State-Space Quantization Design for the Suboptimal Control of Constrained Systems Using Neuromorphic Controllers

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Abstract

During the last few years there has been considerable interest in the use of trainable controllers based upon the use of neuron-like elements, with the expectation being that these controllers can be trained, with relatively little effort, to achieve good performance. However, good performance hinges on the ability of the neural net to generate a "good" control law even when the input does not belong to the training set, and it has been shown that neural-nets do not necessarily generalize well. It has been proposed that this problem can be solved by essentially quantizing the state-space and then using a neural-net to implement a table look-up procedure. However, there is little information on the effect of this quantization upon the controllability properties of the system. In this paper we address this problem by extending the theory of control of constrained systems to the case where the controls and measured states are restricted to finite or countably infinite sets. These results provide the theoretical framework for recently suggested neuromorphic controllers but they are also valuable for analyzing the controllability properties of computer-based control systems.

I. Introduction

During the last few years considerable attention has been focused on the use of neural-net based controllers, the expectation being that these controllers can be trained, with relatively little effort, to achieve good performance. In particular, these controllers could be very useful for complex problems that do not admit a closed-form solution. Such is the case of constrained systems. In this case, of considerable practical importance for applications ranging from aerospace to process control, the problem of steering the system from a given initial condition to a desired target set usually does not admit a linear feedback control law as a solution. Therefore, control engineers have to resort to a number of schemas that include (in increasing order of sophistication) switching between several linear controllers, non-linear controllers and on-line optimization based techniques [1-3]. Clearly, a trainable, Neural-Net based controller could provide a welcomed addition to the handful of techniques available for dealing with constrained systems, with the added bonus that such a controller could achieve good performance even in the face of poor or minimal modeling. As an example, we can mention the celebrated neuromorphic controller used by Anderson [4] to control an inverted pendulum when the control force is restricted to have bounded magnitude.

The basic idea justifying the use of Neural Nets as controllers for dynamical systems is that the controller can be trained, for instance by presenting several instances of input-output pairs, to generate a desired output for a given input. The underlying assumption is that the Neural Net has good *generalization properties*, therefore being capable of generating an *appropriate* output even when the input is not a member of the training set. However, it has been shown [5] that Neural Nets *do not* necessarily generalize well. Therefore, it follows that the *asymptotic stability* properties of systems utilizing neuromorphic controllers are generally unknown and this is a major stumbling block preventing their use.

This difficulty can be solved by realizing the fact that the neural-net essentially implements a look-up table, and that generalization can be achieved by discretizing the input vectors and mapping them to a fixed number of "cells" in such a way that inputs that are "close" in some sense get mapped to the same cell [6]. This idea is based on the idea of "boxes" [7] and has been used several times in connection with neuromorphic controllers. However, none of the work available up to date addresses the effects of this "quantization" upon the controllability properties of the system and the question of how to select a cell size that would allow the system to reach a "desirable" target set.

Clearly, the problem of determining whether or not a given cell "size" will allow the system to reach a given target set is similar to the problem of investigating the controllability properties of a constrained system when the available state measurements are quantized, i.e. when the only information available at a given instant is that the state of the system belongs to a given "cell". However, although the theory of control of constrained systems is well known and the original results due to Lee and Marcus [8] on the controllability of systems under control constraints have been extended in a number of ways to account for different types of constraints (see for example [9]), all these extensions always assume that the set of possible control laws is a dense subset of R^n and that the initial condition of the system is perfectly known. Traditionally, quantization effects have been treated by adding noise sources and non-linear quantizers to the system [10]. This type of analysis provides upper bounds on the errors due to quantization effects, but it is not suitable for extending the theoretical results already known for non-quantized systems to the quantized case. In [1] we presented a theoretical framework capable of handling the case where the control is restricted to a finite or countably infinite set, for systems under both state and control constraints. However, these theoretical results assume that the initial condition of the system is known precisely and therefore cannot handle the case of interest here, where the available measurements are also restricted to a finite or countably infinite set.

In this paper we present basic results on the controllability of constrained discrete time systems using quantized controls and measurements, and we indicate how these results could be applied to the problem of designing suboptimal neuromorphic controllers for constrained systems. As we mentioned before, the main motivation for this paper is to provide a theoretical framework for some recently suggested neuromorphic controllers [6]. However, the results presented here also address the need, grown from the increased use of computer-based controllers in recent years, for a general theory of constrained controllability capable of accommodating the quantization effects that may result from the use of a computer in the feedback loop.

The paper is organized as follows: In section II we introduce the concepts of *quantization* and *quantized controllability.* In section III we use these concepts to show that under very general conditions there exist regions of state space containing initial conditions which can be steered to a desired target set (which without loss of generality can be assumed to be a neighborhood of the origin). In section IV we use an example to illustrate the application of these theoretical results to the problem of designing suboptimal controllers based upon a partition of state-space. Finally, in section V, we summarize our results and we indicate directions for future research.

II. Definitions

Before being able to present the basic results on the controllability of constrained systems with quantized states and controls we need to introduce some concepts. We begin by formalizing the concept of "*quantization*".

- **Def. 1:** Consider a closed set $\mathcal{G} \subseteq R^n$. A family $\mathcal{S}$ of closed sets S_i is called a *closed cover* of $\mathcal{G}$ if $\mathcal{G} = \bigcup_i S_i$

- **Def. 2:** Consider a closed set $\mathcal{G} \subseteq R^n$. A *quantization* Q of G is defined as the set:

$$Q = \{\underline{z}_i : \underline{z}_i \in S_i \text{ and } \{S_i\} \text{ is a closed cover of } \mathcal{G}\}$$

- **Def. 3:** Given a quantization Q of a set $\mathcal{G}$, the *size* of the quantization with respect to some norm $\mathcal{N}$ defined in $\mathcal{G}$ is defined as:

$$\frac{1}{s} = \min_i \{r : S_i \subseteq B(\underline{z}_i, r)\ \forall i\}$$

where $B(\underline{z}_i, r)$ indicates the $\mathcal{N}$-norm ball centered at $\underline{z}_i$ and with radius r. A quantization Q with size s will be denoted as Q_s.

Consider now the case where the sets of the family S that defines a quantization Q have pairwise disjoint interiors (i.e. $\text{int}(S_i) \cap \text{int}(S_j) = \emptyset, i \neq j$). In this case, S induces an equivalence relation in $\mathcal{G}$ as follows:

- **Def. 4:** Consider a closed cover S of $\mathcal{G}$ with pairwise disjoint interiors, and two points $\underline{x}_1, \underline{x}_2 \in \mathcal{G}$. $\underline{x}_1$ and $\underline{x}_2$ are *equivalent modulo* S if $\exists\, i$ such that $\underline{x}_1$ and $\underline{x}_2 \in \text{int}(S_i)$. To complete the partition of $\mathcal{G}$ into equivalence classes, we assign the points that are in $S_i \cap S_j$ (i.e. in the common boundary) *arbitrarily* to either one of the classes. Two points equivalent modulo S will be denoted as $\underline{x}_1 \equiv \underline{x}_2$.

- **Def. 5:** Consider a quantization $\chi_s = \{\underline{z}_i\}$ of a given set $\mathcal{G}$. It follows from Definitions 2 and 4 that for any point $\underline{x} \in \mathcal{G}$ there exists an element $\underline{z} \in \chi_s$ such that $\underline{z} \equiv \underline{x}$. We will define the operator that assigns $\underline{x} \to \underline{z}$ as the *quantization* operator and we will denote it as: $\underline{z} = \chi_s(\underline{x})$.

The following definitions deal with the controllability aspects of the problem and, in particular, with the effects of quantizing the state and control spaces. These definitions will become particularly useful in the second portion of the paper, where a particular algorithm based upon a partition of state-space is analyzed. However, they are also useful outside this context, for instance to analyze the effect of using a computer with a finite word-length in the feedback loop.

Consider the linear, time invariant, discrete system modeled by the difference equation:

$$\underline{x}_{k+1} = A\underline{x}_k + B\underline{u}_k, \qquad k = 0, 1, \ldots \tag{1}$$

with initial condition $\underline{x}_o$ and the constraints:

$$\underline{u}_k \in \Omega \subseteq R^m,\ \underline{x}_k \in \mathcal{G} \subseteq R^n \tag{2}$$

where Ω and $\mathcal{G}$ are closed convex regions containing the origin in their interior.

- **Def. 6:** The system (1) is *Control Quantized Null Controllable* in a region $C \subseteq \mathcal{G}$ if, for any open set $O \subseteq \mathcal{G}$ containing the origin in its interior, there exists a number $s_u(C, O) \in R^+$ such that for all the quantizations Ω_s of Ω with $s \geq s_u$, there exists a sequence of admissible quantized controls $\underline{u}_k \in \Omega_s$ such that the system can be steered from any initial condition $\underline{x}_o \in C$ to O without violating the state constraints.

Consider now the case where state-space is quantized. This situation can be modeled by assuming that rather than having a precisely known initial condition, the only knowledge available is that the initial condition belongs to a given set S_j from a closed cover $\{S_i\}$. This concept is formalized in the next definition.

- **Def. 7:** The system (1) is *State Quantized Null Controllable* in a region $C \subseteq \mathcal{G}$ if, for any open set $O \subseteq \mathcal{G}$ containing the origin in its interior, there exists a number $s_x(C, O) \in R^+$ such that for all the quantizations χ_s of $\mathcal{G}$ with $s \geq s_x$ and for any initial condition $\underline{x}_o \in C$, there exist a *finite* number n, a sequence of admissible controls $\underline{u}_k \in \Omega$, $k = 1, 2 \ldots n$, a point $\underline{z}_n \in O$, and a sequence $\{\underline{z}_k\}$, $\underline{z}_k \in \mathcal{G}$ such that $\underline{x}_k \equiv \underline{z}_k$, $k = 0, 1 \ldots n$, *when* the only information available about the initial condition is that it $\underline{x}_o \equiv \underline{z}_o$, where $\underline{z}_o$ is a given point of the quantization under consideration.

Finally, we consider the case where both state and control space are quantized.

- **Def. 8:** The system (1) is *Completely Quantized Null Controllable* if there exists a number s_o such that (1) is state quantized null controllable *when* the controls are restricted to a quantization Ω_s of Ω with size $s \geq s_o$.

- **Remark:** Note that the situation where both the states and the controls are quantized is particularly important for the case of neuromorphic controllers since, in addition to the state space quantization induced by the "cell" structure, a finite set of control actions is usually required by the controller learning algorithm.

Following previous work in this area [1-3], we proceed now to introduce a restriction on the class of constraints allowed in our problem. As it will become apparent latter, the introduction of this restriction, termed the *constraint qualification hypothesis*, while not affecting significantly the number of real world problems that can be handled by our formalism, introduces more structure into the problem. This additional structure will become essential in showing constrained controllability.

Constraint Qualification Hypothesis

In this paper, we will limit ourselves to constraints of the form:

$$\underline{x} \in \mathcal{G} \triangleq \{\underline{x} \in R^n : (G\underline{x})_i \le \omega_i,\ i = 1 \dots p\}$$

where $G: R^n \to R^p$ is a positive definite, sublinear function, i.e. it has the following properties:

$$\begin{aligned} G(\underline{x})_i &\ge 0,\ i = 1 \dots p\ \forall\, \underline{x} \\ G(\underline{x}) = 0 &\iff \underline{x} = 0 \\ G(\underline{x} + \underline{y})_i &\le G(\underline{x})_i + G(\underline{y})_i,\ i = 1 \dots p\ \forall\, \underline{x}, \underline{y} \\ G(\lambda \underline{x}) &= \lambda G(\underline{x}),\ 0 \le \lambda \le 1 \end{aligned} \tag{3}$$

As examples of constraints that satisfy these conditions we can mention polyhedral and hyperellipsoidal regions.

III. Theoretical Results

In this section we present the basic theoretical results on the null controllability of quantized discrete–time systems. We begin by showing that, under the constraint qualification hypothesis, $G(.)$ induces a norm in $\mathcal{G}$. This norm will be used to find sufficient conditions for quantized null controllability.

- **Lemma 1:** Let:

$$\begin{aligned} v(x) &= \max_{1 \le i \le p} \left\{ \frac{G(\underline{x})_i}{\omega_i} \right\} \\ &= \|W^{-1} G(\underline{x})\|_\infty \triangleq \|\underline{x}\|_{\mathcal{G}} \end{aligned} \tag{4}$$

where $W = diag(\omega_1, \dots, \omega_m)$. Then $v(.)$ defines a norm in R^n and the set $\mathcal{G}$ can be characterized also as:

$$\mathcal{G} = \mathcal{B}(G, \underline{\omega}) \triangleq \{\underline{x} : \|\underline{x}\|_{\mathcal{G}} \le 1\} \tag{5}$$

Proof: The proof follows by noting that the constraint qualification hypothesis implies that $\|.\|_{\mathcal{G}}$ satisfies the condition for a norm.

In the next theorem we consider *control quantized null controllability*.

- **Theorem 1:** Let $\mathcal{G} = \mathcal{B}(G, \underline{\omega})$, where G verifies the constraint qualification hypothesis (3). If:

$$\min_{\underline{u} \in \Omega} \{\|A\underline{x} + B\underline{x}\|_{\mathcal{G}}\} < 1\ \forall\, \underline{x}:\ \|\underline{x}\|_{\mathcal{G}} = 1 \tag{6}$$

then, the system (1) subject to the constraints (2) is control quantized null controllable in $\mathcal{G}$. The proof of the theorem is a straightforward extension of corollary 7-1 in [2].

Condition (6) implies that for any initial condition in the boundary of the admissible region, there exists at least one control that brings the system to its interior. It follows that if the problem of controlling the system (1) to the origin without exceeding the constraints (2) is feasible (as it should be in a well posed problem), then the only effect of condition (6) is to rule out the possibility of the system staying on the boundary for consecutive sampling intervals.

In the next lemmas we introduce a quantity (Λ) that gives a measure of the *maximum amount* that the norm of the present state of the system ($\|\underline{x}\|_{\mathcal{G}}$) can be decreased in one stage. This quantity will be used in Lemma 3 to find an upper bound on the size of the quantizations that guarantee controllability to a given set O.

- **Lemma 2:** Let O be an open set containing the origin it its interior and consider the region $\mathcal{G} - O$. Let:

$$\Lambda = \min_{\underline{x} \in \mathcal{G} - O} \{\lambda : (\frac{1}{\lambda}\underline{x}) \in \partial\mathcal{G}\} \tag{7}$$

where $\partial\mathcal{G}$ denotes the boundary of the set $\mathcal{G}$. Then:

$$\min_{\underline{x} \in \mathcal{G} - O} \{\|\underline{x}\|_{\mathcal{G}} - \min_{\underline{u} \in \Omega} \|A\underline{x} + B\underline{u}\|_{\mathcal{G}}\} > \Lambda \min_{\underline{y} \in \partial\mathcal{G}} \{1 - \min_{\underline{u} \in \Omega} \|A\underline{y} + B\underline{u}\|_{\mathcal{G}}\} \tag{8}$$

Proof: Given any $\underline{x} \in \mathcal{G} - O$ it can be expressed as $\lambda_o \underline{y}_o$ with $\underline{y}_o \in \partial\mathcal{G}$ and $0 < \lambda_o \le 1$. Then:

$$\begin{aligned} \|\underline{x}\|_{\mathcal{G}} - \min_{\underline{u} \in \Omega} \|A\underline{x} + B\underline{u}\|_{\mathcal{G}} &= \|\lambda_o \underline{y}_o\|_{\mathcal{G}} - \min_{\underline{u} \in \Omega} \|A\lambda_o \underline{y}_o + B\underline{u}\|_{\mathcal{G}} \\ &\ge \|\lambda_o \underline{y}_o\|_{\mathcal{G}} - \min_{\underline{u} \in \Omega} \|A\lambda_o \underline{y}_o + B\lambda_o \underline{u}\|_{\mathcal{G}} \\ &= \lambda_o \{\|\underline{y}_o\|_{\mathcal{G}} - \min_{\underline{u} \in \Omega} \|A\underline{y}_o + B\underline{u}\|_{\mathcal{G}}\} \\ &\ge \min_{\substack{\underline{y} \in \partial\mathcal{G} \\ \lambda\underline{y} \in \mathcal{G} - O}} \lambda \{\|\underline{y}\|_{\mathcal{G}} - \min_{\underline{u} \in \Omega} \|A\underline{y} + B\underline{u}\|_{\mathcal{G}}\} \\ &\ge \min_{\substack{\underline{y} \in \partial\mathcal{G} \\ \lambda\underline{y} \in \partial O}} \lambda \{\|\underline{y}\|_{\mathcal{G}} - \min_{\underline{u} \in \Omega} \|A\underline{y} + B\underline{u}\|_{\mathcal{G}}\} \\ &\ge \Lambda \min_{\underline{y} \in \partial\mathcal{G}} \{1 - \min_{\underline{u} \in \Omega} \|A\underline{y} + B\underline{u}\|_{\mathcal{G}}\} \end{aligned} \tag{9}$$

since $\|\underline{y}\|_{\mathcal{G}} = 1$ for $\underline{y} \in \partial\mathcal{G}$ $\diamond$.

- **Remark:** Note that in this lemma we consider a "worst-case" type situation by essentially considering a ray from the origin to the boundary of the constraint set, $\partial\mathcal{G}$, and then, in the last inequality, decoupling the scale factor (Λ) from the orientation. Note also that Λ is a strictly positive function of the target set O which decreases as O gets "smaller".

- **Lemma 3:** Let $\mathcal{G} = \mathcal{B}(G, \underline{\omega})$, where G satisfies (3) and let O be an open set containing the origin in its interior. If

$$1 - \max_{\|\underline{x}\|_{\mathcal{G}} = 1} \left\{ \min_{\underline{u} \in \Omega} \{\|A\underline{x} + B\underline{u}\|_{\mathcal{G}}\} \right\} = \delta < 1 \tag{10}$$

then, for any quantization $\chi_s = \{\underline{z}_i\}$ of $\mathcal{G}$ with size $s \ge s_o \triangleq \frac{1 + \|A\|_{\mathcal{G}}}{\delta\Lambda}$ and for any point $\underline{z}_o \in \mathcal{G} - O$ such that $\underline{z}_o = \chi_s(\underline{x}_o) \in \mathcal{G} - O$, there exists an admissible control $\underline{u}_o \in \Omega$ such that $\|\underline{z}_1\|_{\mathcal{G}} < \|\underline{z}_o\|_{\mathcal{G}}$ where:

1) $\underline{z}_1 = \chi_s(A\underline{x}_o + B\underline{u}_o)$,

2) χ_s is the quantization operator introduced in Definition 5, and

3) $\|A\|_{\mathcal{G}}$ denotes the induced operator norm (i.e. $\|A\|_{\mathcal{G}} = \max_{\|\underline{x}\|_{\mathcal{G}} = 1} \|A\underline{x}\|_{\mathcal{G}}$).

Proof: From the hypothesis and Lemma 2 it follows that:

$$\max_{\underline{u}_o \in \Omega}\{\|\underline{z}_o\|_{\mathcal{G}} - \|A\underline{z}_o + B\underline{u}\|_{\mathcal{G}}\} = \|\underline{z}_o\|_{\mathcal{G}} - \min_{\underline{u}\in\Omega}\{\|A\underline{z}_o + B\underline{u}\|_{\mathcal{G}}\}$$
$$\geq \Lambda\left\{1 - \max_{\underline{x}\in\mathcal{G}}\left\{\min_{\underline{u}\in\Omega}\|A\underline{x} + B\underline{u}\|_{\mathcal{G}}\right\}\right\} = \Lambda\delta \quad (11)$$

Define:

$$\underline{u}_o = \underset{\underline{u}\in\Omega}{\operatorname{argmin}}\{\|A\underline{z}_o + B\underline{u}_o\|_{\mathcal{G}}\}$$
$$\underline{x}_1 = A\underline{x}_o + B\underline{u}_o \quad (12)$$
$$\underline{z}_1 = \chi_s(\underline{x}_1) \triangleq \underline{x}_1 + \delta\underline{x}_1,\ \|\delta\underline{x}_1\|_{\mathcal{G}} \leq \frac{1}{s}$$

Then:

$$\begin{aligned}\|\underline{z}_o\|_{\mathcal{G}} - \|\underline{z}_1\|_{\mathcal{G}} &= \|\underline{z}_o\|_{\mathcal{G}} - \|A\underline{x}_o + B\underline{u}_o + \delta\underline{x}_1\|_{\mathcal{G}} \\ &= \|\underline{z}_o\|_{\mathcal{G}} - \|A\underline{z}_o + B\underline{u}_o - A\delta\underline{x}_o + \delta\underline{x}_1\|_{\mathcal{G}} \\ &\geq \|\underline{z}_o\|_{\mathcal{G}} - \|A\underline{z}_o + B\underline{u}_o\|_{\mathcal{G}} - \|A\delta\underline{x}_o\|_{\mathcal{G}} - \|\delta\underline{x}_1\|_{\mathcal{G}} \\ &\geq \|\underline{z}_o\| - \|A\underline{z}_o + B\underline{u}_o\|_{\mathcal{G}} - \|A\|_{\mathcal{G}}\|\delta\underline{x}_o\|_{\mathcal{G}} - \|\delta\underline{x}_1\|_{\mathcal{G}} \\ &\geq \|\underline{z}_o\|_{\mathcal{G}} - \|A\underline{z}_o + B\underline{u}_o\|_{\mathcal{G}} - \left(\frac{1+\|A\|_{\mathcal{G}}}{s}\right) \\ &\geq \Lambda\delta - \left(\frac{1+\|A\|_{\mathcal{G}}}{s}\right)\end{aligned} \quad (13)$$

Hence, if

$$s > \frac{1+\|A\|_{\mathcal{G}}}{\Lambda\delta} \quad (14)$$

then

$$\|\underline{z}_1\|_{\mathcal{G}} - \|\underline{z}_o\|_{\mathcal{G}} = \mu < 0 \diamond \quad (15)$$

In the next theorem we use the results of Lemma 3 to show that condition (10) is a *sufficient* condition for state–quantized null controllability.

- **Theorem 2:** Let $\mathcal{G} = \mathcal{B}(G,\underline{\omega})$, where G satisfies (3). Then, (10) is a sufficient condition for the system (1) subject to the constraints (2) to be state–quantized null controllable in $\mathcal{G}$.

  Proof: To show state–quantized null controllability, we have to show that for any open set $O \subseteq \mathcal{G}$ containing the origin in its interior, there exists a number s_o such that for all the quantizations χ_s of $\mathcal{G}$ with size $s \geq s_o$, and for any initial condition $\underline{x}_o \in \mathcal{G}$, there exists a sequence of admissible control laws $\mathcal{U} = \{\underline{u}_o, \underline{u}_1 \ldots \underline{u}_n\}$, where n is a *finite* number, such that:

  $$\underline{z}_k = \chi_s(\underline{x}_k) \in \mathcal{G},\ k = 0,1\ldots n$$
  $$\underline{z}_n \in O \quad (16)$$

  Define $s_o \triangleq (\frac{1+\|A\|_{\mathcal{G}}}{\delta\Lambda})$ and consider an arbitrary quantization χ_r with $r \geq s_o$. Let $\underline{x}_o$ be an arbitrary initial condition in $\mathcal{G} - O$. From the definition of quantization, it follows that there exists $\underline{z}_o \in \chi_r$ such that $\underline{x}_o \equiv \underline{z}_o$. Obviously, if $\underline{z}_o \in O$ the theorem is trivial, so lets consider the case where $\underline{z}_o \notin O$. Then, from Lemma 3 it follows that, as long as $\underline{z}_k \notin O$, there exists a sequence $\mathcal{U} = \{\underline{u}_o, \underline{u}_1 \ldots\}$ such that:

  $$\begin{aligned}\|\underline{z}_2\|_{\mathcal{G}} &< \|\underline{z}_1\|_{\mathcal{G}} + \mu \\ \|\underline{z}_3\|_{\mathcal{G}} &< \|\underline{z}_2\|_{\mathcal{G}} + \mu \\ &\vdots \\ \|\underline{z}_m\|_{\mathcal{G}} &< \|\underline{z}_{m-1}\|_{\mathcal{G}} + \mu\end{aligned} \quad (17)$$

  where $\mu < 0$ and $\underline{z}_i = \chi_r(\underline{x}_i) = \chi_r(A\underline{x}_{i-1} + B\underline{u}_{i-1})$. It follows then that there exists n_o such that $\underline{z}_{n_o} \in O \diamond$.

Finally, in the next theorem we show that (10) is a necessary condition for complete quantized null controllability.

- **Theorem 3:** Let $\mathcal{G} = \mathcal{B}(G,\underline{\omega})$, where G satisfies (3). Then, (10) is a sufficient condition for the system (1) subject to the constraints (2) to be completely–quantized null controllable in $\mathcal{G}$.

  Proof: Since $\|B\underline{u}\|_{\mathcal{G}}$ is a continuous function of $\underline{u}$ it follows that there exists r such that $\|B\delta\underline{u}\|_{\mathcal{G}} \leq \frac{\Lambda\delta}{2}$ for all $\delta\underline{u} \in B(0,r) \subseteq \Omega$ where $B(0,r)$ denotes a ball in some arbitrary norm defined in Ω. The proof follows now from the proof of Theorem 2 by substituting $\frac{\Lambda\delta}{2}$ for $\Lambda\delta\ \diamond$.

- **Corollary:** The size of the quantization introduced in Theorems 2 and 3 is inversely proportional to Λ. Hence, as the size of the target set gets smaller, the number of cells increases, while their size decreases. However, note that the target set O is achieved through a sequence of *intermediate* sets $O_i, i = 1,2\ldots,n$ with $O_1 \equiv \mathcal{G}$ and $O_n \equiv O$. Since Λ in (7) can be thought of as a lower bound of the ratio of the norms of the next state of the system and the present state, it follows that to guarantee complete quantized null controllability, it is enough to choose:

  $$\Lambda = \max_i \Lambda_i$$
  $$\Lambda_i = \min_{\underline{x}\in O_i - O_{i+1}}\left\{\lambda : (\frac{1}{\lambda}\underline{x}) \in \partial O_i\right\} \quad (18)$$

- **Remark:** From equation (18) it follows that if the sequence of intermediate stages O_i is chosen so that $\Lambda = \Lambda_i\ \forall i$ (i.e. the sets O_i all have the same "shape") then the number of cells in each set roughly decreases as Λ^n. Alternatively, using the same number of cells at each stage results in a "retina" like structure, having coarser resolution far from the target set and increasingly finer resolution closer to the target. Note that this increased resolution could be achieved essentially having only one set of boxes, whose function adaptively changes with the state of the system.

In this section we presented the basic results on the controllability of constrained discrete time systems when the available state measurements and perhaps also the controls are restricted to finite or countably infinite sets. In the next section we show how to apply these results to an optimal control problem.

IV. Applications to Suboptimal Controllers Design

As a example of the potential applications of our theory to the optimal control of constrained systems, we will use it to address the problem of determining a "cell size" that guarantees controllability to a given target set. Since in this case the quantization of state–space is introduced as an artifact to simplify the search for an optimal trajectory, we will assume that any hardware imposed quantization effects are negligible.

Once a lower bound s_o on the size of the quantizations that guarantee controllability to the desired target set is determined, a suboptimal controller can be implemented as a *table look–up schema* by essentially finding and storing an optimal control law associated with each cell. Moreover, since neural–nets are known to implement table look–up schemas very efficiently [6], it follows that this suboptimal control law could be implemented successfully with a neuromorphic controller, *without* any assumptions on the generalization properties of the neural–net and with *guaranteed* asymptotic stability of the closed–loop system. This idea is formalized in the following conceptual algorithm:

Algorithm L (Optimal control using a Look–up table)

Begin.

1) Determine a lower bound s_o on the size of the quantizations that guarantee controllability to the target set O using (14) and (18). Alternatively, determine the number of boxes to be used with changing resolution, as discussed in the corollary to Theorem 3.

2) Choose a pairwise interior–disjoint closed cover $S = \{S_i\}$ of $\mathcal{G}$ with size $s \geq s_o$. Form a quantization $\chi_s = \{\underline{z}_i\}$ by selecting one representative element from each equivalence class.

3) For each element $\underline{z}_i \in \chi_s$ find the optimal (in some previously defined sense) control law $\underline{u}_i^o$ and store it.

4) While $\underline{z}_i = \chi_s(\underline{x}_i) \notin O$ use as the next control law, the control law associated with $\underline{z}_i$

End.

Next, we show how to apply Algorithm L to a simple example. Since in this paper we are concentrating on the theoretical controllability issue, we will assume that a table look–up procedure is available. The issues concerning the implementation of this procedure by means of a Neural–Net, too extensive to consider here, are left for a future article on the subject.

- *A Simple Example*

Consider the spinning space station with a single axis of symmetry problem [3, 11] The station is controlled by means of a single jet placed on the body and allowed to rotate to any angle in a plane normal to the symmetry axis. Select as state variables the angular velocities around a pair of axes perpendicular to the symmetry axis and assume that the goal is to bring the states from an initial condition $\underline{x}_o$, $\|\underline{x}_o\|_2 = R_x$ to a final state such that $\|\underline{x}_f\|_2 \leq R_f$. This situation can model the case where a sophisticated, non–conventional controller is used to bring a system in minimum time to some region (for instance a region where the constraints are not binding) where some relatively easy to design controller can take over. In the chosen reference frame the system can be represented by:

$$\underline{x}_{k+1} = A\underline{x}_k + B\underline{u}_k$$

with:

$$A = \begin{pmatrix} \cos T & \sin T \\ -\sin T & \cos T \end{pmatrix} \; B = \begin{pmatrix} \sin T & (1-\cos T) \\ (\cos T - 1) & \sin T \end{pmatrix}$$

$$\mathcal{G} = \{\underline{x} \in R^2 : \|\underline{x}\|_2 \leq R_x\} \; \Omega = \{\underline{u} \in R^2 : \|\underline{u}\|_2 \leq 1\} \tag{19}$$

where T is the sampling interval. In this case A is an orthogonal matrix and $BB^T = \alpha^2 I$ where $\alpha^2 = 2(1-\cos T)$, therefore we have:

$$\begin{aligned} \|\underline{x}_{k+1}\|_2^2 &= \|\underline{x}_k\|_2^2 + 2\underline{x}^T A^T B\underline{u} + \underline{u}^T B^T B\underline{u} \\ &= \|\underline{x}_k\|_2^2 + 2\underline{x}^T B^T \underline{u} + \alpha^2 \|\underline{u}\|_2^2 \end{aligned} \tag{20}$$

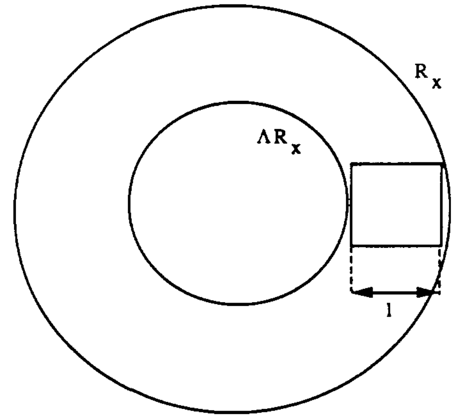

Figure 1: Selecting the Size of the Cells

Hence, by selecting:

$$\underline{u} = \frac{-B\underline{x}_k}{\|B\underline{x}_k\|_2} \tag{21}$$

we have:

$$\begin{aligned} \|\underline{x}_{k+1}\|_2^2 &= \|\underline{x}_k\|_2^2 - 2\alpha\|\underline{x}_k\|_2 + \alpha^2 \\ &= (\|\underline{x}_k\|_2 - \alpha)^2 \end{aligned} \tag{22}$$

From (22) it follows that:

$$\begin{aligned} \delta &= \|\underline{x}_k\|_G - \|\underline{x}_{k+1}\|_G \\ &= \frac{1}{R_x}(\|\underline{x}_k\|_2 - \|\underline{x}_{k+1}\|_2) = \frac{\alpha}{R_x} \end{aligned} \tag{23}$$

Since A is an orthogonal matrix, and since in this case $\|.\|_G$ is simply the euclidian norm scaled by R_x it follows that $\|A\|_G = 1$. Hence, from (14) we have that:

$$\frac{1}{s} \leq \frac{1}{2}\frac{\alpha R_f}{R_x^2} \tag{24}$$

Assume that we want to use a covering formed by square boxes of side l. Then, by choosing the center of each box as the representative element we have that:

$$S_i(l) \subseteq B(\underline{z}_i, \frac{1}{s})_G = B(\underline{z}_i, \frac{R_x}{s}) \iff l = \frac{R_x}{s}\sqrt{2} = \frac{\alpha R_f}{\sqrt{2}R_x} = \frac{\alpha\Lambda}{\sqrt{2}} \tag{25}$$

Moreover, since the norm of the present state of the system can be decreased at each stage by α (in the region $\|\underline{x}\|_2 \geq \alpha$) from Theorem 3 and its corollary it follows that l should be selected (see figure 1) such that:

$$\begin{aligned} l &\leq \alpha \\ l &\leq R_x - \Lambda R_x = R_x(1-\Lambda) \end{aligned} \tag{26}$$

Hence, the region $\|\underline{x}\|_2 \leq \alpha$ (which is the region where the constraints are *not* binding) can be reached, with a *degree of stability* Λ, by using a quantization such that:

$$\frac{\alpha\Lambda}{\sqrt{2}} = R_x(1-\Lambda) \tag{27}$$

or

$$\Lambda = \frac{1}{1+\frac{\alpha}{R_x\sqrt{2}}} \tag{28}$$

In our case a sampling time $T = 2.5$ seconds and a value of $R_x = 20$ yield:

$$\alpha = 1.898 \; \Lambda = 0.937 \; l = 1.258 \tag{29}$$

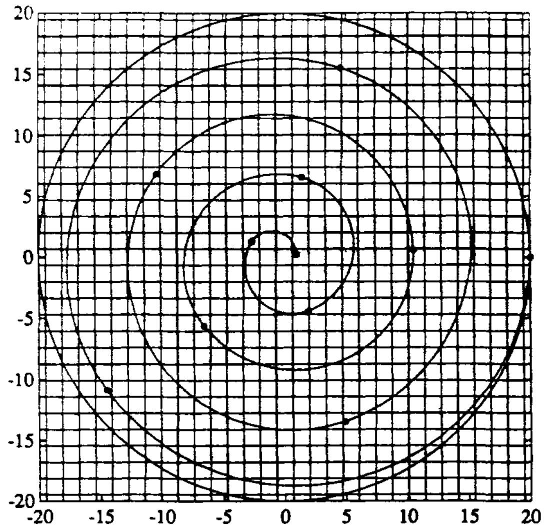

Figure 2: Time Optimal (+) and Quantized (o) Trajectories for the Simple Example

Figure 2 shows the results of applying the algorithm to the system with initial condition $\underline{x}_o^T = (20.0\ 0.0)$. Since in this particular case the time–optimal control law has an explicit expression, we simulated the table look-up by computing at each instance the optimal control law associated with the center of the box that contains the present state of the system. Note the proximity between the quantized and true time–optimal trajectories, indicated respectively by "o" and "+". This proximity suggests that the results of Theorem 2 are overly conservative. In fact, experimenting with this problem we have obtained convergence to the region $\|\underline{x}\|_2 \leq \alpha$ even when $l = \sqrt{2}\alpha$ (the *largest* l such that at least one square box will fit entirely within the target set).

V. Conclusions

During the last few years, there has been considerable interest in the use of trainable controllers based upon the use of neuron like elements. These controllers can be trained, for instance by presenting several instances of *"desirable"* input-output pairs, to achieve good performance, even in the face of poor or minimal modeling. However, the use of neuromorphic controllers has been hampered by the facts that good performance hinges on the ability of the neural-net to generalize the input–output mapping to inputs that are not part of the training set. Through examples [5], it has been shown that neural-nets *do not* necessarily generalize well. Therefore, it follows that the *stability* properties of the closed-loop system are unknown. Moreover, it is conceivable that poor generalization capabilities may result in limit cycles or even in destabilizing control laws. In this paper we address these problems by proposing a neural-net based controller that results in a schema similar to tabular control and then carefully investigating the properties of such a controller. Perhaps the most valuable contribution of this paper results from the qualitative aspects of equation (14), that identify the factors that affect any controller based upon the quantization of state–space (independently of the specific implementation of the look–up schema). Most notably, through the norm of the operator that appears in (14), it is possible to formalize the idea of *"poor"* modeling and to design a *"robust"* controller capable of accommodating modeling errors and disturbances.

There are several questions that remain open. Since one of the main reasons for using neural-net based controllers is their ability to yield good performance with imperfect models, the robustness of these controllers to plant perturbations should be investigated. At this point we are working in a neural-net implementation of the ideas presented in this paper and we are investigating their robustness properties. Future articles are planned to report the results of this line of research. Finally, as we noted in the paper, the results of Theorem 2 that guarantee quantized null controllability can be overly restrictive in some cases, since they result from a "worst-case" type analysis. A relaxed version of these conditions will be highly desirable.

VI. References

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