of matrices and solved using non–differentiable optimization techniques.

The approaches mentioned above provide a solution to the *modified* problem. However, at this time there is no information regarding the gap between the upper bound minimized in the modified problem and the true $\mathcal{H}_2$ cost. Very little work has been done concerning the original problem, which remains, to a large extent, still open. In [10] Rotea and Khargonekar addressed a *simultaneous* $(\mathcal{H}_2/\mathcal{H}_\infty)$ state–feedback control problem and showed that a solution to this problem, when it exists, also solves the mixed $(\mathcal{H}_2/\mathcal{H}_\infty)$ problem. Although this provides some insight into the structure of the problem, there are cases (most notably the case where $B_1 = B_2$) where the simultaneous problem provides little help in solving the original problem. Recently, mixed $(\mathcal{H}_2/\mathcal{H}_\infty)$ control using fixed–order controllers was analyzed using a Lagrange multipliers based approach and necessary conditions for optimality were obtained [11]. However, these conditions involve solving coupled non–linear matrix equations and finding the neutrally stable solution to a Lyapunov equation, which leads to numerical difficulties. Moreover, in [10] it was shown that even in the state–feedback case, the optimal controller must be dynamic, and it is conjectured that in the general case it may have higher order than the plant. This makes a fixed order appproach less attractive, since there is little a priori information on the order of the optimal controller.

In this paper we propose a solution to general discrete–time SISO mixed $(\mathcal{H}_2/\mathcal{H}_\infty)$ problems. Our approach resembles that of Boyd et. al. [12] in the sense that we use the Youla parametrization to cast the problem into a semi–infinite convex optimization form [13]. However, rather than approximately solving this problem by discretizing the constraints, we follow an approach in the spirit of [14] and [15] to show that the problem can be decoupled into a finite dimensional constrained optimization followed by the solution to an unconstrained Nehari approximation problem.

The paper is organized as follows: In section II we introduce the notation to be used and some preliminary results. Section III contains the proposed solution method. The main result of the session shows that the mixed $(\mathcal{H}_2/\mathcal{H}_\infty)$ problem can be solved by solving a sequence of modified problems, each one requiring the solution of a *finite dimensional* convex, constrained optimization problem and an unconstrained Nehari approximation problem. In section IV we present a simple design example. Finally, in section V, we summarize our results and we indicate directions for future research.

2. Preliminaries

2.1 Notation

By l_1 we denote the space of real sequences $\{q_i\}$, equipped with the norm $\|q\|_1 = \sum_{k=0}^{\infty} |q_k| < \infty$. Given a sequence $q \in l_1$ we will denote its Z–transform by $Q(z)$. $\mathcal{P}$ denotes the space of bounded power signals equipped with the seminorm: $\|u\|_{\mathcal{P}}^2 = \lim_{k\to\infty} \frac{1}{2k} \sum_{l=-k}^{l=k} \|u_l\|^2$

$\mathcal{L}_\infty$ denotes the Lebesgue space of complex valued transfer matrices which are essentially bounded on the unit circle with norm

$\|T(z)\|_\infty \triangleq \sup_{|z|=1} \sigma_{max}(T(z))$. $\mathcal{H}_\infty$ ($\mathcal{H}_\infty^-$) denotes the set of stable (antistable) complex matrices $G(z) \in \mathcal{L}_\infty$, i.e. analytic in $|z| \geq 1$ ($|z| \leq 1$). $\mathcal{H}_2$ denotes the space of complex matrices square integrable in the unit circle and analytic in $|z| > 1$, equipped with the norm:

$$\|G\|_2^2 = \frac{1}{2\pi} \oint_{|z|=1} Trace\{G(z)'G(z)\} z dz$$

where $'$ indicates transpose conjugate. The prefix $\mathcal{R}$ denotes real rational transfer matrices. Given $R \in \mathcal{L}_\infty$, $\Gamma_H(R)$ denotes its maximum Hankel singular value. Throughout the paper we will use packed notation to represent state–space realizations, i.e.

$$G(z) = C(zI - A)^{-1}B + D \triangleq \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

Given two transfer matrices $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ and Q with appropriate dimensions, the lower *linear fractional transformation* is defined as:

$$\mathcal{F}_l(T, Q) \triangleq T_{11} + T_{12}Q(I - T_{22}Q)^{-1}T_{21}$$

Finally, for a transfer matrix $G(z)$, $G^{\sim} \triangleq G^T(\frac{1}{z})$.

2.2 Problem Transformation

Assume that the system S has the following state–space realization (where without loss of generality we assume that all weighting factors have been absorbed into the plant):

$$\left(\begin{array}{c|ccc} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ C_3 & D_{31} & D_{32} & D_{33} \end{array}\right) \qquad (S)$$

where D_{13} has full column rank, D_{31} has full row rank, and where the pairs (A, B_3) and (C_3, A) are stabilizable and detectable respectively. It is well known (see for instance [4]) that the set of all internally stabilizing controllers can be parametrized in terms of a free parameter $Q \in \mathcal{H}_\infty$ as:

$$K = \mathcal{F}_l(J, Q) \qquad (1)$$

where J has the following state–space realization:

$$\left(\begin{array}{c|cc} A + B_3F + LC_3 + LD_{33}F & -L & B_3 + LD_{33} \\ \hline F & 0 & I \\ -(C_3 + D_{33}F) & I & -D_{33} \end{array}\right) \qquad (J)$$

where F and L are selected such that $A + B_3F$ and $A + LC_3$ are stable. By using this parametrization, the closed–loop transfer functions $T_{\zeta_\infty w_\infty}$ and $T_{\zeta_2 w_2}$ can be written as:

$$\begin{aligned} T_{\zeta_\infty w_\infty} &= \mathcal{F}_l(T_\infty, Q) = T_{11}^\infty + T_{12}^\infty Q T_{21}^\infty \\ T_{\zeta_2 w_2} &= \mathcal{F}_l(T, Q) = T_{11} + T_{12}QT_{21} \end{aligned} \qquad (2)$$

where $T_i, T_i^\infty \in \mathcal{RH}_\infty$ and where T and T_∞ have the following state–space realizations:

$$T^\infty = \left(\begin{array}{cc|cc} A_F & -B_3F & B_1 & B_3 \\ 0 & A_L & B_1 + LD_{31} & 0 \\ \hline C_1 + D_{13}F & -D_{13}F & D_{11} & D_{13} \\ 0 & C_3 & D_{31} & 0 \end{array}\right)$$

$$T = \left(\begin{array}{cc|cc} A_F & -B_3F & B_2 & B_3 \\ 0 & A_L & B_2 + LD_{32} & 0 \\ \hline C_2 + D_{23}F & -D_{23}F & D_{22} & D_{23} \\ 0 & C_3 & D_{32} & 0 \end{array}\right) \qquad (3)$$

$$A_F = A + B_3F$$
$$A_L = A + LC_3$$

Moreover (see [15]), it is possible to select F and L in such a way that $T_{12}^\infty(z)$ and $T_{21}^\infty(z)$ are inner and co–inner respectively (i.e. $T_{12}^{\infty\sim}T_{12}^\infty = I$, $T_{21}^\infty T_{21}^{\infty\sim} = I$).

Remark 1: For the SISO case, equation (2) reduces to:

$$\begin{aligned} T_{\zeta_\infty w_\infty}(z) &= T_1^\infty(z) + T_2^\infty(z)Q(z) \\ T_{\zeta_2 w_2}(z) &= T_1(z) + T_2(z)Q(z) \end{aligned} \qquad (4)$$

where T_i, T_i^∞, Q are stable transfer functions and where T_2^∞ is inner. Since $\|.\|_\infty$ is invariant under multiplication by an inner function, we have:

$$\|T_{\zeta_\infty w_\infty}\|_\infty = \|T_1^\infty + T_2^\infty Q\|_\infty = \|R + Q\|_\infty \qquad (5)$$

where $R(z) \triangleq T_1^\infty(z)T_2^{\infty\sim}(z)$ has all its poles outside the unit disk. A state–space realization of R in terms of the state–space realization of (S) is given in [15].

By using this parametrization the mixed ($\mathcal{H}_2/\mathcal{H}_\infty$) problem can be now precisely stated as solving:

$$\mu^o = \inf_{Q \in \mathcal{H}_\infty} \|T_{\zeta_2 w_2}\|_2 = \inf_{q_i \in l^1} \left(\sum_{i=0}^{\infty} |t_i|^2\right)^{\frac{1}{2}} \qquad (\mathcal{H}_2/\mathcal{H}_\infty)$$

subject to:

$$\|T_1^\infty(z) + T_2^\infty(z)Q(z)\|_\infty \leq \gamma \qquad (6)$$

where $\{t_i\}$ and $\{q_i\}$ are the coefficients of the impulse responses of $T_{\zeta_2 w_2}$ and Q respectively.

3. Problem Solution

In this section we show that the mixed ($\mathcal{H}_2/\mathcal{H}_\infty$) problem can be solved by solving a sequence of problems, each one requiring the solution of a finite dimensional convex optimization problem and an unconstrained Nehari extension problem.

3.1 A Modified ($\mathcal{H}_2/\mathcal{H}_\infty$) Problem

Since all the solutions to a suboptimal Nehari extension problem of the form $\|R + Q\|_\infty \leq \gamma$ can be parametrized in terms of a free parameter $W(z) \in \mathcal{RH}_\infty, \|W\|_\infty \leq \gamma^{-1}$ problem ($\mathcal{H}_2/\mathcal{H}_\infty$) can be thought of as an optimization problem inside the origin centered γ^{-1}–ball. However, even though the space $\mathcal{H}_\infty$ is complete, it is easily seen that the γ–ball is not compact. Thus a minimizing solution may not exist. Motivated by this difficulty, we introduce the following *modified* mixed ($\mathcal{H}_2/\mathcal{H}_\infty$) problem. Let $\mathcal{H}_\delta = \{Q(z) \in \mathcal{H}_\infty : Q(z) \text{ analytic in} |z| \geq \delta\}$ and define the ($\mathcal{H}_2/\mathcal{H}_\delta$) problem as follows: Given $T_1(z), T_2(z), T_1^\infty(z), T_2^\infty(z) \in \mathcal{RH}_\delta$, find

$$\mu_\delta^o = \min_{Q \in \mathcal{H}_\delta} \|T_{\zeta_2 w_2}\|_2 \qquad (\mathcal{H}_2/\mathcal{H}_\delta)$$

subject to:

$$\|T_1^\infty(z) + T_2^\infty(z)Q(z)\|_\delta \leq \gamma$$

where $\delta < 1$ and $\|Q\|_\delta \triangleq \sup_{|z|=\delta} |Q(z)|$. In section 3.2 we will show that ($\mathcal{H}_2/\mathcal{H}_\delta$), if feasible, always has a minimizing solution. Moreover, this optimal solution is rational (i.e. $Q \in \mathcal{RH}_\delta$).

Remark 2: From the maximum modulus theorem, it follows that any solution Q to ($\mathcal{H}_2/\mathcal{H}_\delta$) is an admissible solution for ($\mathcal{H}_2/\mathcal{H}_\infty$). It follows that μ_δ^o is an upper bound for μ^o.

Remark 3: Problem ($\mathcal{H}_2/\mathcal{H}_\delta$) can be thought as solving problem ($\mathcal{H}_2/\mathcal{H}_\infty$) with the additional constraint that all the poles of the closed–loop system must be inside the disk of radius δ. A parametrization of all achievable closed–loop transfer functions, such that T, T^∞ satisfy this additional constraint can be obtained from (1) by simply changing the stability region from the unit–disk to the δ–disk using the transformation $z = \delta\hat{z}$ before performing the factorization. Furthermore, by combining this transformation with the inner–coinner factorization, the resulting $T_2^\infty(z)$ satisfies $T_2^\infty(\delta z)T_2^\infty(\frac{1}{\delta z}) = 1$.

Next we show that a suboptimal solution to ($\mathcal{H}_2/\mathcal{H}_\infty$), with cost arbitrarily close to the optimum, can be found by solving a sequence of truncated problems, each one requiring consideration of only a *finite* number of elements of the impulse response of $T_{\zeta_2 w_2}$. To establish this result we will show that: i) $(\mathcal{H}_2/\mathcal{H}_\infty)$ can be solved by considering a sequence of modified problems $(\mathcal{H}_2/\mathcal{H}_\delta)$. ii) Given $\epsilon > 0$, a suboptimal solution to $(\mathcal{H}_2/\mathcal{H}_\delta)$ with cost no greater than $\mu^o_\delta + \epsilon$ can be found by solving a truncated problem.

• **Lemma 1:** Consider an increasing sequence $\delta_i \to 1$. Let μ^o and μ_i denote the solution to problems $(\mathcal{H}_2/\mathcal{H}_\infty)$ and $(\mathcal{H}_2/\mathcal{H}_{\delta_i})$ respectively and assume that $\Gamma_H(R) < \gamma$. Then the sequence $\mu_i \to \mu^o$.

Proof: The proof, omitted for space reasons, follows from the maximum modulus theorem and continuity arguments.

Next we show show that, given $\epsilon > 0$, a suboptimal solution to $(\mathcal{H}_2/\mathcal{H}_\delta)$, with cost μ^ϵ_δ such that $\mu^o_\delta \le \mu^\epsilon_\delta \le \mu^o_\delta + \epsilon$ can be found by solving a truncated problem.

• **Lemma 2:** Let $\epsilon > 0$ be given. Then, there exists $N(\epsilon, \delta)$ such that if $Q \in \mathcal{H}_\delta$ satisfies the constraint $\|R+Q\|_\delta \le \gamma$ then it also satisfies $\sum_{i=N}^{\infty} |t_k|^2 \le \epsilon^2$, where t_k denote the coefficients of the impulse response of $T_{\zeta_2 w_2} = T_1 + T_2 Q$.

Proof: Since $Q \in \mathcal{H}_\delta$, $T_{\zeta_2 w_2}$ is analytic in $|z| \ge \delta$ and:

$$t_k = \frac{1}{2\pi j}\oint_{|z|=\delta} T_{\zeta_2 w_2}(z) z^{k-1} dz \tag{7}$$

Hence

$$|t_k| \le \|T_{\zeta_2 w_2}\|_\delta \delta^k$$
$$\sum_{i=N}^{\infty} |t_k|^2 \le \frac{\|T_{\zeta_2 w_2}\|^2_\delta \delta^{2N}}{1-\delta^2} \tag{8}$$

If Q satisfies $\|R+Q\|_\delta \le \gamma$, since $\|.\|_\delta$ is submultiplicative, we have:

$$\begin{aligned} \|T_{\zeta_2 w_2}(z)\|_\delta &\le \|T_1\|_\delta + \|T_2\|_\delta \|Q\|_\delta \\ &\le \|T_1\|_\delta + \|T_2\|_\delta(\gamma + \|R\|_\delta) \stackrel{\Delta}{=} K \end{aligned} \tag{9}$$

The desired result follows by selecting $N \ge N_o = \frac{1}{2}\frac{\log\epsilon^2(1-\delta^2)-\log K^2}{\log\delta}$ ◇

• **Lemma 3:** Consider the following optimization problem:

$$\min_{Q \in \mathcal{H}_\delta} \left(\sum_{i=0}^{N-1} |t_i|^2\right)^{\frac{1}{2}} = \|\underline{t}_1 + \tau \underline{q}\|_2^2 \qquad (\mathcal{H}_2/\mathcal{H}^\epsilon_\delta)$$

subject to:

$$\|R+Q\|_\delta \le \gamma$$

where:

$$\underline{t}_1 \stackrel{\Delta}{=} (t_{1o} \; \dots \; t_{1N-1})'$$
$$\tau = \begin{pmatrix} t_{2o} & 0 & \dots & 0 \\ t_{21} & t_{20} & \dots & 0 \\ \vdots & & \ddots & \\ t_{2N-1} & \dots & & t_{2o} \end{pmatrix} \tag{10}$$
$$\underline{q}^o \stackrel{\Delta}{=} (q_o \; \dots \; q_{N-1})'$$

and where q_k, t_{k_i} denote the k^{th} element of the impulse response of $Q(z), T_i(z)$ respectively. Let Q^* and $T^*_{\zeta_2 w_2}$ denote the optimal solution and define $\mu^\epsilon_\delta = \|T^*_{\zeta_2 w_2}\|_2$. Then $\mu^o_\delta \le \mu^\epsilon_\delta \le \mu^o_\delta + \epsilon$

Proof: $\mu^o_\delta \le \mu^\epsilon_\delta$ is immediate from the definition of μ^o_δ. Denote by $T^\epsilon_{\zeta_2 w_2}$ and $T^\delta_{\zeta_2 w_2}$ the solution to problems $(\mathcal{H}_2/\mathcal{H}^\epsilon_\delta)$ and $(\mathcal{H}_2/\mathcal{H}_\delta)$ respectively and let t^ϵ_i, t^δ_i be the corresponding impulse responses. Then:

$$\begin{aligned} (\mu^\epsilon_\delta)^2 = \|T^\epsilon_{\zeta_2 w_2}\|^2_2 &= \sum_{i=0}^{\infty} |t^\epsilon_i|^2 = \sum_{i=0}^{N-1} |t^\epsilon_i|^2 + \sum_{i=N}^{\infty} |t^\epsilon_i|^2 \\ &\le \sum_{i=0}^{N-1} |t^\delta_i|^2 + \epsilon^2 \le \sum_{i=0}^{\infty} |t^\delta_i|^2 + \epsilon^2 = (\mu^o_\delta)^2 + \epsilon^2 \le (\mu^o_\delta + \epsilon)^2 \end{aligned} \quad ◇$$

By combining the results of Lemmas 1, 2 and 3, the following result is now apparent:

• **Lemma 4:** Consider an increasing sequence $\delta_i \to 1$. Let μ^o and $\mu^\epsilon_{\delta_i}$ denote the solution to problems $(\mathcal{H}_2/\mathcal{H}_\infty)$ and $(\mathcal{H}_2/\mathcal{H}^\epsilon_{\delta_i})$ respectively. Then the sequence $\mu^\epsilon_{\delta_i}$ has an accumulation point $\hat{\mu}_\epsilon$ such that $\mu^o \le \hat{\mu}_\epsilon \le \mu^o + \epsilon$.

3.2 The $\mathcal{H}_\infty$ Performance Constraint

In the last section we showed that $(\mathcal{H}_2/\mathcal{H}_\infty)$ can be solved by solving a sequence of truncated problems. In principle these problems have the form of a semi-infinite optimization problem, and can be approximately solved by discretizing the unit-circle and applying outer approximation methods (see [13]). In this section we show that each problem $(\mathcal{H}_2/\mathcal{H}^\epsilon_\delta)$ can be *exactly* solved by solving a finite dimensional convex optimization problem and an unconstrained Nehari approximation problem. Moreover, since the solution to this Nehari approximation problem is rational, it follows that the solution to $(\mathcal{H}_2/\mathcal{H}^\epsilon_\delta)$ is also rational. The key to establish this result is to note that: i) the objective function of the truncated problem involves only the first N terms of the impulse response of Q and ii) If the first N terms of the impulse response of Q are fixed, the existence of Q such that $\|R+Q\|_\delta \le \gamma$ is equivalent to a finite dimensional convex constraint.

• **Theorem 1:** Let $R \stackrel{\Delta}{=} \left(\begin{array}{c|c} A_R & b_R \\ \hline c_R & d_R \end{array}\right) \in \mathcal{RH}^-_\infty$,with McMillan degree n, and $Q_F = \sum_{i=0}^{N-1} q_i z^{-i}$ be given. Then there exist $Q_R \in \mathcal{RH}_\infty$, such that $\|R + Q_F + z^{-N} Q_R\|_{\mathcal{H}_\infty} \le \gamma$, iff $\|\mathcal{Q}\|_2 \le \gamma$ where $\mathcal{Q}$, a symmetric matrix affine in the coefficients of Q_F, has the following form:

$$\mathcal{Q} = W^{\frac{1}{2}} \begin{pmatrix} I & 0 \\ 0 & \mathcal{H}' \end{pmatrix} L_c^{\frac{1}{2}}$$
$$Lc = \begin{pmatrix} L^C_{11} & L^C_{12} \\ L^{C\prime}_{12} & L^C_{22} \end{pmatrix}$$
$$L^C_{11} = L^C_o$$
$$L^C_{12} = -\left((A'_R)^{N-1} c'_R \quad (A'_R)^{N-2} c'_R \dots \quad c'_R\right)$$
$$L^C_{22} = I_N$$
$$W'^{\frac{1}{2}} W^{\frac{1}{2}} = \begin{pmatrix} L^0_o & \mathcal{A} \\ \mathcal{A}' & I \end{pmatrix}$$
$$\mathcal{A} = \left(A_R^{-N} b_R \quad A_R^{-(N-1)} b_R \dots A_R^{-1} b_R\right)$$
$$\mathcal{H} = \begin{pmatrix} h_N & h_{N-1} & \dots & \dots & h_1 \\ & h_N & h_{N-1} & \dots & h_2 \\ & & \ddots & & \\ & & & h_N & h_{N-1} \\ & & & & h_N \end{pmatrix}$$
$$h_i = q_{N-i} + b'_R (A'_R)^{N-1-i} c'_R \qquad 1 \le i \le N-1$$
$$h_N = q_0 + d_R$$

where L^0_o and L^C_o are the solutions to the following Lyapunov equations:

$$A_R L^0_o A'_R - L^0_o = b_R b'_R$$
$$A'_R L^C_o A_R - L^C_o = (A'_R)^N c'_R c_R (A_R)^N$$

Proof: See [15].

Combining Lemma 3 and Theorem 1 yields the main result of this section:

• **Theorem 2:** A suboptimal solution to $(\mathcal{H}_2/\mathcal{H}_\delta)$, with cost $\mu_\delta \le \mu^\epsilon_\delta \le \mu_\delta + \epsilon$ is given by $Q^o = Q^o_F + z^{-N} Q^o_R$ where $Q^o_F = \sum_{i=0}^{N-1} q_i z^{-i}$, $\underline{q}^o = (q_o \dots q_{N-1})'$ solves the following finite dimensional convex optimization problem:

$$\underline{q}^o = \underset{\substack{\underline{q} \in R^N \\ \|\mathcal{Q}\|_2 \le \gamma}}{\operatorname{argmin}} \|\underline{t}_1 + \tau \underline{q}\|_2$$

and Q_R solves the unconstrained Nehari approximation problem

$$Q^o_R(\hat{z}) = \underset{Q_R \in \mathcal{RH}_\infty}{\operatorname{argmin}} \|R(\hat{z}) + Q^o_F + \hat{z}^{-N} Q_R(\hat{z})\|_\infty$$

where R is defined in (5), $\underline{t}_1, \tau$ are defined in (10), N is selected according to Lemma 2, and $z = \delta\hat{z}$

3.3 Synthesis Algorithm

Combining Theorem 2 and Lemma 4, it follows that a suboptimal solution to $(\mathcal{H}_2/\mathcal{H}_\infty)$, with cost arbitrarily close to the optimum, can be found using the following iterative algorithm.

0) *Data:* An increasing sequence $\delta_i \to 1, \epsilon > 0, \nu > 0$.

1) Solve the unconstrained $\mathcal{H}_2$ problem (using the standard $\mathcal{H}_2$ theory) . Compute $\|T_{\zeta_\infty w_\infty}\|_\infty$. If $\|T_{\zeta_\infty w_\infty}\|_\infty \leq \gamma$ stop, else set $i = 1$.

2) For each i, find a suboptimal solution to problem $(\mathcal{H}_2/\mathcal{H}_{\delta_i})$ proceeding as follows:

   2.1) Let $z = \delta_i \hat{z}$ and consider the system $S(\hat{z})$

   2.2) Perform the factorization (2) to obtain $T_i(\hat{z}), T_i^\infty(\hat{z})$.

   2.3) Compute N from Lemma 2.

   2.4) Find $\hat{Q}(\hat{z})$ using Theorem 2.

3) Let $Q = \hat{Q}(\frac{z}{\delta_i}), K = F_l(J, Q)$. Compute $\|T_{\zeta_\infty w_\infty}(z)\|_\infty$. If $\|T_{\zeta_\infty w_\infty}(z)\|_\infty \geq \gamma - \nu$ stop, else set $i = i + 1$ and go to 2.

Remark 4: At each stage the algorithm produces a feasible solution to $(\mathcal{H}_2/\mathcal{H}_\infty)$, with cost μ_i which is an upper bound of the optimal cost μ^o.

4. A Simple Example

Consider the simple system shown in figure 2, consisting of two unity masses coupled by a spring with constant $0.5 \leq k \leq 2$ but otherwise unknown. A control force acts on body 1 and the position of body 2 is measured, resulting in a non–colocated sensor actuator problem that embodies many of the pathologies and challenges present in realistic problems, such as control of complex aircraft and large space structures. This system has been used as a benchmark during the last few years at the American Control Conference [16–17] to highlight the issues and trade–offs involved in robust control design.

Assume that it is desired to design an internally stabilizing controller subject to the following performance specifications: i) the closed-loop system must be stable for all possible values of the uncertain parameter $k \in [0.5, 2]$. ii) the energy of the control action u in response to a white noise disturbance acting on m_2 should be minimized.

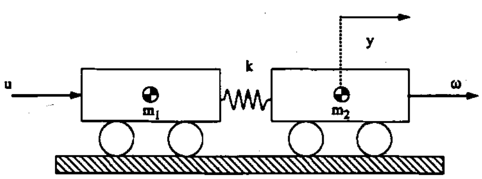

Figure 2: The ACC Robust Control Benchmark Problem.

In order to fit the problem into the $\mathcal{H}_\infty$ framework, the uncertain spring constant k is modeled as $k = k_o + \Delta$ (with $k_o = 1.25$ and $|\Delta| \leq 0.75$) and, following a standard procedure [18], Δ is "pulled out" of the system. The problem can be stated now as the problem of minimizing $\|T_{yw}\|_2$ over the set of all internally stabilizing controllers, subject to the constraint $\|T_{\zeta v}\|_\infty \leq \frac{4}{3}$. The system, with the uncertainty "pulled out", can be represented by the following state space realization:

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_o & k_o & 0 & 0 \\ k_o & -k_o & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In order to fit the problem into our framework, the system was discretized using sample and hold elements at the inputs and outputs, with a sampling time of 0.1 seconds. Finally, to remove the ill-conditioning caused by the poles on the unit circle, a bilinear transformation was used, constraining the poles of the closed–loop system to lie inside the $|z| \leq 0.95$ disk (i.e. $\delta = 0.95$) and the proposed design procedure was used with $\|T_{\zeta v}\|_\delta \leq 1.6$ and $N = 100$, resulting in a controller with 205 states.

Figure 3 shows the control action in response to an impulse disturbance acting on m_2 for the optimal $\mathcal{H}_\infty$ central controller, the optimal $\mathcal{H}_2$ and the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controllers, with the corresponding bode plots of $T_{\zeta v}$ shown in figure 4. These results are summarized in Table 1.

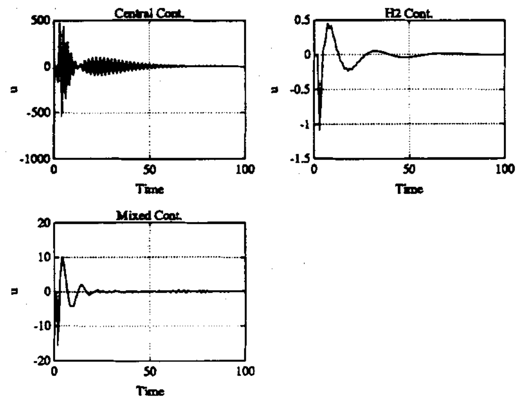

Figure 3: T_{wu} Impulse Response for the $\mathcal{H}_\infty$, $\mathcal{H}_2$ and $\mathcal{H}_2/\mathcal{H}_\infty$ Cont.

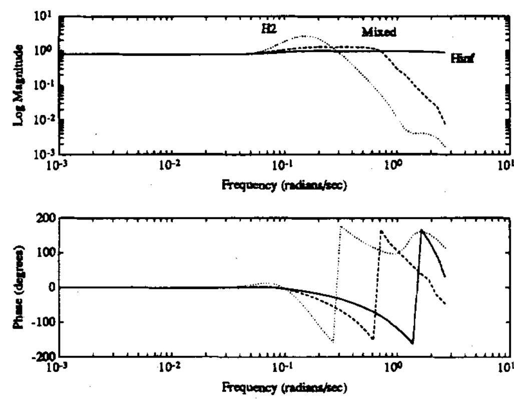

Figure 4: $T_{\zeta v}$ Frequency Response for the $\mathcal{H}_\infty$, $\mathcal{H}_2$ and $\mathcal{H}_2/\mathcal{H}_\infty$ Cont.

	$\|T_{\zeta v}\|_\infty$	$\|T_{uw}\|_2$
$\mathcal{H}_2$	2.604	1.5760
$\mathcal{H}_\infty$	0.9977	1085.2
$\mathcal{H}_2/\mathcal{H}_\infty$	1.292	22.6493

Table 1. $\|T_{\zeta v}\|_\infty$ and $\|T_{uw}\|_2$ for the example

Note that the actual value $\|T_{\zeta v}\|_\infty$ obtained with the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller is 1.29. This is due to the fact that $\|T_{\zeta v}\|_\delta$ is an upper bound of $\|T_{\zeta v}\|_\infty$.

Table 2 shows a comparison between the optimal mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller and several reduced order controllers. It is interesting to notice that the controller can be reduced to 10^{th} order with virtually no performance loss. Further reduction to a 3^{rd} order controller only entails about 10% increase in the $\mathcal{H}_2$ cost. These results seem to support the conjecture of [11] that the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem results in controllers having higher dimension than the plant.

	$\|T_{\zeta v}\|_\infty$	$\|T_{uw}\|_2$
$\mathcal{H}_2/\mathcal{H}_\infty$	1.292	22.6493
10 ord.	1.281	22.8842
3 ord.	1.292	24.8594

Table 2. $\|T_{\zeta v}\|_\infty$ and $\|T_{uw}\|_2$ for reduced order controllers

5. Conclusions

In this paper we provide a sub–optimal solution to discrete–time mixed ($\mathcal{H}_2/\mathcal{H}_\infty$) problems. Unlike previous approaches, our method yields a global minimum of the actual $\mathcal{H}_2$ cost rather than of an upper bound and it is not limited to cases where either the disturbances or the regulated outputs coincide. Although here we considered only the simpler case of a one–block $\mathcal{H}_\infty$ problems, we anticipate that the results will extend naturally to the 4–block case.

Perhaps the most severe limitation of the proposed method is that may result in very large order controllers (roughly $2N$), necessitating some type of model reduction. Hence, at this time, the proposed approach provides an analysis tool to establish the limits of performance of the plant, rather than a practical design tool. The example of section 4 suggests that substantial order reduction can be accomplished without performance degradation. Research is currently under way addressing this issue.

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Appendix: Some Numerical Considerations

In this appendix we give an alternative for computing Q. Since this alternative expression does not involve increasing powers of A_R it is preferable in cases where N is large or A_R has large eigenvalues. From [15] it can be shown that:

$$
\begin{aligned}
L_c &= \begin{pmatrix} L_o^c & Y \\ Y' & I_N \end{pmatrix} = \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} \begin{pmatrix} W_{oR} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ Y' & I \end{pmatrix} \\
L_o &= \begin{pmatrix} I & \mathcal{A} \\ 0 & \mathcal{H} \end{pmatrix} \begin{pmatrix} A_R^{-N} L_o^0 A_R'^{N} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ \mathcal{A}' & \mathcal{H}' \end{pmatrix} \\
&= \begin{pmatrix} A_R^{-N} & \mathcal{A} \\ 0 & \mathcal{H} \end{pmatrix} \begin{pmatrix} L_o^0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_R^{-N} & 0 \\ \mathcal{A}' & \mathcal{H}' \end{pmatrix}
\end{aligned} \tag{C1}
$$

where:

$$
Y = -\begin{pmatrix} (A_R')^{N-1} c_R' & (A_R')^{(N-2)} c_R' & \dots & c_R' \end{pmatrix} \tag{C2}
$$

and $W_{oR} \triangleq L_o^c - YY'$ satisfies:

$$
A_R' W_{oR} A_R - W_{oR} = c_R' c_R
$$

(i.e. W_{oR} is the observability grammian of A_R). Since the spectral radius of $L_o L_c$ is invariant under a similarity transformation, it follows that Q can be replaced by:

$$
Q = \begin{pmatrix} L_o^{0\frac{1}{2}} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_R'^{-N} & A_R'^{-N} Y \\ \mathcal{A}' & \mathcal{A}'Y + \mathcal{H}' \end{pmatrix} \begin{pmatrix} W_{oR}^{\frac{1}{2}} & 0 \\ 0 & I \end{pmatrix} \tag{C3}
$$

where the only terms that contain powers $A_R^i, i = 1 \dots N$ are in $\mathcal{H}' + \mathcal{A}'Y$. Defining $\hat{H} \triangleq H + b_R'(A_R')^{-1}Y + b_R'(A_R')^{-1} c_R' e_N' = \begin{pmatrix} \hat{h}_1 & \dots & \hat{h}_N \end{pmatrix}$, yields:

$$
\begin{aligned}
\hat{h}_i &= q_{N-i} \qquad 1 \leq i \leq N-1 \\
\hat{h}_N &= q_0 + d_R
\end{aligned} \tag{C4}
$$

Hence, we have that

$$
\mathcal{H}' + \mathcal{A}'Y = \begin{pmatrix} \hat{h}_N & \hat{h}_{N-1} & \dots & \dots & \hat{h}_1 \\ & \hat{h}_N & \hat{h}_{N-1} & \dots & \hat{h}_2 \\ & & \ddots & & \\ & & & \hat{h}_N & \hat{h}_{N-1} \\ & & & & \hat{h}_N \end{pmatrix} + \begin{pmatrix} c_R A_R^{-1} b_R & c_R A_R^{-2} b_R & \dots & \dots & c_R A_R^{-N} b_R \\ & c_R A_R^{-1} b_R & c_R A_R^{-2} b_R & \dots & c_R A_R^{-(N-1)} b_R \\ & & \ddots & & \\ & & & c_R A_R^{-1} b_R & c_R A_R^{-2} b_R \\ & & & & c_R A_R^{-1} b_R \end{pmatrix} \tag{C5}
$$

which does not contain increasing powers of A_R.