Norm Based Optimally Robust Control of Constrained Discrete Time Linear Systems

M. Sznaier[†] and *A. Sideris*[‡]

† Electrical Engineering Dept., University of Central Florida, Orlando, Fl 32816

‡ Electrical Engineering Dept., Caltech 116-81, Pasadena, CA 91125

Abstract

Most realistic control problems involve both some type of time-domain constraints and model uncertainty. However, the majority of controller design procedures currently available focus only on one aspect of the problem, with only a handful of method capable of simultaneously addressing, albeit in a limited fashion, both issues. In this paper we propose a simple design procedure that takes explicitly into account both time domain constraints and model uncertainty. Specifically, we use a operator norm approach to define a simple robustness measure for constrained systems. The available degrees of freedom are then used to optimize this measure subject to additional performance specifications. We believe that the results presented here provide a useful new approach for designing controllers capable of yielding good performance under substantial uncertainty while meeting design constraints.

I. Introduction

A substantial number of control problems can be summarized as the problem of designing, with minimal design effort, a controller capable of achieving acceptable performance under system uncertainty and design constraints. This statement looks deceptively simple, but even in the case where the system under consideration is linear, the problem is far from solved.

Several methods have been proposed recently to deal with constrained control problems [1], but as a rule, they assume exact knowledge of the dynamics involved, ruling out cases where good qualitative models of the plant are available but the numerical values of various parameters are unknown or even change during operation.

On the other hand, during the last decade a considerable amount of time has been spent analyzing the question of whether some relevant properties of a system (most notably asymptotic stability) are preserved under the presence of unknown perturbations. This research effort has led to procedures for designing controllers, termed "robust controllers", capable of achieving desirable properties under various classes of plant perturbations while, at the same time, satisfying frequency-domain constraints. However, these design procedures cannot accommodate directly time domain constraints.

Recently, some progress has been made in this direction. By using a parametrization of all stabilizing linear controllers in terms of a stable transfer matrix Q, the problem of finding the "best" linear controller can be formulated as the constrained optimization problem of minimizing a weighted ∞-norm over the set of suitable Q. In this formulation, additional specifications can be imposed by further constraining the problem. The resulting optimization problem has been solved using convex programming [2] and constrained non-differentiable optimization [3]. However, although the methods based upon the Q-parametrization are effective when the specifications are easily expressed in terms of the frequency response, they can handle time-domain constraints only in a very conservative fashion. Further, these methods rely on rough approximations to transform the problem into a finite dimensional optimization, which in some cases leads to badly conditioned problems and numerical difficulties. An additional drawback is that the order of the resulting controller is not bounded a priori and can conceivably be extremely high. A different approach has been pursued in [4, 5] where time-constraints over a finite horizon are incorporated into an H_∞ optimal control problem which is then transformed into a finite dimensional optimization problem. Presently, the main drawback of this method is its inability to handle constraints over an *infinite horizon*.

In a recent paper [6] we proposed to approach time-domain constrained systems using an operator norm-theoretic approach and we introduced a simple robustness measure that indicates how well the family of systems under consideration satisfies a given set of time-domain constraints. This approach has the advantage of yielding a simple design procedure that takes explicitly into account time-domain constraints and model uncertainty. In this paper we apply the operator-norm approach to the problem of designing simple controllers capable of maximizing this robustness measure subject to additional performance specifications of the form of upper bounds on a quadratic performance index. This design philosophy reflects the fact that in most applications the larger effort is incurred in identifying the model and deriving uncertainty bounds. The proposed design method clearly identifies the trade-offs between uncertainty and performance, displaying the admissible level of uncertainty for a given level of performance. Hence, in addition to providing a very simple design methodology for dealing with uncertain constrained systems, our method is also valuable as an analysis tool in a pre-design stage, to indicate whether an additional identification effort is required.

The paper is organized as follows: In section II we introduce a simple robustness measure and we formally define the *optimally robust constrained linear control problem*. The main result of this section is a necessary and sufficient condition that guarantees the constrained stability of a family of systems. In Section III we show that a suboptimal solution to the optimally robust control problem can be found by considering an auxiliary Discrete Linear Quadratic Problem. In section IV we present a controller design algorithm and several examples of application. Finally, in section V we summarize our results and we indicate directions for future research.

II. Definitions and Statement of the Problem

In this section we introduce several required concepts and a formal definition of the optimally robust constrained linear control problem. We begin by introducing the concept of *constrained stability:*

- **Def. 1:** Consider the linear, time invariant, discrete time, autonomous system modeled by the difference equation:

$$\underline{x}_{k+1} = A\underline{x}_k,\ k = 0, 1\dots \qquad (S^*)$$

subject to the constraint $\underline{x} \in \mathcal{G} \subseteq R^n$ where $A \in R^{n*n}$ and where $\underline{x}$ indicates x is a vector quantity. The system (S^*) is *Constraint Stable* (C-stable) if for *any* point $\underline{\tilde{x}} \in \mathcal{G}$, the trajectory $\underline{x}_k(\underline{\tilde{x}})$ originating in $\underline{\tilde{x}}$ remains in $\mathcal{G}$ for all k.

We proceed to introduce now a restriction on the class of constraints allowed in our problem. This restriction introduces more structure into the problem, playing a key role in deriving necessary and sufficient conditions for constrained stability.

Constraint Qualification Hypothesis

In this paper, we will limit ourselves to constraints of the form:

$$\underline{x} \in \mathcal{G} = \{\underline{x}: G(\underline{x}) \leq \underline{\omega}\} \tag{1}$$

where $\underline{\omega} \in R^p, \omega_i \geq 0$, the inequalities should be interpreted on a component by component sense and where $G: R^n \rightarrow R^p$ has the following properties:

$$\begin{aligned} G(\underline{x})_i &\geq 0,\ i = 1 \dots p\ \forall\ \underline{x} \\ G(\underline{x}) = 0 &\iff \underline{x} = 0 \\ G(\underline{x}+\underline{y})_i &\leq G(\underline{x})_i + G(\underline{y})_i,\ i = 1 \dots p\ \forall\ \underline{x}, \underline{y} \\ G(\lambda \underline{x}) &= \lambda G(\underline{x}),\ 0 \leq \lambda \leq 1 \end{aligned} \tag{2}$$

As examples of constraints that satisfy these conditions we can mention [7] polyhedral and hyperellipsoidal regions. The next Lemma shows that $G(.)$ induces a norm and characterizes the region $\mathcal{G}$ in terms of this norm.

- **Lemma 1:** [6] Let:

$$v(x) = \max_{1 \leq i \leq p} \left\{ \frac{G(\underline{x})_i}{\omega_i} \right\} = \|W^{-1}G(\underline{x})\|_\infty \triangleq \|\underline{x}\|_{\mathcal{G}} \tag{3}$$

where $W = diag(\omega_1, \dots, \omega_m)$. Then $v(.)$ defines a norm in R^n and the admissible region $\mathcal{G}$ can be characterized as the $\mathcal{G}$-norm unity ball, i.e.: $\mathcal{G} = \{\underline{x}: \|\underline{x}\|_{\mathcal{G}} \leq 1\}$.

Next, we take into account model uncertainty, extending the concept of constrained stability to a family of systems and, in Theorem 1, we define a quantitative way of measuring the "size" of the smallest destabilizing perturbation.

- **Def. 2:** Consider the system (S^a). Let the perturbed system (S^a_Δ) be defined as:

$$\underline{x}_{k+1} = (A + \Delta)\underline{x}_k \tag{S^a_Δ}$$

where $\Delta \in \mathcal{D} \subseteq R^{n*n}$. The system (S^a) is *Robustly Constraint Stable* (RC-stable) with respect to the set $\mathcal{D}$ if (S^a_Δ) is C-stable for all perturbation matrices $\Delta \in \mathcal{D}$.

- **Theorem 1:** [6] The system (S^a) is RC-stable with respect to the set $\mathcal{D}$ **iff**:

$$\|A + \Delta\|_{\mathcal{G}} \leq 1 \forall\ \Delta \in \mathcal{D} \tag{4}$$

where $\|.\|_{\mathcal{G}}$ denotes the induced operator norm, i.e.:

$$\|A + \Delta\|_{\mathcal{G}} = \max_{\|\underline{x}\|_{\mathcal{G}}=1} \{\|(A+\Delta)\underline{x}\|_{\mathcal{G}}\} \tag{5}$$

From this theorem, it follows that the "robustness" of the system can be characterized in terms of the size of the smallest destabilizing perturbation as follows.

- **Def. 3:** *The Constrained Stability Measure*, $\varrho^{\mathcal{N}}_{\mathcal{G}}$, is defined as:

$$\varrho^{\mathcal{N}}_{\mathcal{G}} = \min_{\Delta \in \mathcal{D}} \{\|\Delta\|_{\mathcal{N}}: \|A + \Delta\|_{\mathcal{G}} = 1\} \tag{6}$$

where $\mathcal{N}$ denotes a suitable norm defined over the perturbation set $\mathcal{D}$. If $\|A\|_{\mathcal{G}} \geq 1$ then we define $\varrho^{\mathcal{N}}_{\mathcal{G}} \triangleq 0$. Finally, if $\|A + \Delta\|_{\mathcal{G}} < 1 \forall\ \Delta \in \mathcal{D}$ then $\varrho^{\mathcal{N}}_{\mathcal{G}} \triangleq \max_{\Delta \in \mathcal{D}} \|\Delta\|_{\mathcal{N}}$.

- **Remark 1:** Let the set $\mathcal{D}_s$ be the intersection of $\mathcal{D}$ with the origin-centered ball of radius $\varrho^{\mathcal{N}}_{\mathcal{G}}$, i.e.:

$$\mathcal{D}_s = \{\Delta \in \mathcal{D}: \|\Delta\|_{\mathcal{N}} \leq \varrho^{\mathcal{N}}_{\mathcal{G}}\}$$

Then from definition 3 it follows that the perturbed system S^a_Δ is constrained stable for *all* perturbations $\Delta \in \mathcal{D}_s$.

Definition 3 is quite general since in principle no conditions are imposed over the set $\mathcal{D}$. However, in the general case nothing can be stated about the properties of $\varrho^{\mathcal{N}}_{\mathcal{G}}$ which could conceivably be a *non-continuous* function of A. In the sequel we will show that under some assumptions that are commonly verified in practice, $\varrho^{\mathcal{N}}_{\mathcal{G}}$ is a *continuous, concave* function of the dynamics matrix A.

- **Theorem 2:** Assume that the perturbation set $\mathcal{D}$ is a *closed cone* with vertex at the origin [8], i.e: $\Delta^o \in \mathcal{D} \iff \alpha\Delta^o \in \mathcal{D}\ \forall\ 0 \leq \alpha$. Then $\varrho^{\mathcal{N}}_{\mathcal{G}}$ is a *continuous* function of A.

- **Theorem 3:** Assume that $\mathcal{D}$ is a cone with vertex at the origin. Then $\varrho^{\mathcal{N}}_{\mathcal{G}}$ is a concave function of A

The proof of these theorems are given in Appendix A. Note that the class of sets considered in these theorems includes as a particular case sets of the form:

$$\mathcal{D} = \left\{ \Delta : \Delta = \sum_1^m \mu_i E_i;\ \mu_i \in \mathcal{R}, E_i \text{ given} \right\}$$

which has been the object of much interest lately ([9] and references therein).

2.1 Optimally Robust Constrained Linear Control Problem

Given:

1) A family of linear time invariant, stabilizable discrete time systems, represented by:

$$\underline{x}_{k+1} = (A + \Delta)\underline{x}_k + B\underline{u}_k,\ k = 0, 1 \dots,\ \underline{x}_o \in \mathcal{G}\ \Delta \in \mathcal{D} \tag{S}$$

2) A nominal performance specification of the form:

$$J(\underline{x}_o, \underline{u}) = \sum_{k=0}^{\infty} (\underline{x}_k \quad \underline{u}_k)' H \begin{pmatrix} \underline{x}_k \\ \underline{u}_k \end{pmatrix} \leq \gamma \tag{7}$$

where:

$$H = \begin{pmatrix} Q & M \\ M' & R \end{pmatrix} > 0$$

Find: a *constant* feedback matrix F such that the nominal closed-loop system is constrained stable and $\varrho^{\mathcal{N}}_{\mathcal{G}}$ is maximized subject to the constraint

$$J_F(\underline{x}_o, F) \triangleq J(\underline{x}_o, -F\underline{x}) \leq \gamma \tag{8}$$

Note that in general, the solution F_o to this optimal control problem will depend on the initial condition $\underline{x}_o$. This difficulty can be solved by assuming that the the initial condition is a *random variable* uniformly distributed over the $\mathcal{G}$-norm unity ball and taking the expectation of the performance index (8). Hence we have the modified performance specification:

$$\mathcal{J}(F) = \mathbf{E}_{\underline{x}_o} \{J_F(\underline{x}_o, F)\} \leq \gamma \tag{9}$$

Since the nominal closed-loop matrix is stable, we have that [10]:

$$\sum_{k=0}^{\infty} (\underline{x}_k \quad \underline{u}_k)' H \begin{pmatrix} \underline{x}_k \\ \underline{u}_k \end{pmatrix} \Big|_{\underline{u}_k = -F\underline{x}_k} = \underline{x}_o' P \underline{x}_o$$

where $P > 0$ is the unique solution to the following Lyapunov equation:

$$(A - BF)'P(A - BF) - P = -Q - F'RF + F'M' + MF \tag{10}$$

Hence it follows that (9) is equivalent to:

$$\mathcal{J}(F) = Tr\{PV\} \leq \gamma \tag{11}$$

where $V = \mathbf{E}\{\underline{x}_o \underline{x}_o'\}$. Throughout this paper we will refer to the problem defined by maximizing $\varrho^{\mathcal{N}}_{\mathcal{G}}$ subject to the conditions (10) and (11) as the ***Suboptimally Robust Constrained Control Problem***.

III. Main Results

In principle the Suboptimally Robust Constrained Control Problem could be solved using *non-smooth* optimization techniques (see for instance [11] for a description of several techniques) since ϱ_G^N is usually non-differentiable. Although we plan to explore this approach in the future, we expect that, given the form of the objective function and the constraints, it will result in a fairly complex design procedure. In this paper we will use a different approach. First, we will formulate a LQR problem equivalent to the problem of maximizing the constrained robustness measure ϱ_G^N. Then, we will use a homotopy-like procedure to find a suboptimal solution to our problem. This approach has the advantage of resulting in a very simple design procedure that uses tools commonly available to the control engineer. Furthermore, it displays clearly the trade-offs between optimal performance and robustness, indicating whether an additional identification effort is required and providing a better understanding of the physical limitations of the design.

We begin by showing the equivalence of the constrained robust control problem with a particular LQR problem. To this effect, we will investigate the discrete inverse Linear Quadratic Regulator Problem. Although the continuous-time domain version of this problem has been extensively investigated (see for example [12-13] and references therein), there are no counterparts to these results for the discrete time case. In the following theorem we show that given any stabilizing full-state feedback control law F, there exist a Discrete Time Linear Quadratic Problem that has F as a solution.

- **Theorem 4:** Consider the system:

$$\underline{x}_{k+1} = A\underline{x}_k + B\underline{u}_k \tag{12}$$

Let F be such that $A_{cl} = A - BF$ is asymptotically stable. Then there exist matrices $Q > 0, R > 0$, and M such that $\underline{u}_k = -F\underline{x}_k$ is the solution to the discrete LQR problem:

$$\min_{\underline{u}} J(\underline{x}_o, \underline{u}) = \sum_{k=0}^{\infty} (\underline{x}_k \quad \underline{u}_k)' H \begin{pmatrix} \underline{x}_k \\ \underline{u}_k \end{pmatrix} \tag{13}$$

where:

$$H = \begin{pmatrix} Q & M \\ M' & R \end{pmatrix}, \; H > 0$$

Moreover, a suitably choice for Q and M is:

$$\begin{aligned} M' &= (R + B'P_oB)F - B'P_oA \\ Q &= P_o - A'P_oA + F'(B'P_oB + R)F \end{aligned} \tag{14}$$

where $P_o > 0$ is selected such that:

$$(A - BF)'P_o(A - BF) - P_o < 0 \tag{15}$$

and where $R > 0$ is selected such that $H > 0$.

Proof: The proof, omitted for space reasons, is based upon noting that since $A - BF$ is stable then P_o exists. Furthermore, with the choice of Q and M, F is precisely the solution to the LQR problem (13). Finally, it can be shown from Finsler's Theorem [14] that $R > 0$ can be selected so that $H > 0$ ⋄.

- **Corollary:** Let F be a state feedback matrix such that:

$$\varrho_G^N(F) \geq \varrho > 0 \tag{16}$$

Then, there exist a LQR problem that has F as a solution. Furthermore, let Q^*, M^* and R^* be the weighting matrices for this problem and let $Q(\epsilon), M(\epsilon), R(\epsilon)$ be continuous functions of ϵ such that the corresponding LQR problem is well defined and such that $Q(\epsilon_o) = Q^*, M(\epsilon_o) = M^*$ and $R(\epsilon_o) = R^*$. Then:

$$\varrho_G^N(\epsilon) \triangleq \varrho_G^N(F(\epsilon))$$

where $F(\epsilon)$ denotes the solution to the LQR problem defined by $Q(\epsilon), M(\epsilon)$ and $R(\epsilon)$ is continuous at $\epsilon = \epsilon_o$.

IV. Controller Design Algorithm

From the results of Theorem 4 and its corollary it follows that there exist a LQR problem equivalent (in the sense of yielding the same solution) to the problem of finding the full state feedback matrix that maximizes the constrained robustness measure. Moreover, the constrained robustness measure of the closed-loop system is a *continuous* function of the weighting matrices of the equivalent LQR problem. Hence, a ***suboptimal*** solution to the Optimally Robust Constrained Control Problem can be found using an ***homotopy-like*** procedure as follows:

Controller Design Algorith

Begin:

1) Solve the maximally robust constrained problem:

$$F_e = \underset{F \in \Re^{m \times n}}{\operatorname{argmax}} \, \varrho_G^N \tag{17}$$

Note that the assumption that $\mathcal{D}$ is a closed-cone guarantees that this problem reduces to the well-posed problem of finding the maximum of a continuous concave function. A further discussion of the properties of this problem as well as examples of solution methods can be found in [6].

2) Use the results of Theorem 4 to find an equivalent LQR problem. Let the weighting matrices for this equivalent problem be Q_e, M_e and R_e.

3) Consider the following LQR optimization problem:

$$\min_{F \in \Re^{m \times n}} \{\mathcal{J}_\epsilon(F) = Tr\{PV\}\} \tag{18}$$

subject to:

$$(A - BF)'P(A - BF) - P + Q_\epsilon + F'R_\epsilon F - F'M'_\epsilon - MF_\epsilon = 0$$

$$\begin{aligned} Q_\epsilon &\triangleq \epsilon Q + (1 - \epsilon)Q_e \\ R_\epsilon &\triangleq \epsilon R + (1 - \epsilon)R_e \\ M_\epsilon &\triangleq \epsilon M + (1 - \epsilon)M_e \end{aligned} \tag{19}$$

where $0 \leq \epsilon \leq 1$ is a *relaxation parameter.*

4) Sweep step: solve (18) (using standard LQR theory) for a sequence of values $0 \leq \epsilon_1, \ldots \leq \epsilon_n \ldots \leq 1$ until a solution F^* such that the closed-loop system satisfies the given performance specification $\mathcal{J}(F^*) \leq \gamma$ is found.

5) (Optional) Improve the design using the solution obtained in step 4 as the initial guess for a non-smooth optimization algorithm.

End.

A Simple Example:

Consider the following system:

$$A = \begin{pmatrix} 0 & 1 \\ 0.505 & -0.51 \end{pmatrix} \; B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathcal{G} = \{\underline{x} : \|\underline{x}\|_2 \leq 1\}$$

with the performance specification:

$$J(\underline{x}_o, F) = \sum_{k=0}^{\infty} \underline{x}'_k Q \underline{x}_k + \underline{u}'_k R \underline{u}_k \leq 6$$

$$Q = \begin{pmatrix} 0.2 & 0 \\ 0 & 1 \end{pmatrix}, \; R = (20)$$

The open-loop system has poles at $s_1 = 0.5$ and $s_2 = -1.01$. Assume that the perturbation set is such that changes the position of the poles while maintaining constant their sum, i. e:

$$\mathcal{D} = \left\{ \Delta : \Delta = \mu E,\ E \triangleq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ \mu \in \Re \right\} \tag{20}$$

Note that $\|E\|_2 = 1$ hence $\|\Delta\|_2 = |\mu|$.

Step 1: In this case, it is easily seen that the solution to the maximally robust control problem (17) can be computed by solving a matrix dilation problem [15]. Rewrite the dynamics matrix as:

$$A = \begin{pmatrix} x_1 & x_2 \\ a_1 & a_2 \end{pmatrix}$$

where x_i denote elements that can be modified using state-feedback. Since matrix dilations are norm-increasing we have that:

$$\begin{aligned} \|A + \mu E\|_2 &\geq \max\{\|(\, a_1 \quad a_2 + \mu\,)\|_2\} \\ &= \sqrt{a_1^2 + (a_2 + \mu)^2} \end{aligned} \tag{21}$$

Define now:

$$\begin{aligned} \mu^0 &= \operatorname{argmin}\{|\mu|, \mu \in \Re : a_1^2 + (a_2 + \mu)^2 = 1\} \\ &= \sqrt{(1 - a_1^2)} - |a_2| \end{aligned} \tag{22}$$

From (20) and (21) it follows that $\|A + \mu^o E\|_2 \geq 1$ which implies that $\varrho_2(F) \leq \mu^o$ for all F. Furthermore, from the definition of μ^o (22) it follows that if F is selected such that $x_1 = x_2 = 0$, then $\varrho_2(F) = \mu^o$. Hence, this choice of F yields the solution to (22). In this particular example we have:

$$F^o = (\, 0 \quad 1\,),\ \varrho_2 = 0.3531$$

Step 2: Use Theorem 4 with $R_e = 1$ to find an equivalent LQR problem.

Step 3: Assuming that the initial condition $\underline{x}_o$ is uniformly distributed over the unity $\mathcal{G}$-ball we have that the initial state covariance V is given by $V = \frac{1}{2}I$. Hence the modified performance index is given by:

$$\mathcal{J}(F) = Tr\left\{\frac{1}{2}P\right\}$$

where P satisfies the Riccati equation (19).

Step 4: Consider a fixed ϵ. Let F_ϵ be the solution to (18) and A_ϵ the corresponding *closed-loop* matrix, i.e:

$$A_\epsilon = A - BF_\epsilon = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} + \mu \end{pmatrix} \tag{23}$$

In order to carry-out the sweep step, we need to compute the corresponding value of the robustness measure. This computation can be performed using standard results on matrix dilations [15] as follows: The set Υ of numbers μ such that $\|A_\epsilon\|_2 \leq 1$ can be parametrized as:

$$\Upsilon = \left\{\mu : \mu = -a_{22} - y a_{11} z + (1 - y^2)^{\frac{1}{2}} w (1 - z^2)^{\frac{1}{2}}\right\} \tag{24}$$

where:

$$\begin{aligned} y &= \frac{a_{21}}{(1 - a_{11}^2)^{\frac{1}{2}}} \\ z &= \frac{a_{12}}{(1 - a_{11}^2)^{\frac{1}{2}}} \\ w &\in \Re,\ |w| \leq 1 \end{aligned} \tag{25}$$

From (24) it follow that the constrained stability margin of A_ϵ is given by:

$$\varrho_2(\epsilon) = |a_{22} + y a_{11} z - (1 - y^2)^{\frac{1}{2}} w (1 - z^2)^{\frac{1}{2}} \operatorname{sign}(a_{22} + y a_{11} z)| \tag{26}$$

Figure 1 shows $\varrho_2(\epsilon)$ and $\mathcal{J}(F)$ as the relaxation parameter is continuously changed from 0 to 1. For $\epsilon = 0$ we recover the maximally robust controller, which yields a robustness measure $\varrho_{\text{max. robust}} = 0.3531$ and a cost $J(F) = 18$. For $\epsilon = 1$ we recover the optimal controller in the sense of J which yields a cost $J(F) = 5.35$, but with a robustness measure $\varrho_{\text{min cost}} = 0.017$ (i.e. the closed-loop system can become constrained unstable for perturbations such that $|\mu| \geq 0.017$. Finally, for $\epsilon = 0.4$ we get $F = (\,-0.0942 \quad 0.3308\,)$, $J(F) = 6$ and $\varrho_2(F) = 0.16$ which satisfies the performance specifications.

Step 5: In this case a simple two-dimensional grid shows that the optimal solution is achieved with $\tilde{F} = (\,-0.0941 \quad 0.3257\,)$ which yields $J(\tilde{F}) = 6$ and $\varrho_2(\tilde{F}) = 0.1564$.

Remark 2: Note that the value yielded by step 4 is whithin approximately 2% of the true optimum. It is also interesting to note that in this case although a perturbation such that $|\mu| = 0.16$ can be tolerated by our controller, it will destabilize (not only in the constrained sense but *also* in the classical sense) the closed-loop system obtained using the LQR methodology.

A Realistic Problem:

Consider the problem of controlling an F-100 jet engine. The system at intermediate power, sea level static and Power Lever Angle (PLA) = 83° can be represented by the following discrete time model [1]:

$$A = \begin{pmatrix} 0.8907 & 0.0474 & -0.0980 & 0.3616 & 0.0689 \\ 0.0237 & 0.9022 & -0.0202 & 0.1057 & 0.0311 \\ 0.0233 & -0.0149 & 0.8167 & 0.2255 & 0.0296 \\ 0.0 & 0.0 & 0.0 & 0.7788 & 0.0 \\ -0.0979 & 0.3532 & 0.3662 & 0.6489 & 0.0296 \end{pmatrix} \quad B = \begin{pmatrix} 0.0213 & -0.3704 \\ 0.0731 & -0.1973 \\ -0.0357 & -0.5538 \\ 0.2212 & 0.0 \\ 0.0527 & -3.9068 \end{pmatrix}$$

$$\mathcal{G} = \{\underline{x} : |G\underline{x}| \leq \underline{\omega}\} \qquad G = I, \qquad \underline{\omega} = \begin{pmatrix} 50.0 \\ 64.0 \\ 20.0 \\ 5.0 \\ 18.1 \end{pmatrix}$$

Assume that the performance specification is given in terms of the following weighting matrices, (selected using Bryson's rule [16]):

$$Q = diag(50, 64, 20, 5, 18.1)^{-1}$$
$$R = diag(31, 200)^{-1}$$

Finally, assume that that the perturbations are unstructured (i.e. all the elements of the dynamics matrix are subject to perturbations) and that the induced norm $\|.\|_{\mathcal{G}}$ is also used in the set $\mathcal{D}$.

Step 1: Since the perturbations are unstructured, it can be shown [6] that:

$$\varrho_{\mathcal{G}} = 1 - \|A\|_{\mathcal{G}}$$

Hence, the problem of finding the feedback gain that maximizes $\varrho_{\mathcal{G}}$ is *equivalent* to the problem of minimizing $\|A_B F\|_{\mathcal{G}}$. Solving this problem using L. P. ([6]) yields:

$$F_{\text{max. robust.}} = \begin{pmatrix} 0 & 0 & -0.2338 & 0 & 0 \\ -0.0256 & -0.0904 & -0.0969 & -0.4072 & -0.0832 \end{pmatrix}$$

$$\|A - BF\|_{\mathcal{G}} = 0.986,\ \varrho_{\mathcal{G}} = 1 - \|A_B F\|_{\mathcal{G}} = 0.014$$

Step 2: Use Theorem 4 with $R = 10 I_2$ to get the equivalent LQR problem.

Step 3: Elementary calculations show that a random variable uniformly distributed over the unity $\mathcal{G}$-ball has covariance $V = \frac{7}{15} WW'$ where $W = diag(w_1 \ldots w_p)$. Hence, the modified performance index is given by $\mathcal{J}(F) = Tr\{PV\}$ where P satisfies (19).

Step 4: Figures 2 a and b show the constrained stability measure and the performance index as the relaxation parameter ϵ is continuously changed from 0 to 1. Note that the system ceases to be constrained stable for values of $\epsilon > 0.92$. For $\epsilon = 0.92$ we obtain:

$$F = \begin{pmatrix} 0.0118 & 0.0079 & -0.2212 & 0.1050 & 0.0009 \\ -0.0230 & -0.0927 & -0.1094 & -0.369 & -0.0418 \end{pmatrix}$$

$$\|A - BF\|_{\mathcal{G}} = 1,\ \mathcal{J}(F) = 0.3286 \cdot 10^3$$

Step 5: Using non-smooth optimization, a local optimum is achieved at:

$$\tilde{F} = \begin{pmatrix} 0.0151 & 0.0036 & -0.2166 & 0.1066 & -0.0013 \\ -0.0286 & -0.0014 & -0.1115 & -0.3675 & -0.0367 \end{pmatrix}$$

which yields $\|A - B\tilde{F}\|_{\mathcal{G}} = 0.9915$. Note that this value is within 1% of the value yielded by the approximate solution of step 4.

V. Conclusions

Most realistic control problems involve both some type of time-domain constraints and certain degree of model uncertainty. However, the majority of control design methods currently available focus only on one aspect of the problem.

In this paper we propose to address this type of problems using a constraint induced operator norm approach. Specifically, we define a robustness measure that indicates how well the family of systems under consideration satisfies a given set of constraints. Then, we use the available degrees of freedom to maximize this robustness measure over the set of static state-feedback controllers that satisfy a given performance specification. In the first part of the paper we cast this problem into the form of a constrained optimization problem. In the second part we propose a simple homotopy-type algorithm to find an approximate solution, which can be further refined, if so desired, by using non-smooth optimization. The proposed algorithm has a number of appealing features. In particular:

- Presents the advantages of modern control techniques, i.e. it is an algorithmic approach, capable of dealing with multivariable systems and guaranteeing optimality in some previously defined sense.
- Provides a systematic approach that deals explicitly with the constraints. Hence we can expect a considerable simplification of the design phase.
- Displays, by means of the robustness measure, the trade-offs between model uncertainty and design constraints. For instance, it indicates when an additional system identification effort is required in the predesign phase in order to satisfy the design constraints.
- Finally, the numerical examples seem to indicate that the approximate solution generated by step 4 is very close to at least a local minimum. This suggest that several design alternatives could be quickly explored using the approximate solution, while leaving the more time consuming non-smooth optimization to refine the final design. However, more research should be carried-out to substantiate this point.

The main drawback of the proposed solution method is the fact that it yields suboptimal solutions. Note that the equivalent LQR problem introduced in Theorem 4 and used in step 2 of the design procedure is not unique. Hence, the use of different equivalent LQR problems leads in principle to different solutions at step 4, which in turn could result in different local minima. As we mentioned before, this problem could be solved using an homotopic continuation method rather than the method proposed here, and keeping track of the bifurcations. However, we expect this approach to be fairly complex. Alternatively, since the transformation used in step 2 of the algorithm is a *local* transformation, the approximate and exact solution are close for small values of the relaxation parameter ϵ. Hence, the proposed method can be improved by alternating sweeping steps with non-smooth optimization and a recalculation of the equivalent LQR problem. Note that this is essentially equivalent to an homotopic continuation method. Research relating to the properties of this modified algorithm is currently been pursued.

Finally, we are currently looking into a method based upon a parametrization, in terms of a stable transfer function Q, of all the dynamic controllers that achieve a specific performance level. Although more complex that the method presented here we expect this alternative to result in convex minimization problems, hence guaranteed to have only a global minimum. However, at this stage we feel that the drawbacks of the method proposed here are offset by its relative simplicity (most of the design is essentially a LQR step which can be carried-out very efficiently with tools commonly available) and by the additional information that it provides about the trade-offs between performance and uncertainty.

Appendix A: Proof of Theorems 2 and 3

The following 2 lemmas are introduced without proof.

- **Lemma 2:** Consider the system (S^*). Assume that the perturbation set D is a *cone with vertex at the origin* [8], i.e. $\Delta^o \in \mathcal{D} \iff \alpha\Delta^o \in \mathcal{D}\ \forall\, 0 \le \alpha$ and that (S^*) is constraint stable (i.e. $\|A\|_{\mathcal{G}} < 1$). Let:

$$\Delta^o = \underset{\Delta\in\mathcal{D}}{\operatorname{argmin}} \{\|\Delta\|_{\mathcal{N}}: \|A+\Delta\|_{\mathcal{G}} = 1\} \tag{A1}$$

and consider a sequence $A^i \to A$ such that $\|A^i\|_{\mathcal{G}} < 1$. Finally, define the sequence λ^i as:

$$\lambda^i = \min_{\lambda\in\Re^+} \{\lambda: \|A^i + \lambda\Delta^o\|_{\mathcal{G}} = 1\} \tag{A2}$$

Then the sequence λ^i has an accumulation point at 1.

- **Lemma 3:** Let $\rho_1 > 0, \rho_2 > 0$ and $0 \le \lambda \le 1$ be given numbers. Consider the following sets:

$$\begin{aligned} \rho_1 B\Delta &= \{\Delta \in \mathcal{D}: \|\Delta\|_{\mathcal{N}} \le \rho_1\} \\ \rho_2 B\Delta &= \{\Delta \in \mathcal{D}: \|\Delta\|_{\mathcal{N}} \le \rho_2\} \\ \rho B\Delta &= \{\Delta \in \mathcal{D}: \|\Delta\|_{\mathcal{N}} \le \rho \stackrel{\Delta}{=} \lambda\rho_1 + (1-\lambda)\rho_2\} \end{aligned} \tag{A3}$$

Then:

$$\rho B\Delta \subseteq \lambda\rho_1 B\Delta + (1-\lambda)\rho_2 B\Delta$$

Proof of Theorem 2

Assume that $\varrho_{\mathcal{G}}^{\mathcal{N}}$ is *not* continuous. Then, given $\epsilon > 0$, for every $\delta > 0$ there exist A_δ such that $\|A_\delta - A\|_{\mathcal{G}} \le \delta$ and $|\varrho_{\mathcal{G}}^{\mathcal{N}}(A_\delta) - \varrho_{\mathcal{G}}^{\mathcal{N}}| > \epsilon$. Furthermore, it is easily seen that the sequence $\varrho_{\mathcal{G}}^{\mathcal{N}^i}$ is bounded. Hence, there exist a sequence $A^i \to A$ such that $\varrho_{\mathcal{G}}^{\mathcal{N}^i} \to \tilde{\varrho} \ne \varrho_{\mathcal{G}}^{\mathcal{N}}$. Let:

$$\Delta^i = \underset{\Delta\in\mathcal{D}}{\operatorname{argmin}} \{\|\Delta\|_{\mathcal{N}}: \|A^i+\Delta\|_{\mathcal{G}} = 1\} \tag{A4}$$

From (A4) it follows that $\|\Delta^i\|_{\mathcal{G}} \le 1 + \|A^i\|_{\mathcal{G}}$. It follows then that the sequence Δ^i is bounded and therefore, since $R^{n\times n}$ with a finite dimensional matrix norm is complete and since $\mathcal{D}$ is a closed set, it has an accumulation point $\tilde{\Delta}$ (Bolzano Weierstrass) and a convergent subsequence $\tilde{\Delta}^i \to \tilde{\Delta}$ such that $\|A + \tilde{\Delta}\|_{\mathcal{G}} = 1$. Furthermore, from the definition of Δ^o it follows that

$$\tilde{\varrho} = \|\tilde{\Delta}\|_{\mathcal{N}} > \|\Delta^o\|_{\mathcal{N}} = \varrho_{\mathcal{G}}^{\mathcal{N}} \tag{A5}$$

Hence, for i large enough,

$$\|\tilde{\Delta}^i\|_{\mathcal{N}} > \|\Delta^o\|_{\mathcal{N}} \tag{A6}$$

Applying Lemma 2, we have that there exist a sequence $\lambda^i \to 1$ such that:

$$\lambda^i = \min_{\lambda\in\Re^+} \{\lambda: \|A^i + \lambda\Delta^o\|_{\mathcal{G}} = 1\} \tag{A7}$$

From (A6) and since $\lambda^i \to 1$ it follows that for i large enough

$$\begin{aligned} \|\lambda^i\Delta^o\|_{\mathcal{N}} &< \|\tilde{\Delta}^i\|_{\mathcal{N}} \\ \|A^i + \lambda^i\Delta^o\|_{\mathcal{G}} &= 1 \end{aligned} \tag{A8}$$

and, since $\mathcal{D}$ is a cone, $\lambda^i\Delta^o \in \mathcal{D}$, which contradicts (A4). The proof is completed by noting that since *all* finite dimensional matrix norms are equivalent (Theorem 5.4.4, [17]) then continuity in the $\|.\|_{\mathcal{G}}$ norm implies continuity in any other norm defined over $R^{n\times n}$ $\diamond$.

Proof of Theorem 3:

Given two matrices F_1 and F_2, consider a convex linear combination $F = \lambda F_1 + (1-\lambda)F_2$. Then:

$$\begin{aligned}\max_{\Delta\in\rho B\Delta} \|A + BF + \Delta\|_G &\leq \max_{\substack{\Delta_1\in\rho_1 B\Delta\\ \Delta_2\in\rho_2 B\Delta}} \|\lambda(A + BF_1 + \Delta_1) \\ &\quad + (1-\lambda)(A + BF_2 + \Delta_2\|_G \\ &\leq \lambda \max_{\Delta_1\in\rho_1 B\Delta} \|A + BF_1 + \Delta_1\|_G \\ &\quad + (1-\lambda) \max_{\Delta_2\in\rho_2 B\Delta} \|A + BF_2 + \Delta_2\|_G\end{aligned} \qquad (A9)$$

Consider now the case where $\rho_1 = \varrho_G^N(F_1)$ and $\rho_2 = \varrho_G^N(F_2)$. Then it follows from the definition of ϱ_G^N that both maximizations in the right hand side of (A9) yield 1 and therefore:

$$\max_{\Delta\in\rho B\Delta} \|A + BF + \Delta\|_G \leq 1 \qquad (A10)$$

Hence, from the definition of ϱ_G^N:

$$\varrho_G^N[\lambda F_1 + (1-\lambda)F_2] \geq \varrho = \lambda\varrho_G^N(F_1) + (1-\lambda)\varrho_G^N(F_2) \diamond$$

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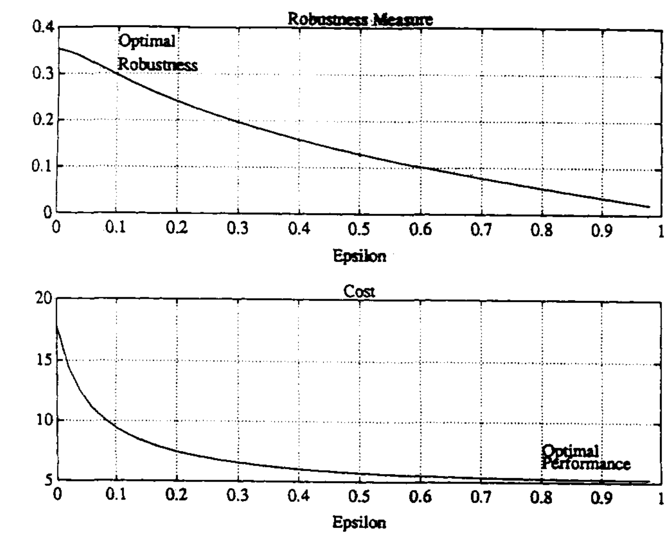

Figure 1: Constrained Robustness Measure and Nominal Performance for Example 1

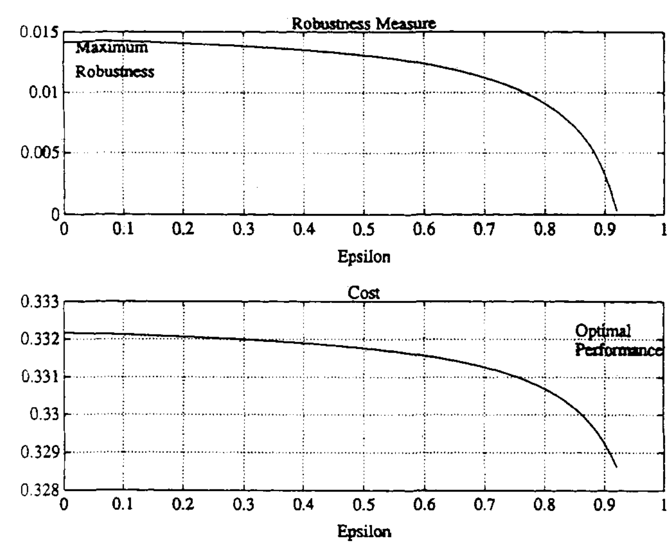

Figure 2: Constrained Robustness Measure and Nominal Performance for Example 2