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A Mixed l_∞/H_∞ Optimization Approach to Robust Controller Design

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Abstract

Most realistic control problems involve both mixed time/frequency domain performance requirements and model uncertainty. However, the majority of controller design procedures currently available focus only on one aspect of the problem. In this paper we propose a design procedure for minimizing the maximum amplitude of a regulated error to a specified input while, at the same time, addressing model uncertainty through bounds on the H_∞ norm of a relevant transfer function. This problem is of interest in optimal tracking applications where the objective is to achieve minimum tracking error while, at the same time, maintaining an adequate robustness level. We show that for the SISO case the problem can be decoupled into a finite dimensional constrained optimization and an unconstrained Nehari approximation problem.

1. Introduction

A substantial number of control problems can be summarized as the problem of designing a controller capable of achieving acceptable performance under system uncertainty and design constraints. This statement looks deceptively simple, but even in the case where the system under consideration is linear, the problem is far from solved. During the last decade a large research effort led to procedures for designing robust controllers, capable of achieving desirable properties under various classes of plant uncertainties while, at the same time, satisfying frequency–domain constraints. However, these design procedures cannot accommodate directly time domain performance specifications.

Recently, some progress has been made in this direction [1–4]. By using a parametrization of all stabilizing linear controllers in terms of a stable transfer matrix Q, the problem of finding the "best" linear controller can be formulated as the constrained optimization problem of minimizing a weighted ∞–norm over the set of suitable Q. In this formulation, additional specifications can be imposed by further constraining the problem. The resulting optimization problem has been solved using convex programming [1] and constrained nondifferentiable optimization [2]. However, although these methods are effective when the specifications are easily expressed in terms of the frequency response, presently they can handle time–domain specifications in a conservative fashion, through the use of several approximations. A different approach has been pursued in [3–4], where time–domain constraints over a finite horizon are incorporated into an H_∞ optimal control problem which is then transformed into a finite dimensional optimization problem. However, at this stage constraints over an infinite horizon can be handled only indirectly. Finally, in [5] and [6] the problems of finding an internally stabilizing compensator that minimizes the maximum error to l_∞ bounded disturbances and to a fixed, given signal was solved. However, these designs cannot accommodate frequency–domain specifications.

In this paper we address the problem of finding an internally stabilizing compensator that minimizes the maximum amplitude of the error to a fixed given input subject to constraints upon the H_∞ norm of a relevant transfer function. This problem, which can be though as the dual of the problem proposed in [3–4], is of particular interest for optimal tracking problems where the objective is to achieve minimum error magnitude, while at the same time maintaining an adequate robustness level against model uncertainty. We believe that the results presented here will provide a useful new approach for addressing more realistic control design problems including a combination of time–domain and frequency–domain specifications.

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2. Problem Formulation

2.1 Notation

By $\mathcal{L}_\infty$ we denote the Lebesgue space of complex valued transfer matrices which are essentially bounded on the unit circle with norm $\|T(z)\|_{H_\infty} \triangleq \sigma_{\max}(T(e^{jw}))$. H_∞ (H_∞^-) denotes the set of stable (antistable) complex matrices $g(z) \in \mathcal{L}_\infty$, i.e analytic in $z \geq 1$ ($z \leq 1$). $\mathcal{RH}_\infty$ ($\mathcal{RH}_\infty^-$) denotes the subset of H_∞ (H_∞^-) formed by real rational transfer matrices. l_∞ denotes the space of bounded real sequences $\{e_k\}$ equipped with the norm $\|e\|_{l_\infty} \triangleq \sup_k |e_k|$. To avoid confusion, we will denote the H_∞ norm of a transfer function as $\|.\|_{H_\infty}$ and the l_∞ norm of a sequence as $\|.\|_{l_\infty}$. Throughout the paper we will use packed notation to represent state–space realizations, i.e.

$$G(z) = C(zI - A)^{-1}B + D \triangleq \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

Finally, for a transfer matrix $G(z)$, $G^{\sim} \triangleq G'(\frac{1}{z})$.

2.2 Statement of the Problem

Consider the system represented by the block diagram 1, where S represents the system to be controlled; the scalar signals w, r and u represent an exogenous disturbance, a *known, fixed* signal, and the control action respectively; and where ζ, e and y represent the outputs subject to frequency domain performance constraints, the tracking error to the signal r and the measurements respectively. Note that w and ζ include fictitious signals used to assess stability in the presence of model uncertainty. Then, the basic problem that we address in this paper is the following:

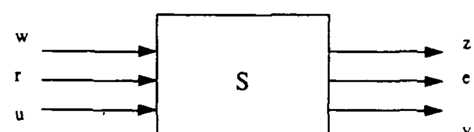

Fig. 1. The Generalized Plant

- **Mixed l_∞/H_∞ Control Problem:** Given the nominal system (S), with frequency–domain performance specifications of the form:

$$\|W(z)T_{\zeta w}(z)\|_\infty \leq \gamma \qquad (P)$$

where $W(z)$ is a suitable weighting function (used for instance to give different weights to different frequencies), find an internally stabilizing controller $u(z) = K(z)y(z)$ such that the maximum amplitude of the regulated output e due to r is minimized subject to the performance specifications (P)

3. Problem Solution

In this section we show that the mixed l_∞/H_∞ optimization problem can be decoupled into a constrained convex finite dimensional optimization and an unconstrained Nehari extension problem.

3.1 Problem Transformation

Assume that the system S has the following state-space realization (where w.l.o.g we assume that all weighting factors have been absorbed into the plant):

$$\left(\begin{array}{c|ccc} A & B_{1f} & B_{1t} & B_2 \\ \hline C_f & D_{ff} & D_{ft} & D_{f2} \\ C_t & D_{tf} & D_{tt} & D_{t2} \\ C_2 & D_{2f} & D_{2t} & D_{22} \end{array}\right) \qquad (S)$$

where D_{f2} has full column rank, D_{2f} has full row rank, and where the pairs (A, B_2) and (C_2, A) are stabilizable and detectable respectively. It is well known (see for instance [8]) that the set of all internally stabilizing controllers can be parametrized in terms of a free parameter $Q \in \mathcal{RH}_\infty$ as $K = \mathcal{F}_l(J, Q)$ where J has the following state-space realization:

$$\left(\begin{array}{c|cc} A + B_2F + LC_2 + LD_{22}F & -L & B_2 + LD_{22} \\ \hline F & 0 & I \\ -(C_2 + D_{22}F) & I & -D_{22} \end{array}\right) \qquad (1)$$

where F and L are selected such that $A + B_2F$ and $A + LC_2$ are stable. Furthermore, the closed-loop transfer functions $T_{\zeta w}$ and T_{er} can be written as:

$$\begin{aligned} T_{\zeta w} &= \mathcal{F}_l(T_f, Q) = T_{11} + T_{12}QT_{21} \\ T_{er} &= \mathcal{F}_l(T_t, Q) = T_{11}^t + T_{12}^tQT_{21}^t \end{aligned} \qquad (2)$$

where $T_i, T_i^t \in \mathcal{RH}_\infty$ and where T_f and T_t have the following state-space realizations:

$$T_f = \left(\begin{array}{cc|cc} A_F & -B_2F & B_{1f} & B_2 \\ 0 & A_L & B_{1f} + LD_{2f} & 0 \\ \hline C_f + D_{f2}F & -D_{f2}F & D_{ff} & D_{f2} \\ 0 & C_2 & D_{2f} & 0 \end{array}\right)$$

$$T_t = \left(\begin{array}{cc|cc} A_F & -B_2F & B_{1t} & B_2 \\ 0 & A_L & B_{1t} + LD_{2t} & 0 \\ \hline C_t + D_{t2}F & -D_{t2}F & D_{tt} & D_{t2} \\ 0 & C_2 & D_{2t} & 0 \end{array}\right) \qquad (3)$$

$A_F = A + B_2F$, $A_L = A + LC_2$

Moreover (see the Appendix), it is possible to select F and L in such a way that $T_{12}(z)$ and $T_{21}(z)$ are inner and co-inner respectively (i.e. $T_{12}^{\sim}T_{12} = I$, $T_{21}T_{21}^{\sim} = I$). Note that for the SISO case, equation (2) reduces to:

$$\begin{aligned} T_{\zeta w}(z) &= t_1(z) + t_2(z)q(z) \\ T_{er}(z) &= t_1^r(z) + t_2^r(z)q(z) \end{aligned} \qquad (4)$$

where t_i, t_i^r, q are stable transfer functions and where t_2 is inner. Since $\|\cdot\|_{H_\infty}$ is invariant under multiplication by an inner function we have:

$$\|T_{\zeta w}\|_{H_\infty} = \|t_1 + t_2q\|_{H_\infty} = \|R + q\|_{H_\infty} \qquad (5)$$

where $R(z) \triangleq t_1(z)t_2^{\sim}(z)$ has all its poles outside the unit disk. A state-space realization of R in terms of the state-space realization of (S) is given in the Appendix.

3.2 l_∞ Optimization Analysis

In this section we show that the minimization of $\|e_k\|_{l_\infty}$ subject to the constraints (P) requires considering only a finite number N of elements of the sequence $\{e_k\}$.

- **Theorem 1:** Dahleh and Pearson, [6]

Let $t_2^r(z)$ have n distinct zeros a_k outside the open unit disk. Then:

$$\begin{aligned} \mu^* &= \min_{K stab} \|e\|_{l_\infty} \\ &= \max_{\alpha_j} \left[\sum_{i=1}^{n} \alpha_i Re\{t_1^r(a_i)\} + \sum_{i=1}^{n} \alpha_{i+n} Im\{t_1^r(a_i)\} \right] \end{aligned} \qquad (6)$$

subject to:

$$\sum_{j=0}^{\infty} \left| \sum_{i=1}^{n} \alpha_i Re\{a_i^{-j}\} + \sum_{i=1}^{n} \alpha_{i+n} Im\{a_i^{-j}\} \right| \leq 1 \qquad (7)$$

Furthermore, let:

$$r_j \triangleq \sum_{i=1}^{n} \alpha_i Re\{a_i^{-j}\} + \sum_{i=1}^{n} \alpha_{i+n} Im\{a_i^{-j}\} \qquad (8)$$

Then, the optimal error e_k satisfies the following condition:

$$|e_k| = \begin{cases} \mu^*, & \text{if } r_k \neq 0; \\ \leq \mu^*, & \text{if } r_k = 0. \end{cases}$$

Remark 1: Note that the optimal solution may have infinitely many terms such that $|e_i| = \mu^*$.

- **Theorem 2:** Assume that $r \in l_1$ and that the mixed optimization problem is feasible. Let q^*, e^* denote the solution. Then, there exist a finite number N such that:

$$\begin{aligned} &\mu_c = \|e^*\|_{l_\infty} = \sup_{0 \leq k \leq N-1} |e_k^*| \triangleq \|\underline{t}_1 + \mathcal{T}\underline{q}^*\|_\infty \\ &|e_k^*| < \mu_c \qquad k \geq N \end{aligned} \qquad (9)$$

where:

$$\underline{t}_1 \triangleq (t_{1o} \; \dots \; t_{1N-1})'$$
$$\mathcal{T} = \begin{pmatrix} t_{2o}^r & 0 & \dots & 0 \\ t_{21}^r & t_{2o}^r & \dots & 0 \\ \vdots & & \ddots & \\ t_{2N-1}^r & \dots & & t_{2o}^r \end{pmatrix}, \; \underline{q}^* \triangleq \begin{pmatrix} q_o \\ q_1 \\ \vdots \\ q_{N-1} \end{pmatrix} \qquad (10)$$

and where t_k denotes the k^{th} element of the impulse response of $t(z)$ (i.e. $t(z) = \sum_0^\infty t_k z^{-k}$)

Proof: Let ρ denote the spectral radius of $e^*(z)$. Since T_{er} is stable and $r \in l_1$, then $\rho < 1$ and:

$$e_k = \frac{1}{2\pi j} \oint_C e^*(z) z^{k-1} dz \qquad (11)$$

where C is a circle with radius $r > \rho$. Let $1 > \delta > \rho$. From (11) it follows that:

$$|e_k| \leq K_e \delta^k \tag{12}$$

where:

$$K_e \geq \sup_{z=\delta e^{j\theta}} |e^*(z)| \triangleq \|e\|_{H_\infty,\delta}$$

A suitable K_e can be found from (4) as follows. Since $\|.\|_{H_\infty,\delta}$ is submultiplicative we have:

$$\begin{aligned} \|e(z)\|_{H_\infty,\delta} &\leq \|T_{er}\|_{H_\infty,\delta}\|r\|_{H_\infty,\delta} \\ &\leq (\|t_1^r\|_{H_\infty,\delta} + \|t_2^r\|_{H_\infty,\delta}\|q^*\|_{H_\infty,\delta})\|r\|_{H_\infty,\delta} \end{aligned} \tag{13}$$

From the performance constraint (P) we have:

$$\|t_1 + t_2 q^*\|_{H_\infty} = \|R + q^*\|_{H_\infty} \leq \gamma$$

Hence:

$$\|q^*\|_{H_\infty} \leq \gamma + \|R\|_{H_\infty} \triangleq \gamma_q \tag{14}$$

Since q^* is analytical outside the disk of radius ρ, it follows that for any $\epsilon > 0$, δ, $\rho < \delta < 1$ can be chosen such that:

$$\|q\|_{H_\infty,\delta} \leq (1+\epsilon)\gamma_q \triangleq \gamma_\epsilon \tag{15}$$

Finally, substituting (15) in (13) yields:

$$K_e = (\|t_1^r\|_{H_\infty,\delta} + \|t_2^r\|_{H_\infty,\delta}\gamma_\epsilon)\|r\|_{H_\infty,\delta} < \infty \tag{16}$$

Let μ_u and μ_c denote the solution to the unconstrained (found using Theorem 1) and constrained problems respectively. It follows that if N is selected such that:

$$K_e \delta^N < \mu_u \tag{17}$$

then, for $l > N$:

$$\mu_c = \min_{K \text{ stab}, \|T_{\zeta w}\| \leq \gamma} \left\{ \max_k |e_k| \right\} \geq \mu_u > K_e \delta^l \geq |e_l| \tag{18}$$

Hence, the optimal value μ_c is determined only by the first N terms of the sequence $\{e_k\}$ $\diamond$.

Remark 2: Note that although in principle the Theorem requires $r \in l_1$, it can be applied even if r is a step function, since in this case the pole at $z = 1$ can be absorbed into the plant, forcing a controller with integral action.

3.3 The H_∞ Performance Constraint

From section 3.2, it follows that the mixed l_∞/H_∞ control problem has the form of the following constrained optimization problem:

$$\min_Q \epsilon$$

subject to:

$$\begin{aligned} \|e\|_{l_\infty} &\leq \epsilon \\ \bar{\sigma}(T_{\zeta w}(z)) &\leq \gamma \, \forall z \in \tau \end{aligned}$$

where τ denotes the unit circle in the z-plane. Note that this can be cast into the form of a semi-infinite optimization problem, since, from Theorem 2 it follows that for each $z \in \tau$ there are at most N constraints. In [1-2] a similar problem was solved using a double approximation: first Q was approximated by a finite response filter and then a *finite* set of frequency points was used, hence replacing the semi-infinite constraints by a (large) finite number of single frequency constraints. In [9] we proposed to solve a similar problem approximating Q by an FIR and then using outer approximation methods [10] to solve the resulting semi-infinite optimization problem. In this paper we will use a different approach to find an *exact* solution to the problem. Let q_i denote the terms of the impulse response of $q(z)$. The key observation to the method is that only the first N terms of this expansion appear in the l_∞ optimization. We will use this observation to decompose the problem into a finite-dimensional convex optimization problem followed by the solution, based upon the use of state-space methods [11], of an *unconstrained* Nehari approximation problem.

• **Lemma 1:** Consider the following Sylvester equation

$$A_R' Y A_q - Y = c_R' e_N' \tag{19}$$

where A_R is a non-singular anti-stable matrix and where:

$$A_q = \begin{pmatrix} 0 & \dots & 0 & 0 \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}, \quad e_N = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \tag{20}$$

Then, the solution Y to (19) can be explicitly calculated as:

$$Y = -\left((A_R')^{N-1} c_R' \quad (A_R')^{(N-2)} c_R' \quad \dots \quad c_R' \right) \tag{21}$$

Proof: The proof follows by successive right multiplications of (19) by the columns of the identity.

• **Theorem 3:** Let $q_F = \sum_{i=0}^{N-1} q_i z^{-i}$ be given. Then, the condition that there exist $q_R \in \mathcal{RH}_\infty$ such that $\|R + q\|_{H_\infty} \leq \gamma$, where $q = q_F + z^{-N} q_R$ is equivalent to a *convex* constraint of the form $\|Q\|_2 \leq \gamma$ where Q is a symmetric matrix that is a linear function of the coefficients of q_F.

Proof: Let $q_F \triangleq \sum_{i=0}^{N-1} q_i z^{-i}$ and define $z^{-N} q_R \triangleq q - q_F$. Then:

$$\|T_{\zeta w}\|_{H_\infty} = \|R + q_F + z^{-N} q_R\|_{H_\infty} \tag{22}$$

Let $G \triangleq R + q_F$. It follows that, given q_F, there exist $q_R \in \mathcal{RH}_\infty$ such that $\|T_{\zeta w}\|_{H_\infty} \leq \gamma$ *iff* the corresponding *unconstrained* 1 block Nehari approximation problem has a solution, i.e. if:

$$\begin{aligned} \min_{q_R \in \mathcal{RH}_\infty} \|G + z^{-N} q_R\|_{H_\infty} &= \min_{q_R \in RH_\infty^-} \|z^{-N} G^\sim + q_R\|_{H_\infty} \\ &= \Gamma_H(z^{-N} G^\sim) \leq \gamma \end{aligned} \tag{23}$$

where Γ_H indicates the maximum Hankel singular value and where we used the facts that z^N is an inner function and that the best stable approximation to a given function coincides with the best antistable approximation to its conjugate. In order to compute Γ_H we need a space-state realization for the stable part of $z^{-N} G^\sim$. Let $G_1 \triangleq R^\sim z^{-N}$ and denote the space-state realization of R given in the Appendix by:

$$R = \left(\begin{array}{c|c} A_R & b_R \\ \hline c_R & d_R \end{array} \right)$$

Standard space-state manipulations [11] yield:

$$R^\sim = \left(\begin{array}{c|c} (A_R')^{-1} & -(A_R')^{-1} c_R' \\ \hline b_R'(A_R')^{-1} & d_R' - b_R'(A_R')^{-1} c_R' \end{array} \right) \quad z^{-N} = \left(\begin{array}{c|c} A_q & e_1 \\ \hline e_N' & 0 \end{array} \right)$$

where $e_1' = (1 \ \dots \ 0)$, $e_N' = (0 \ \dots \ 1)$. Hence:

$$G_1 = \left(\begin{array}{cc|c} (A_R')^{-1} & -(A_R')^{-1} c_R' e_N' & 0 \\ 0 & A_q & e_1 \\ \hline b_R'(A_R')^{-1} & d_R' e' n - b_R'(A_R')^{-1} c_R' e_N' & 0 \end{array} \right) \tag{24}$$

Finally, the similarity transformation $T = \begin{pmatrix} I_N & Y \\ 0 & I_N \end{pmatrix}$ where Y is the unique solution to the Sylvester equation:

$$A'_R Y A_q - Y = c'_R e'_N$$

yields:

$$G_1 = \left(\begin{array}{cc|c} (A'_R)^{-1} & 0 & Ye_1 \\ 0 & A_q & e_1 \\ \hline b'_R(A'_R)^{-1} & d'_R e'_N - b'_R(A'_R)^{-1}(c'_R e'_N + Y) & 0 \end{array}\right) \quad (25)$$

Since A_R is antistable, A_R^{-1} is stable. Hence $P_+[G_1] = G_1$. Similarly:

$$G_2 \triangleq z^{-N} q_F^{-} = \sum_{i=1}^{N-1} q_{N-i} z^{-i} = \left(\begin{array}{c|c} A_q & e_1 \\ \hline c_q & 0 \end{array}\right) \quad (26)$$

where $c_q = (\, q_{N-1} \ \ \dots \ \ q_0\,)$. Hence:

$$\mathcal{G} \triangleq P_+[G_1 + G_2] = \left(\begin{array}{cc|c} (A'_R)^{-1} & 0 & Ye_1 \\ 0 & A_q & e_1 \\ \hline b'_R(A'_R)^{-1} & H & 0 \end{array}\right) \quad (27)$$

where:

$$H \triangleq c_q + d'_R e'_N - b'_R(A'_R)^{-1}(c'_R e'_N + Y) \triangleq (\, h_1 \ \ \dots \ \ h_N\,) \quad (28)$$

Finally, note that Y can be computed explicitly by using Lemma 1. Substituting (21) in (27) and (28) yields:

$$\mathcal{G} = P_+[G_1 + G_2] = \left(\begin{array}{cc|c} (A'_R)^{-1} & 0 & -(A'_R)^{-1}c'_R \\ 0 & A_q & e_1 \\ \hline b'_R(A'_R)^{-1} & H & 0 \end{array}\right)$$
$$h_i = q_{N-i} + b'_R(A'_R)^{N-1-i}c'_R \qquad 1 \le i \le N-1$$
$$h_N = q_0 + d_R \quad (29)$$

In order to compute the approximation error we need to compute the observability and controllability grammians of $\mathcal{G}$. Although in principle this requires the solution of 2 Lyapunov equations, with coefficients that are functions of q_F, we will show that the particular structure of the problem allows for computing these solutions explicitly. For the controllability grammian we have:

$$\begin{pmatrix} (A'_R)^{-1} & 0 \\ 0 & A_q \end{pmatrix} \begin{pmatrix} L^C_{11} & L^C_{12} \\ L'^C_{12} & L^C_{22} \end{pmatrix} \begin{pmatrix} (A'_R)^{-1} & 0 \\ 0 & A_q \end{pmatrix}' - \begin{pmatrix} L^C_{11} & L^C_{12} \\ L'^C_{12} & -L^C_{22} \end{pmatrix}$$
$$= -\begin{pmatrix} (A'_R)^{N-1}c'_R c_R (A_R)^{N-1} & -(A'_R)^{N-1}c'_R e'_1 \\ -e_1 c_R (A_R)^{N-1} & e_1 e'_1 \end{pmatrix} \quad (30)$$

Solving for each of the blocks of the grammian yields:

$$L^C_{11} = L^C_o$$
$$L^C_{12} = -(\,(A'_R)^{N-1}c'_R \ \ (A'_R)^{N-2}c'_R \dots \ \ c'_R\,) = Y \quad (31)$$
$$L^C_{22} = I_N$$

where L^C_o is the solution of the following Lyapunov equation:

$$A'_R L^C_o A_R - L^C_o = (A'_R)^N c'_R c_R (A_R)^N \quad (32)$$

and where the expression for L^C_{12} was obtained from the corresponding equation by successive right multiplications by e_i.

Note that the controllability grammian of $\mathcal{G}$ is independent of q_F. Similarly, for the observability grammian we have:

$$\begin{pmatrix} (A'_R)^{-1} & 0 \\ 0 & A_q \end{pmatrix}' \begin{pmatrix} L^0_{11} & L^0_{12} \\ L'^0_{12} & L^0_{22} \end{pmatrix} \begin{pmatrix} (A'_R)^{-1} & 0 \\ 0 & A_q \end{pmatrix} - \begin{pmatrix} L^0_{11} & L^0_{12} \\ L'^0_{12} & -L^0_{22} \end{pmatrix}$$
$$= -\begin{pmatrix} (A_R)^{-1}b_R b'_R (A'_R)^{-1} & (A_R)^{-1}b_r H \\ H' b'_R (A'_R)^{-1} & H'H \end{pmatrix} \quad (33)$$

Solving for each of the blocks of the grammian yields:

$$L^0_{11} = L^0_o$$
$$L^0_{12} = \mathcal{A}\mathcal{H}' \quad (34)$$
$$L^0_{22} = \mathcal{H}\mathcal{H}'$$

where:

$$\mathcal{H} \triangleq \begin{pmatrix} h_N & h_{N-1} & \dots & \dots & h_1 \\ & h_N & h_{N-1} & \dots & h_2 \\ & & \ddots & & \\ & & & h_N & h_{N-1} \\ & & & & h_N \end{pmatrix} \quad (35)$$
$$\mathcal{A} = \left(A_R^{-N} b_R \ \ A_R^{-(N-1)} b_R \dots A_R^{-1} b_R \right)$$

and where L^0_o is the solution to the following Lyapunov equation:

$$A_R L^0_o A'_R - L^0_o = b_R b'_R \quad (36)$$

(i.e. the controllability grammian for R) which is independent from q_F. Finally, note that

$$\mathrm{L}_o = \begin{pmatrix} L^0_o & \mathcal{A}\mathcal{H}' \\ \mathcal{H}\mathcal{A}' & \mathcal{H}\mathcal{H}' \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \mathcal{H} \end{pmatrix} \begin{pmatrix} L^0_o & \mathcal{A} \\ \mathcal{A}' & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \mathcal{H}' \end{pmatrix} \quad (37)$$

Let

$$W'^{\frac{1}{2}} W^{\frac{1}{2}} \triangleq \begin{pmatrix} L^0_o & \mathcal{A} \\ \mathcal{A}' & I \end{pmatrix} \quad (38)$$

Then:

$$L_o = \begin{pmatrix} I & 0 \\ 0 & \mathcal{H} \end{pmatrix} W'^{\frac{1}{2}} W^{\frac{1}{2}} \begin{pmatrix} I & 0 \\ 0 & \mathcal{H}' \end{pmatrix} \quad (39)$$

Hence, from (32) and (40) we have that:

$$L_c^{\frac{1}{2}} L_o L_c^{\frac{1}{2}} = Q'Q$$
$$Q \triangleq W^{\frac{1}{2}} \begin{pmatrix} I & 0 \\ 0 & \mathcal{H}' \end{pmatrix} L_c^{\frac{1}{2}} \quad (40)$$

From Nehari Theorem ([12]) it follows that:

$$\|T_{\zeta w}\|_{H_\infty} \le \gamma \iff \rho^{\frac{1}{2}}\left(L_c^{\frac{1}{2}} L_o L_c^{\frac{1}{2}}\right) \le \gamma \iff \|Q\|_2 \le \gamma \quad (41)$$

where ρ indicates the spectral radius. Since Q is a linear function of the coefficients of q_F it follows that the constraint (41) is convex in the variables q_i ◇.

Remark 3: Note that $\begin{pmatrix} L^0_o & \mathcal{A} \\ \mathcal{A}' & I \end{pmatrix}$ is positive definite, since from (36) it can be easily shown that:

$$L^0_o - \mathcal{A}\mathcal{A}' = A_R{}^{-N} L^0_o A_R^{-N} > 0$$

The following result is now obvious:

• **Theorem 4:** $q^o = q^o_F + z^{-N} q^o_R$ solves the mixed l_∞/H_∞ control problem iff $\underline{q}^o = (\, q_o \dots q_{N-1}\,)'$ solves the following finite dimensional convex optimization problem:

$$\underline{q}^o = \operatorname*{argmin}_{\substack{\underline{q} \in R^N \\ \|Q\|_2 \le \gamma}} \|\underline{t}_1 + \tau \underline{q}\|_\infty$$

and q_R solves the unconstrained Nehari approximation problem

$$q^o_R = \operatorname*{argmin}_{q_R \in \mathcal{RH}_\infty} \|R + q_R\|_{H_\infty}$$

where R is defined in (5).

Based upon the results of Theorem 4, the mixed optimization problem can be solved using the following iterative algorithm.

3.4 Synthesis Algorithm

Begin

1) Find t_i, t_i^r and R using the formulas given in the Appendix.
2) Compute μ_u using Theorem 1 and N from (17) using an initial guess for δ. Alternatively, use an initial guess N_o. Solve (31) for L_c and (36) for L_o^0. Compute $W^{\frac{1}{2}}$ from (38)
3) Solve the following convex optimization problem:

$$\min_{\substack{\underline{q} \in R^N \\ \|\underline{Q}\|_2 \leq \gamma}} \|\underline{t}_1 + \tau \underline{q}\|_\infty$$

4) Solve the unconstrained Nehari approximation problem:

$$\min_{q_R \in \mathcal{RH}_\infty} \|\mathcal{G} + q_R\|_{H_\infty}$$

5) Compute $\|q\|_{H_\infty,\delta}$ and check (15).

   If (15) holds then the optimal solution has been found

   Else

   Increase N and go to step 3

End.

• **Theorem 5:** Consider a monotonically increasing sequence $\mathcal{N} = \{N_1, N_2 \ldots N_i \ldots\}$ and let $\mu_i = \|e\|_{l_\infty}$ be the peak value of e when using the controller obtained using N_i in the algorithm. Then the sequence μ_i has a limit μ.

Proof: The proof follows by noting that $\{\mu_i\}$ is nondecreasing (since we add more constraints), bounded above (by the value of the unconstrained H_∞ controller. Hence μ_i has a limit.

Remark 4: Theorem 5 shows that by taking N large enough we can get arbitrarily close to the optimal solution. However, Theorem 1 shows that for the unconstrained case the solution to the l_∞ problem may have infinitely many elements of $\{e_k\}$ achieving the peak value. Hence, it follows that if the H_∞ constraint is not tight, the proposed algorithm may require several iterations involving large optimization problems. These problems are addressed in the next section.

3.5 A Non–Iterative Suboptimal Algorithm

In this section we present a non-iterative algorithm that yields a suboptimal solution to the problem. This suboptimal solution allows more control on the location of the closed–loop poles and therefore on the settling time. Consider the following modified problem, where we require all the poles of the closed-loop system to be inside a given disk †:

$$\min \|e\|_{l_\infty}$$

subject to:

$$\|T_{\zeta w}\|_{H_\infty} \leq \gamma$$
$$\rho(T) \leq \delta$$

where $\delta < 1$ is given and $\rho(T)$ denotes the spectral radius of the closed-loop system. From the Maximum Modulus Theorem it follows that an upper bound of the solution can be minimized by solving the following auxiliary minimization problem:

$$\min \|e\|_{l_\infty}$$

subject to:

$$\|T_{\zeta w}\|_{H_\infty,\delta} \leq \gamma$$
$$\rho(T) \leq \delta$$

† The idea of constraining the pole locations was suggested by Dr. A. Sideris and Mr. H. Rotstein

since $\|T_{\zeta w}\|_{H_\infty} \leq \|T_{\zeta w}\|_{H_\infty,\delta}$. Note that the factorizat introduced in section 3.1 can be used to solve the auxiliary minimization problem by using the change of variable $z = \delta\hat{z}$. Furthermore, from (18) it follows that the auxiliary minimization problem requires considering only N elements of the sequence $\{e_k\}$, where N is selected such that:

$$K_e\delta^N \leq \mu_u$$
$$K_e = \left(\|t_1^r(\hat{z})\|_{H_\infty} + \|t_2^r(\hat{z})\|_{H_\infty}\left(\gamma + \|R(\hat{z})\|_{H_\infty}\right)\right)\|r(\hat{z})\|_{H_\infty} \quad (42)$$

4. A Simple Example

Consider the problem of minimizing the step response error for the non–minimum phase system shown in figure 2 subject to the robustness constraint $\|T_{\zeta w}\|_{H_\infty} \leq \gamma$. Note that in this case $T_{\zeta w} = T$, the complementary sensitivity function and $T_{er} = 1 - T = S$. Assume that settling time considerations require $\rho \leq 0.8$ Then, the change of variable $z = 0.8\hat{z}$ yields:

$$P(\hat{z}) = \frac{\hat{z} - 2.5}{\hat{z} - 1.25} \quad (43)$$

The inner factorization of the Appendix yields:

$$t_1 = \frac{-0.3060\hat{z} + 0.7650}{(\hat{z}-0.8)(\hat{z}-0.4)}, \; t_2 = \frac{(0.4\hat{z}-1)(0.8\hat{z}-1)}{(\hat{z}-0.4)(\hat{z}-0.8)}$$
$$R = t_1 t_2^{\sim} = \frac{-0.9562}{\hat{z} - 1.25}$$
$$S = \frac{(\hat{z}-1.25)}{(\hat{z}-0.8)(\hat{z}-0.4)}\left(\hat{z} + 0.356 - (0.32\hat{z} - 0.8)q\right) \quad (44)$$
$$e(\hat{z}) = S(\hat{z})r(\hat{z}) = \frac{\hat{z}(\hat{z}+0.356)}{(\hat{z}-0.8)(\hat{z}-0.4)} - \frac{\hat{z}(0.32\hat{z}-0.8)}{(\hat{z}-0.4)(\hat{z}-0.8)}q \triangleq H(\hat{z}) + U(\hat{z})q$$

In this case the unconstrained minimum achievable value of $\|T\|_{H_\infty} = \Gamma_H(R) = 1.7$ Since $U(\hat{z})$ has a single non–minimum phase zero at $\hat{z} = 2.5$, direct application of Theorem 1 yields:

$$\mu_u = \max_\alpha H(2.5) = 2\alpha$$

subjet to:

$$\sum_{k=0}^{\infty} \alpha(2.5)^{-k} = \frac{\alpha}{0.6} \leq 1$$

hence μ_u=1.2. Assume that $\gamma = 2$. Then we have:

$$\|S(\hat{z})\|_{H_\infty} = \|1 - T\|_{H_\infty}$$
$$\leq 1 + \|T\|_{H_\infty} \leq 1 + \gamma \quad (45)$$
$$\|e(\hat{z})\|_{H_\infty} \leq \|S\|_{H_\infty}\|r\|_{H_\infty} = 12$$

From (17) it follows that N should be selected such that:

$$12\left(\frac{1}{1.25}\right)^N \leq \mu_u = 1.2$$

hence it suffices to consider only $N = 10$ terms of $\{e_k\}$.

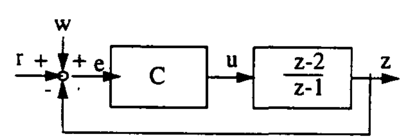

Figure 2. Block Diagram for the Example

Figure 3 shows the step response of the system using the l_∞/H_∞ controller versus the step response of the system using the optimal H_∞ controller. Note that the l_∞/H_∞ controller reduces the maximum error from 1.68 to 1.28, at the price of longer settling time. Research is currently underway to extend the present formalism to accommodate shaping of the step response.

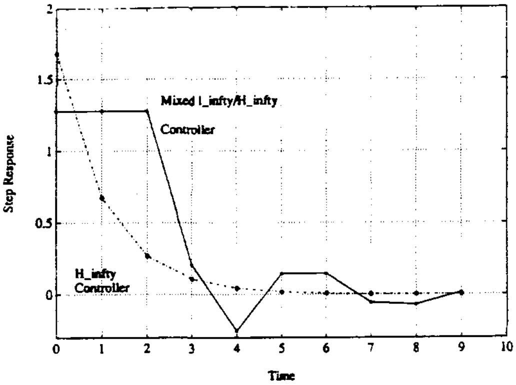

Fig. 3. H_∞ vs. l_∞/H_∞ step response

5. Conclusions

In this paper we address the problem of finding an internally stabilizing compensator that minimizes the maximum amplitude of the error to a fixed given input subject to constraints upon the H_∞ norm of a relevant transfer function. This problem can be thought as the problem of designing a controller capable of guaranteeing an adequate robustness level agains dynamic uncertainty while using the extra available degrees of freedom to optimize a time–domain performance. Although here we considered only the simpler case of a one–block problem, we anticipate that the results will extend naturally to the general 4–block case.

Perhaps the most severe limitation of the proposed method is that may result in very large order controllers (twice the number of time elements of $\{e_k\}$ considered), necessitating some type of model reduction. Research is currently under way addressing this issue and pursuing the extension of the formalism to allow more control on the shape of the error response.

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Appendix: Factorization Formulas for the One–Block Case

In this Appendix we present the State–Space formulas for the Inner and Co–Inner factorizations used in section 3.1. These formulas are the discrete–time counterpart of the continuous time formulas presented in [8] and [11]. Assume that (P) has the following State–Space realization:

$$\left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right) \qquad (P)$$

where D_{12} has full column rank with $(D_{12}\ \ D\perp)$ unitary, D_{21} has full row rank with $(D_{21}\ \ \tilde{D}_\perp)$ unitary, and where the pairs (A, B_2) and (C_2, A) are stabilizable and detectable respectively. Furthermore, assume that the following conditions hold:

$$\begin{pmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{pmatrix} \text{ has full column rank for all } \omega$$

$$\begin{pmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{pmatrix} \text{ has full row rank for all } \omega.$$

Note that for the one block case the hypothesis implies $D_{12} = D_{21} = I$, which can be assumed without loss of generality (by redefining the inputs if necessary). Selecting F and L as:

$$\begin{aligned} F &= -(B_2' X B_2 + I)^{-1}(C_1 + B_2' X A) \\ L &= -(B_1 + AYC_2')(I + C_2 Y C_2')^{-1} \end{aligned} \qquad (A1)$$

where $X, Y > 0$ are the solution to the following Riccati equations:

$$\begin{aligned} A'XA - (C_1 + B_2'XA)'(I + B_2'XB_2)^{-1}(C_1 + B_2'XA) + C_1'C_1 &= X \\ AYA' - (B_1 + AYC_2')(I + C_2YC_2')^{-1}(B_1' + C_2YA') + B_1B_1' &= Y \end{aligned} \qquad (A2)$$

yields T_{12} and T_{21} such that $T_{12}^{\sim}T_{12} = R_B$ and $T_{21}T_{21}^{\sim} = R_L$ where $R_B = (I + B_2'XB_2)$ and $R_L = (I + C_2YC_2')$. Setting $\hat{T}_{12} = T_{12}R_B^{\frac{-1}{2}}$, $\hat{T}_{21} = R_L^{\frac{-1}{2}}T_{21}$ and redefining Q as $R_B^{\frac{1}{2}}QR_L^{\frac{1}{2}}$ yields the desired result. Finally, it can be shown though some lengthy computations that:

$$T_{12}^{\sim}T_{11}T_{21}^{\sim} \triangleq R = \left(\begin{array}{c|c} (A_F')^{-1} & (A_F')^{-1}C_F' \quad -XL \\ \hline -(A_F^{-1}\hat{B}_2)' & R_B^{\frac{1}{2}} \quad 0 \end{array}\right) \times$$

$$\left(\begin{array}{c|c} (A_L')^{-1} & (A_L')^{-1}\hat{C}_2' \\ \hline -D_{11}B_L'(A_L')^{-1} - C_1Y & \begin{array}{c} D_{11}R_L^{\frac{1}{2}} \\ R_L^{\frac{1}{2}} \end{array} \\ 0 & \end{array}\right) \qquad (A3)$$