Robust Controller Design for the Benchmark Problem Using Mixed l_∞/H_∞ Optimization

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Abstract

A successful controller design paradigm must take into account both model uncertainty and performance specifications. Model uncertainty can be addressed using the $\mathcal{H}_\infty$ robust control framework. However, this framework cannot accommodate the realistic case where in addition to robustness considerations, the system is subject to time domain specifications although some progress has been recently made in this direction [1-2]. We recently proposed a design procedure to explicitly incorporate time-domain specifications into the $\mathcal{H}_\infty$ framework [3]. In this paper we apply this design procedure to the simple flexible structure used as a benchmark in the 1990- 1992 ACC, with the goal of minimizing the peak control effort due to disturbances while satisfying settling time and robustness specifications. The results show that there exist a severe trade-off between peak control action, settling time and robustness to model uncertainty.

1. Description of the Problem

The issues involved in controlling systems subject to model uncertainty and constraints can be illustrated by the simple ACC Benchmark Problem proposed by Wie and Bernstein [4]. The system, shown in figure 1, consists of two unity masses coupled by a spring with constant $0.5 \leq k \leq 2$ but otherwise unknown. A control force acts on body 1 and the position of body 2 is measured, resulting in a non-colocated sensor actuator problem. In this paper we present a design for the following simplified version of problem # 4: design a stabilizing controller to meet the following performance specifications: i) the closed-loop system must be stable for all possible values of the uncertain parameter k. ii) the peak of the control action following a unit impulse disturbance ω acting on m_2 should be minimized; and iii) for the same disturbance the displacement y of m_2 has a settling time of about 15 seconds.

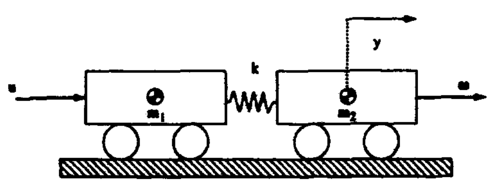

Figure 1: The ACC Robust Control Benchmark Problem.

2. The Mixed $l_\infty/\mathcal{H}_\infty$ Optimization Approach

Consider the system represented by the block diagram 2, where the scalar signals v, w and u represent an exogenous disturbance, a *known, fixed* signal, and the control action respectively; ζ and ψ represent the outputs subject to frequency and time domain performance specifications respectively; and y represents the measurements available to the controller. Note that v and ζ include fictitious signals used to assess stability in the presence of model uncertainty. Then, the *mixed $l_\infty/\mathcal{H}_\infty$ problem* can be stated as follows:

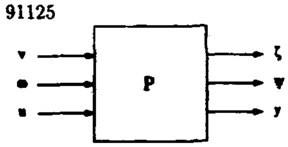

Figure 2: Block Diagram of the Generalized Plant.

Given a system (S) subject to frequency-domain performance specifications of the form:

$$\|T_{\zeta v}\|_{\mathcal{H}_\infty} \leq \gamma \tag{P}$$

find an internally stabilizing controller

$$u(z) = K(z)y(z) \tag{C}$$

such that the maximum amplitude of the regulated output ψ due to ω is minimized subject to the performance specifications (P). In [3] we showed that this problem can be decoupled into a constrained convex finite dimensional optimization and an unconstrained Nehari extension problem. By using the Youla parametrization [5] the set of all closed-loop transfer matrices achievable by an internally stabilizing compensator can be parametrized in terms of a free parameter $Q \in \mathcal{RH}_\infty$ as:

$$\begin{aligned} T_{\zeta v}(z) &= t_1(z) + t_2(z)q(z) \\ T_{\psi w}(z) &= t_1^\psi(z) + t_2^\psi(z)q(z) \end{aligned} \tag{1}$$

where $t_i, t_i^\psi \in \mathcal{RH}_\infty$. Moreover, it is possible to select the parametrization in such a way that $t_2(z)$ is inner. Hence, the problem has been transformed into a a constrained convex optimization problem in the free parameter $q \in \mathcal{RH}_\infty$. To solve this problem we will decouple it into a *finite dimensional* convex optimization and an unconstrained Nehari approximation problem. This is achieved by i) expanding the free parameter q into a power series and ii) observing that only the first N (where N depends on the problem but can be determined before hand) terms of this expansion appear in the optimization of the time response. These results are summarized in the following theorem (see [3] for a complete description):

• **Theorem 1:** $q^o = q_F^o + z^{-N} q_R^o$ solves the mixed $l_\infty/\mathcal{H}_\infty$ control problem iff $\underline{q}_F^o = (q_o \dots q_{N-1})'$ solves the following finite dimensional convex optimization problem:

$$\underline{q}_F^o = \operatorname*{argmin}_{\substack{\underline{q} \in R^N \\ \|Q\|_2 \leq \gamma}} \|\underline{t}_1 + \tau \underline{q}\|_\infty \tag{2}$$

and q_R^o solves the unconstrained Nehari approximation problem

$$q_R^o = \operatorname*{argmin}_{q_R \in \mathcal{RH}_\infty} \|R + q_R\|_{\mathcal{H}_\infty} \tag{3}$$

where:

$$\begin{aligned} \underline{t}_1 &\triangleq (t_{1o} \ \dots \ t_{1N-1})' \\ \tau &= \begin{pmatrix} t_{2o}^\tau & 0 & \dots & 0 \\ t_{21}^\tau & t_{20}^\tau & \dots & 0 \\ \vdots & & \ddots & \\ t_{2N-1}^\tau & \dots & & t_{2o}^\tau \end{pmatrix} \end{aligned} \tag{4}$$

t_{ik} denotes the k^{th} element of the impulse response of $t_i(z)$ (i.e. $t_i(z) = \sum_0^\infty t_{ik} z^{-k}$) and where:

$$Q = W^{\frac{1}{2}} \begin{pmatrix} I & 0 \\ 0 & \mathcal{H}' \end{pmatrix} L_c^{\frac{1}{2}}$$

$$Lc = \begin{pmatrix} L_{11}^C & L_{12}^C \\ L_{12}^{C\prime} & L_{22}^C \end{pmatrix}$$

$$L_{11}^C = L_o^C$$

$$L_{12}^C = -\left((A_R')^{N-1} c_R' \quad (A_R')^{N-2} c_R' \ldots \quad c_R' \right)$$

$$L_{22}^C = I_N$$

$$W'^{\frac{1}{2}} W^{\frac{1}{2}} = \begin{pmatrix} L_o^0 & \mathcal{A} \\ \mathcal{A}' & I \end{pmatrix}$$

$$\mathcal{A} = \left(A_R^{-N} b_R \quad A_R^{-(N-1)} b_R \ldots A_R^{-1} b_R \right) \tag{5}$$

$$\mathcal{H} = \begin{pmatrix} h_N & h_{N-1} & \ldots & \ldots & h_1 \\ & h_N & h_{N-1} & \ldots & h_2 \\ & & \ddots & & \\ & & & h_N & h_{N-1} \\ & & & & h_N \end{pmatrix}$$

$$h_i = q_{N-i} + b_R'(A_R')^{N-1-i} c_R' \qquad 1 \leq i \leq N-1$$

$$h_N = q_0 + d_R$$

$$R = t_1(z) t_2^{\sim}(z) \triangleq \left(\begin{array}{c|c} A_R & b_R \\ \hline c_R & d_R \end{array} \right)$$

and L_o^0 and L_o^C are the solutions to the following Lyapunov equations:

$$\begin{aligned} A_R L_o^0 A_R' - L_o^0 &= b_R b_R' \\ A_R' L_o^C A_R - L_o^C &= (A_R')^N c_R' c_R (A_R)^N \end{aligned} \tag{6}$$

3. Problem Transformation

In order to fit the problem into the mixed $l_\infty/\mathcal{H}_\infty$ framework a number of transformations are required. First, in order to use the $\mathcal{H}_\infty$ framework, the uncertain spring constant k is modeled as $k = k_o + \Delta$ (with $k_o = 1.25$ and $\|\Delta\| \leq 0.75$) and, following a standard procedure, Δ is "pulled out" of the system, as shown in figure 3. Second, the system is discretized using sample and hold elements at the inputs and outputs, with a sampling time of 0.1 seconds. The problem can be stated now as the problem of minimizing the peak control effort u_{pk} over the set of all internally stabilizing controllers, subject to the settling time and $\|T_{\zeta v}\|_{\mathcal{H}_\infty} \leq \frac{4}{3}$ constraints.

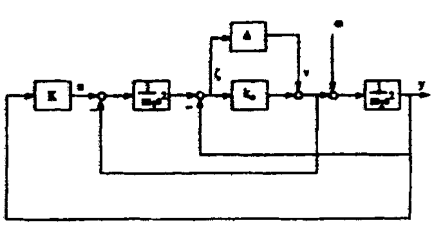

Figure 3:Block Diagram with the Uncertainty "Pulled Out" of the System.

Finally, note that the system has poles on the $j\omega$-axis (or equivalently, the discretized system has poles on the unit circle), which prevents direct application of the $\mathcal{H}_\infty$ methodology. This ill-posedness is removed by using the change of variable $\hat{z} \triangleq \frac{z}{\rho}$, where $\rho > 1$, which amounts to contracting the unit circle. This is similar to the bilinear transformations used in [6–7]. Note that the maximum modulus theorem guarantees that $\|T_{\zeta v}(\hat{z})\|_\infty \geq \|T_{\zeta v}(z)\|_\infty$.

4. Results

By solving the optimization problem (2) (with N=150), we found that for the discrete time system, the minimum value of the peak control action, $\|u_k\|_{l_\infty}$ subject to the constraints $\|T_{y\zeta}\|_\infty \leq \frac{4}{3}$ is slightly less than 1. Hence, for the discretized version of the BMP, the specifications are achievable, although they may require a very large order controller (see [3] for details). It should be noted that the settling time constraint is binding. By relaxing this constraint, the specifications are achievable with low order controllers. Figure 4 shows the impulse and frequency responses achieved with the following second order controller:

$$Ac = \begin{pmatrix} 1.7404 & -0.7769 \\ 0.9950 & 0 \end{pmatrix} \quad Bc = \begin{pmatrix} 0.9975 \\ 0 \end{pmatrix}$$

$$Cc = \begin{pmatrix} -1.1347 & 1.0044 \end{pmatrix} \quad Dc = 4.1150$$

Although $\|T_{\zeta v}\|_\infty = 1.43$ a simple analysis shows that the closed-loop system is stable for all $0.5 \leq k \leq 2$.

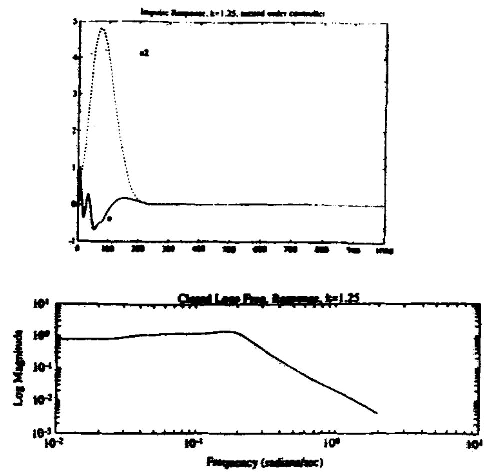

Figure 4: Impulse and Frequency Responses

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