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$\mathcal{H}_2$ Control with Time Domain Constraints: Theory and an Application. [1]

Mario Sznaier,
Department of Electrical Engineering,
The Pennsylvania State University, University Park, PA 16802
email msznaier@frodo.ee.psu.edu

Abstract

In this paper we study the problem of minimizing the $\mathcal{H}_2$ norm of a given transfer function subject to time–domain constraints on the time response of a different transfer function to a given test signal. The main result of the paper shows that this problem admits a minimizing solution in $\overline{\mathcal{RH}_2}$. Moreover, rational solutions with performance arbitrarily close to optimal can be found by constructing families of approximating problems. Each one of these problems entails solving a finite–dimensional quadratic programming problem whose dimension can be determined before hand. These results are illustrated and experimentally validated by designing a controller for an active vision application.

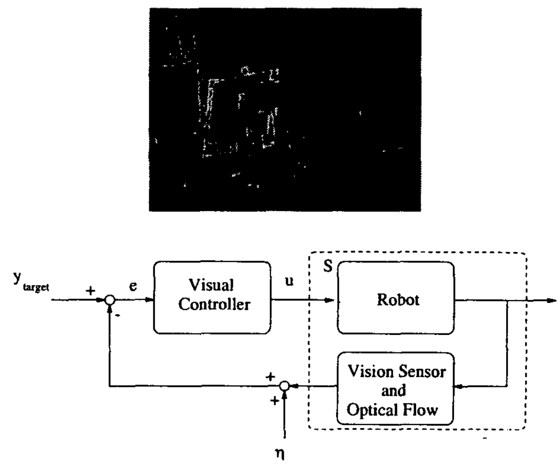

Figure 1: (Top) The experimental setup, (Bottom) Block diagram of a visual tracking system.

1 Introduction

In many cases the objective of a control system design can be stated simply as synthesizing an internally stabilizing controller that minimizes the response to some given, fixed exogenous inputs [2, 11, 16, 17, 6, 3, 1].

In general, a realistic control problem is likely to involve specifications on both the energy and peak values of the output. Consider for example the problem of smooth tracking of a non–cooperative target, illustrated in the block diagram shown in Figure 1 (b). Here the goal is to internally stabilize the plant and to track target motions, y_{target}, using as measurements images possibly corrupted by noise.

Figure 2 shows experimental results obtained with an optimal $\mathcal{H}_2$ controller for a step displacement of the target of 25 pixels. Note that the tracking error settles to ± 4 pixels (within the experimental measurement error) in approximately one second. However, the control action has large oscillations, leading to jerky motions that create significant stress on the pan and tilt unit. Our goal is to design a controller that substantially decreases the peak value of the control action and the oscillations in the error response, while achieving comparable tracking performance in terms of the RMS value of the error.

LQR control subject to input constraints has been addressed in [4, 18] using ellipsoidal invariant sets. However, these methods are potentially conservative, due to the choice of invariant sets and are restricted to the state feedback case. Alternatively, these problems can be addressed using receding horizon type methods ([13, 14, 7]). However, stability considerations require the on–line solution of a constrained optimization problem, which limits the applicability of the method in situations like the one above, with relatively fast sampling times (33 ms). $\mathcal{H}_2$ control problems with time–domain constraints

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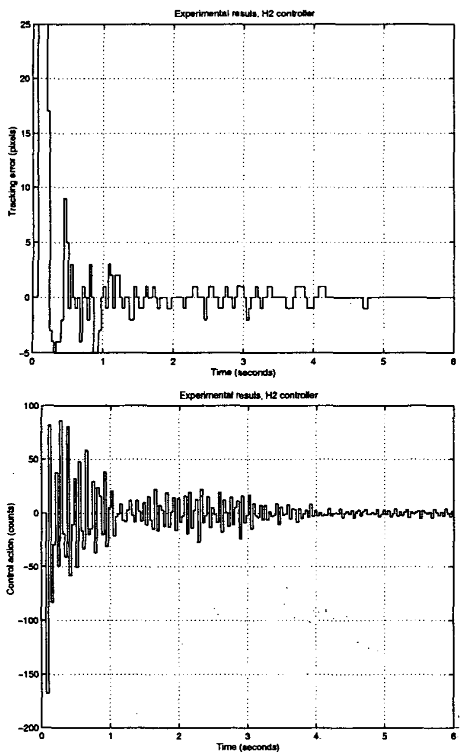

Figure 2: (Top) Tracking error to a step input (experimental) (Bottom) Control action.

can be addressed by recasting them into a mixed $\mathcal{H}_2/\ell^1$ optimization form and elegantly solved using the methods proposed in [9]. However, this is a worst–case type approach that guarantees satisfaction of the time–domain constraints for all signals in the ℓ^∞–unit ball. Thus, these controllers are potentially very conservative for applications such as the active vision problem discussed above, where the specifications are given in terms of the response to a few test signals, representing the typical patterns of motion of the target.

In this paper, motivated by the results in [10] we propose a solution to MIMO discrete time $\mathcal{H}_2$ problems subject to time domain constraints given in terms of the response to a set of fixed, given signals. The main result shows that these problems can be solved, with arbitrary precision, using Quadratic Programming.

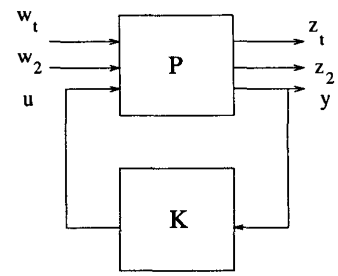

Figure 3: The $\mathcal{H}_2$ with time domain constraints setup

2 Preliminaries

2.1 The $\mathcal{H}_2$ with time domain constraints problem

Consider the system shown in Figure 3, where the signals $w_t \in R^{n_{w_t}}$ and $w_2 \in R^{n_{w_2}}$ represent known test signals and exogenous disturbances, respectively, and where $z_t \in R^{n_{z_t}}$ and $z_2 \in R^{n_{w_2}}$ represent regulated outputs. Our goal is to find an internally stabilizing control law $u = Ky$, $u \in R^{n_u}$, $y \in R^{n_y}$ that minimizes the $\mathcal{H}_2$ norm of the closed loop transfer function from w_2 to z_2, subject to time domain constraints on the response of some of the elements of z_t to test signals $w_t \in \mathcal{W}_t$, of the form:

$$|z_{t_i}(k)| \leq \phi_i(k)$$

where $\{\phi_i(k)\}$ is a given ℓ^∞ sequence. A typical choice for $\phi_i(.)$ is

$$\begin{aligned} \phi(k) &= M, \; k = 0, 1, \ldots, k_1 \\ \phi(k) &= Ma^{(k-k_1)}, \; k_1 \leq k, \; 0 < a < 1 \end{aligned} \quad (2\text{-}1)$$

This sequence imposes constraints on the maximum overshoot (M) and forces exponential decay of the output after time k_1.

In the sequel we will assume without loss of generality (by using superposition if necessary) that the test signals in the set $\mathcal{W}_t$ are of the form $w_t^j(k) = [0 \;\; 0 \;\; \ldots \;\; w_j(k) \;\; \ldots \;\; 0]^T$. Moreover, by using weighting functions and absorbing these weights in the generalized plant (see [16] for details)) it can also be assumed that $w_j(k)$ is an impulse.

Let $T(\lambda)$ and $S(\lambda)$ denote the closed loop transfer matrices from w_2 to z_2 and from w_t to z_t respectively, obtained when connecting a stabilizing controller from y to u. Using the Youla Parameterization, the set of all such transfer matrices can be parameterized by:

$$\begin{aligned} T &= T^{11} + T^{12} Q T^{21} \\ S &= S^{11} + S^{12} Q S^{21} \end{aligned} \quad (2\text{-}2)$$

where $Q \in \mathcal{H}_2^{n_u \times n_y}$, $T^{11} \in \ell_1^{n_{z_2} \times n_{w_2}}$, $T^{12} \in \ell_1^{n_{z_2} \times n_u}$, $T^{21} \in \ell_1^{n_y \times n_{w_2}}$, $S^{11} \in \ell_1^{n_{z_t} \times n_{w_t}}$, $S^{12} \in \ell_1^{n_{z_t} \times n_u}$, and $S^{21} \in \ell_1^{n_y \times n_{w_t}}$. Moreover, by suitable selecting the parametrization, without loss of generality it can be assumed that the transfer matrices T^{ij} and S^{ij} are analytic inside the disk $|\lambda| \leq \frac{1}{a} < \rho$. In order to stress the dependence on Q, the notations $T(Q)$ and $S(Q)$ are sometimes used in the sequel.

The parameterization allows for precisely stating the $\mathcal{H}_2$ with time–domain constraints problem as:

Problem 1 *Given sequences $\{\phi_{ij}(k)\}$ of the form (2-1), find the optimal value of the performance measure:*

$$\mu \doteq \inf_{Q \in \mathcal{H}_2^{n_u \times n_y}} \|T^{11} + T^{12} Q T^{21}\|_{\mathcal{H}_2}^2 \quad (2\text{-}3)$$

subject to

$$\begin{aligned} &\|S(Q)_{ij}\|_{\phi_{ij},\infty} \doteq \|\tfrac{S(Q,k)_{ij}}{\phi_{ij}(k)}\|_{\ell^\infty} \leq 1 \\ &k = 0, 1, 2, \cdots, \ \{ij\} \in \mathcal{I} \end{aligned} \quad (2\text{-}4)$$

and the corresponding controller Q_{opt}, where $\mathcal{I}$ denotes the set of input–output pairs subject to time domain constraints.

In the sequel we solve Problem 1 by constructing sequences of super and sub–optimal controllers, $\{\underline{Q^i}\}$ and $\left\{\overline{Q^i}\right\}$, such that $\|T(\underline{Q^i})\|_2 \uparrow \mu$ and $\left\|T(\overline{Q^i})\right\|_2 \downarrow \mu$ respectively. Moreover, these controllers can be found by solving finite–dimensional quadratic programming problems. In order to establish these facts, we need the following result, showing that the components of every feasible controller Q that are relevant to the time–domain constraints are bounded in the ℓ^∞ sense.

Lemma 1 *Assume that $S_i^{12}(\lambda), S_j^{21}(\lambda)$ have full row and column rank on $|\lambda|$=1. Then all feasible controllers satisfy $\|\tilde{Q}_{ij}\|_{\ell^\infty} \leq M_{ij}$, where M_{ij} depends only on the problem data.*

Proof: Omitted for space reasons, follows from Wiener Gelfand's theorem.

3 Problem Solution

In this section we show that Problem 1 can be solved by solving two modified $\mathcal{H}_2/\ell^\infty$ problems, providing suboptimal and a super–optimal solutions respectively. Both problems can be reduced to finite dimensional quadratic programming, and in the limit their respective solutions strongly converge, in the $\mathcal{H}_2$ topology, to the solution of the original problem.

3.1 Problem Transformation

It is a standard result (see for instance [15], pag. 194) that the parameterization of all stabilizing controllers can be selected (by redefining Q if necessary), so that T^{12} and T^{21} are inner and co-inner respectively. Thus, there exist $T^{12\perp}$, $T^{21\perp}$ such that $\begin{bmatrix} T^{12} & T^{12\perp} \end{bmatrix}$ and $\begin{bmatrix} T^{21} \\ T^{21\perp} \end{bmatrix}$ are unitary. Let

$$\begin{aligned} R^{11} &\doteq T^{12\sim} T^{11} T^{21\sim} \\ R^{12} &\doteq T^{12\sim} T^{11} T^{21\perp\sim} \\ R^{21} &\doteq T^{12\perp\sim} T^{11} T^{21\sim} \\ R^{22} &\doteq T^{12\perp\sim} T^{11} T^{21\perp\sim} \end{aligned} \quad (3\text{-}1)$$

Through straightforward but tedious operations it can be shown that with this choice of the parametrization, $R^{ij} \in \mathcal{RH}_2{}^\perp$. Since the $\mathcal{H}_2$ norm is invariant under pre (post) multiplication by unitary matrices, we have that

$$\|T^{11} + T^{12} Q T^{21}\|_{\mathcal{H}_2}^2 = \left\| \begin{bmatrix} R^{11sp} & R^{12} \\ R^{21} & R^{22} \end{bmatrix} \right\|_{\mathcal{H}_2}^2 + \|D^{R^{11}} + Q\|_{\mathcal{H}_2}^2 \quad (3\text{-}2)$$

where R^{11sp} and $D^{R^{11}}$ denote the strictly proper part of R^{11} and its feed through term respectively. Thus Problem 1 may be reformulated as follows.

Problem 2 *Find the optimal value of the performance measure*

$$\inf_{Q \in \mathcal{H}_2^{n_u \times n_y}} \|Q\|_{\mathcal{H}_2}^2$$

$$\text{subject to } \left\| \left[S^{11} + S^{12}(Q - D^{R^{11}}) S^{21} \right]_{rs} \right\|_{\phi_{rs},\infty} \leq 1 \quad (3\text{-}3)$$

Problem 2 is a convex infinite-dimensional problem, for which no closed-form solution is known to exist. In this paper, a solution will be computed by taking the limit of the solution to some finite-dimensional minimization problems. In the sequel, we will assume without loss of generality (by redefining S^{11} as $S^{11} - S^{12} D^{R^{11}} S^{21}$ if necessary) that $D^{R^{11}} = 0$.

3.2 Computation of super–optimal solutions

In this section, a sequence of finite dimensional convex optimization problems is introduced. The n-th problem has $\mathcal{O}(n)$ variables, and its optimal cost μ^n satisfies $\mu^n \leq \mu$. The sequence of problems approximates Problem 1 in the sense that $\mu^n \to \mu$ and the partial solutions converge to the optimal solution (in the $\mathcal{H}_2$ norm) as $n \to \infty$.

Using the projection operator $\mathcal{P}_n$, consider the optimization problem

Problem 3 *Find the optimal value of the performance measure:*

$$\underline{\mu}^n = \inf_{Q \in \mathcal{H}_2{}^{n_u \times n_y}} \|Q\|^2_{\mathcal{H}_2}$$

$$\text{subject to } \|\mathcal{P}_n(S^{11} + S^{12}QS^{21})_{rs}\|_{\phi_{rs},\infty} \leq 1. \tag{3-4}$$

Problem 3 can be thought of as a finitely–many constraints approximation to the original problem, where the constraints are enforced only over a finite horizon n. In the sequel we show that this problem is equivalent to a finite dimensional quadratic programming problem.

Lemma 2
Problem 3 is equivalent to:

$$\underline{\mu}^n = \min_{[Q^n(0) \cdots Q^n(n-1)]} \sum_{i=0}^{n-1} \|Q^n(i)\|^2_F \tag{3-5}$$

subject to:

$$\left\| \mathcal{P}_n \left[\begin{array}{l} S^{11}(\lambda) + \\ S^{12}\left(\sum_{i=0}^{n-1} Q^n(i)\lambda^i\right) S^{21} \end{array} \right]_{rs} \right\|_{\phi_{rs},\infty} \leq 1$$

Proof: Follows from the fact that for any feasible $Q \in \mathcal{H}_2{}^{n_u \times n_y}$ we have that $Q^n = \mathcal{P}_n(Q)$ is also feasible and yields a lower cost.

Theorem 1 *Assume that there exists $\hat{Q} \in \mathcal{H}_2{}^{n_u \times n_y}$ such that $\|(S^{11} + S^{12}\hat{Q}S^{21})_{rs}\|_{\phi_{rs},\infty} \leq 1$. Then $\underline{\mu}^n \uparrow \mu$ and $\|Q^n - Q_{opt}\|_{\mathcal{H}_2} \to 0$, where $Q_{opt} \in \mathcal{H}_2{}^{n_u \times n_y}$ is the solution to Problem 1.*

Proof: Omitted for space reasons, follows by establishing that Q^n is a Cauchy sequence and its limit Q^* is feasible.

3.3 Computation of sub–optimal solutions

Theorem 1 shows that a solution to Problem 1 can be obtained by solving a sequence of quadratic programming problems. However, it does not furnish information on how to select n to achieve some desired error bound. To solve this difficulty, in this section we introduce a sequence of suboptimal solutions converging to the optimal from above. Solutions to Problem 1 with arbitrary accuracy can then be found by computing upper and lower bounds of μ until the difference between these bounds is as small as desired.

Consider the following finitely many variables approximation to Problem 1:

Problem 4

$$\overline{\mu}^n = \min_{[Q^n(0) \cdots Q^n(n-1)]} \sum_{i=0}^{n-1} \|Q^n(i)\|^2_F$$

$$s.t.$$

$$\left\| \left[S^{11}(\lambda) + S^{21}(\lambda)Q^n(\lambda)S^{21}(\lambda)\right]_{rs} \right\|_{\phi_{rs},\infty} \leq 1$$

where $Q^n(\lambda) = \sum_{i=0}^{n-1} Q^n(i)\lambda^i$

Theorem 2 *Assume that there exists $\hat{Q} \in \mathcal{H}_2{}^{n_u \times n_y}$ such that $\|S(\hat{Q})\|_{\phi,\infty} \leq 1$. Then $\overline{\mu}^n \downarrow \mu$ and $\|Q^n - Q_{opt}\|_{\mathcal{H}_2} \to 0$, where $Q_{opt} \in \mathcal{H}_2{}^{n_u \times n_y}$ is the solution to Problem 1.*

In principle, Problem 4 is a semi–infinite dimensional quadratic programming problem, since it has an infinite number of constraints. However, as we show in the sequel, under mild conditions only finitely many of these constraints are active.

Theorem 3 *Let $\mathcal{I}$ denote the set of pairs (r, s) such that $S(Q)_{rs}$ is subject to time–domain constraints. Denote by S^{12}_r and S^{21}_s the r^{th} row and s^{th} columns of S^{12} and S^{21}, and assume that S^{12}_r and S^{21}_s have full row and column rank on $\lambda = 1$ respectively for all pairs $(r, s) \in \mathcal{I}$. Then Problem 4 is equivalent to:*

$$\overline{\mu}^n = \min_{[Q^n(0) \cdots Q^n(n-1)]} \sum_{i=0}^{n-1} \|Q^n(i)\|^2_F$$

subject to

$$\left\|\mathcal{P}_{N_1} \left[S^{11} + S^{12}Q^n(\lambda)S^{21}\right]_{rs}\right\|_{\phi_{rs},\infty} \leq 1 \tag{3-6}$$

$$\left|\left(V_i^R Q^n V_j^L\right)_{ij}(k)\right| \leq M_Q, \quad \begin{array}{l} k = 0, \ldots, N_2 - 1 \\ (ij) \in \mathcal{I} \end{array} \tag{3-7}$$

$$Q^n(\lambda) = \sum_{i=0}^{n-1} Q^n(i)\lambda^i \tag{3-8}$$

where M_Q, $N_1(n)$ and $N_2(n)$ are constants that depend only on the problem data and the length of the

FIR Q, and V_i^R,V_j^L are unimodular matrices such that:

$$S_1^{12} = [\,\tilde{S}^{12}(\lambda) \;\; 0 \;\; \dots \;\; 0\,]\, V_R(\lambda)$$
$$S_1^{21} = V_L(\lambda) \begin{bmatrix} \tilde{S}^{21}(\lambda) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3-9}$$

<u>Proof</u> For notational simplicity, let $\tilde{Q}_{ij} = \left(V_i^R Q^n V_j^L\right)_{ij}$ and $\tilde{S}_{ij} = \tilde{S}_i^{12}\tilde{S}_j^{21}$. Since V_i^R, V_j^L and Q^n are polynomial matrices, it follows that there exist some $N_2(n)$ such that $\tilde{Q}_{ij}(k) = 0$, for all $k \geq N_2$ and $(i,j) \in \mathcal{I}$. From Lemma 1 we have that every feasible controller satisfies a bound of the form

$$|\tilde{Q}_{ij}(k)| \leq M_{ij}$$

Thus defining $M_Q = \max\{M_{ij}\}$ renders the additional constraint (3-7) redundant at the optimum. Moreover, since the Youla parametrization is chosen so that S^{ij} is analytic in $|\lambda| \leq \frac{1}{a} < \rho$, there exists $N_3(n, N_2)$ (that can be precomputed a-priori) such that $|S_{rs}^{11}(k)| + \|(I - \mathcal{P}_{(k-N_2+1)})\tilde{S}_{rs}\|_{\ell^1} * M_Q \leq \phi_{rs}(k)$ for all $k \geq N_3$. The proof follows now by noting that for all $k \geq N_1 = \max\{N_2, N_3\}$ we have that

$$
\begin{aligned}
|(S^{11} + S^{12}QS^{21})_{rs}(k)| &= |(S_{rs}^{11} + \tilde{S}_{rs}\tilde{Q}_{rs})(k)| \\
&\leq |S_{rs}^{11}(k)| + \sum_{l=0}^{N_2-1} |\tilde{S}_{rs}(k-l)||\tilde{Q}ij(l)| \\
&\leq |S_{rs}^{11}(k)| + \|(I - \mathcal{P}_{(k-N_2+1)})\tilde{S}_{rs})\|_{\ell^1} M_{rs} \\
&\leq \phi_{rs}(k)
\end{aligned} \tag{3-10}
$$

i.e., all the constraints are satisfied for $k \geq N_1$.

4 An Active Vision Application:

In this section we illustrate the advantages of the proposed method by designing a controller for the active vision application described in section 1. The system under consideration, shown in Figure 1, consists of a BiSight stereo head, equipped with Hitachi KP-M1 Cameras and Fujinon H10X11EMPX-31 motorized lenses, mounted on a Unisight pan/tilt platform. The head and lenses are controlled by a 10 channel $\delta - \tau$ controller and the image processing required to capture the images and locate the target is performed using a Datacube MaxSPARC S250 hosted by a Dual Processor Sun Ultra 2 workstation.

Obtaining a model suitable for controller design requires identifying the overall transfer functions from the command input u to the pan and tilt unit to the corresponding displacements y_p and y_t of the target in the image, measured in pixels. To this effect the system was sequentially excited in each axis with a step input of amplitude 67 encoder units (roughly corresponding to an angular displacement of 1.5^o) and the position of a target (originally located at the center of the image) was measured. In addition, the experimental noise level was determined by repeatedly measuring the location of the target in the absence of input (see [12] for details). Using this data, the control oriented identification algorithm developed in [5] was run, followed by a model reduction step, leading to the following transfer functions:

$$
\begin{aligned}
\hat{G}_{pan}(z) &= \frac{0.0359z^6+0.0419z^5+0.1289z^4-0.0468z^3}{1.0000z^6-0.3585z^5+0.3282z^4-0.1777z^3} \\
&\quad \frac{-0.0366z^2+0.0002z+0.0389}{0.1762z^2-0.0424z+0.0345} \\
\hat{G}_{tilt}(z) &= \frac{0.0597z^2+0.1109z+0.0954}{1.0000z^2+0.3585z+0.0595}
\end{aligned} \tag{4-1}
$$

The overall transfer function from the command input (in encoder counts) to the pan and tils axes tracking error in pixels is given by:

$$G_{pan}(z) = \hat{G}_{pan}(z) \times \frac{1}{z^3},\; G_{tilt}(z) = \hat{G}_{tilt}(z) \times \frac{1}{z^3}$$

where the factor $\frac{1}{z^3}$ models the delay due to the time required by the image processing algorithms to find the target in each frame.

In the sequel, for the sake of briefness, we concentrate in the controller design for the pan axis, since the design for the tilt one follows exactly along the same lines. In order to recast the problem into the form (2-4) (which involves the impulse rather than the step response of the generalized plant) the plant was augmented with integrators at the disturbance and, following the internal model principle, control inputs.

The goal is to design a controller that achieves a RMS value of the tracking error comparable to that achieved by the optimal $\mathcal{H}_2$ controller, while at the same time avoiding the large control action and oscillatory responses noted in the introduction. To this effect, we first carried–out a design where the control action in response to a step displacement of the target of 25 pixels was bounded by $\|u\|_{\ell^\infty} \leq 50$ (roughly $\frac{1}{3}$ of the control action used by the optimal

$\mathcal{H}_2$ controller). Note that in this case Theorem 3 is not directly applicable since S^{12} has a zero at $z = 1$ due to the integrator at the control input. However, as we show next the upper bound of the cost can still be computed using finite–dimensional optimization.

Consider the Youla parametrization obtained by selecting $K = \mathcal{F}_\ell(J, Q)$ with

$$J = \left(\begin{array}{c|c} A_j & B_j \\ \hline C_j & D_j \end{array} \right) \tag{4-2}$$

where

$$A_j = \begin{pmatrix} 0.475 & -0.415 & 0.080 & -0.730 & -0.584 & -0.188 & -0.411 & -3.499 \\ 0.676 & -0.112 & 0.552 & 0.280 & 0.130 & -0.089 & -0.166 & -0.274 \\ -0.001 & 0.717 & 0.394 & -0.414 & -0.253 & -0.293 & -0.194 & -0.846 \\ -0.003 & 0.00 & 0.355 & -0.488 & -0.204 & -0.159 & -0.430 & -1.467 \\ -0.008 & 0.001 & 0.077 & 0.155 & -0.879 & -0.731 & -1.042 & -4.924 \\ -0.003 & 0.001 & 0.026 & -0.146 & 0.654 & -0.367 & -0.417 & -1.685 \\ -0.001 & 0.00 & 0.011 & -0.059 & -0.141 & 0.542 & -0.667 & -0.649 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.385 & -0.079 \end{pmatrix} \tag{4-3}$$

$$B_j = \begin{pmatrix} -0.056 & 5.110 \\ 0.015 & 0.442 \\ 0.031 & 1.248 \\ 0.020 & 2.226 \\ 0.056 & 7.305 \\ 0.096 & 2.474 \\ 0.288 & 1.000 \\ 0.275 & 0.000 \\ 1.346 & 0.000 \end{pmatrix}, \quad C_j = \begin{pmatrix} -0.008 & 0 \\ 0.001 & 0 \\ 0.071 & 0 \\ -0.395 & 0 \\ -0.944 & 0 \\ -0.538 & 0 \\ -0.934 & 0 \\ -0.838 & 0 \\ -4.538 & -0.743 \end{pmatrix}^T$$

$$D_j = \begin{pmatrix} 0 & 6.716 \\ 1.000 & 0 \end{pmatrix} \tag{4-4}$$

It can be easily verified that this choice renders T^{12} and T^{21} inner and co–inner respectively. Moreover, the controller corresponding to the following Q:

$$Q_{FIR} = 0.7022 + 0.2593z^{-1} + 0.0194z^{-2} + 0.0076z^{-3} + 0.0492z^{-4} + 0.0729z^{-5} + 0.0522z^{-6} + 0.0172z^{-7} \tag{4-5}$$

is feasible and yields $\|T_{ew}\|_2 = 2.13$. Since $\|Q\|_{\ell^\infty} \leq \|Q\|_2$, it follows that the optimal solution to Problem 4 satisfies $|q_k| \leq 2.13 \doteq M_Q$. Finally, direct computations show that for the choice of Youla parametrization given above we have that: $|S^1(k)| + \|(I - \mathcal{P}_k)S^2\|_{\ell^1} M_Q \leq 2$ for all $k \geq 12$. Thus, it follows that $N = 12$ is a suitable horizon for the upper-bound computation. The corresponding controller was found by solving Problem 4 using the projection–based method implemented in Matlab's `quadprog` command for medium–sized problems [8]. The corresponding tracking error settles very quickly, with little overshoot, but the control action oscillates, settling down after 13 samples. To remove this oscillation, we carried out a second design, imposing the constraints: (i) $|u(k)| \leq 50$, and (ii) $|u(k)| \leq 1,\ k \geq 9$. The resulting 28^{th} order controller was reduced to 10^{th} order by using Hankel norm model reduction (the optimal $\mathcal{H}_2$ controller for this problem has order 9). This controller achieves an error response virtually identical to that of design 1, while, as shown in Figure 4 avoiding oscillations in the control action.

Finally, for benchmarking purposes we also designed a PID controller, empirically tuned to minimize the peak of the control action while maintaining a comparable settling time. In this case extensive trial and error iterations were needed to bring the control action down to 60 encoder units and no combination of the parameters was found that further reduced this action, subject to the settling time constraint.

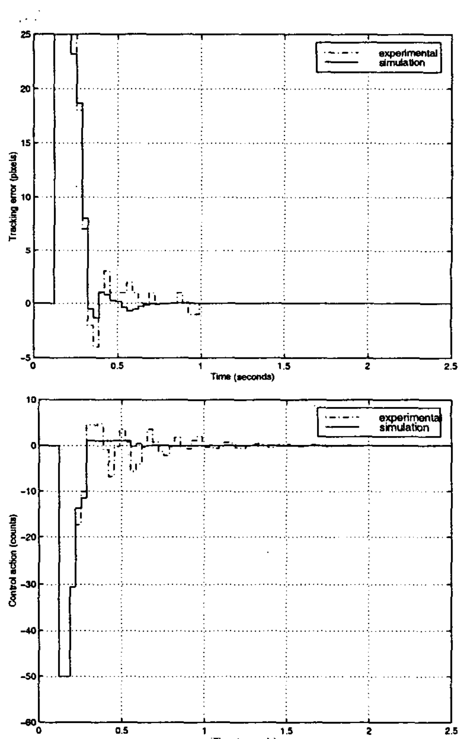

Figure 4: Response of the constrained controller (design 2): (top) Tracking error (bottom) Control action.

method	contr. order	$\mathcal{H}_2$ cost
opt. $\mathcal{H}_2$	9	1.99
Design 1	10	2.13
Design 2	10	2.13

Table 1: $\mathcal{H}_2$ cost for different controllers

5 Conclusions

In this paper we consider the problem of optimizing the $\mathcal{H}_2$ norm of a given system subject to additional specifications given in terms of the response to a given test signal. The main result shows that this problem admits a solution in $\overline{\mathcal{RH}_2}$. Moreover, suboptimal solutions can be obtained by solving sequences of finite–dimensional quadratic programming problems until the gap between upper and lower bounds of the solution is smaller than a pre–specified tolerance. Additional results show that the sequence of controllers thus obtained converges strongly to the optimal solution. These results were illustrated with a practical example arising in the context of active vision. Similar results are available for the case of continuous–time systems and can be obtained by contacting the author.

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