Second IEEE Conference on Control Applications, September 13 - 16, 1993 Vancouver, B.C.

A Time–Domain Penalty Function Approach to Mixed H_2/H_∞–Control Using Parameter Optimization Methods

Ewald Schömig[1]
Department of
Electrical Engineering, FT-10
University of Washington
Seattle, Washington 98195

Mario Sznaier[2]
Department of
Electrical Engineering,
The Pennsylvania State University
University Park, PA 16802

Uy-Loi Ly[3]
Department of
Aero- and Astronautics, FS-10
University of Washington
Seattle, Washington 98195

Abstract

In this paper we consider the problem of minimum nominal H_2–norm with H_∞–constraints for systems with multiple operating points. The performance measure is defined as a weighted sum of the corresponding nominal H_2–norms while robust stability of the individual closed–loop systems is defined in terms of a H_∞-bound for each plant condition. In this paper we define a new time–domain scalar cost function $J_\infty(t_f)$ representing the H_∞–bounds in an overall cost function for the mixed H_2/H_∞–design. $J_\infty(t_f)$ is, for finite time t_f, a penalty function and, for $t_f \rightarrow \infty$, a barrier function. Using $J_\infty(t_f)$, the mixed H_2/H_∞–design problem results in an unconstrained optimization problem, that, for $t_f \rightarrow \infty$, recovers the original objective of minimizing the performance measure subject to the H_∞–bounds. The resulting optimization problem is smooth and hence standard gradient–based software can be applied. The class of controllers considered includes proper and strictly proper LTI controllers with fixed structure and/or fixed order.

1. Introduction

In the past few years, the mixed H_2/H_∞–control problem has been the object of much interest, since it allows the incorporation of robust stability into the LQG framework.

Robust H_2/H_∞–performance still remains, to a large extent, an unsolved problem. An approach based upon parameter optimization methods can be found in [14], where necessary conditions for this problem with fixed order controllers have been derived.

Alternatively, a "Nominal Performance with Robust Stability (NPRS)" problem can be formulated, where the controller yields a desired performance level for the nominal system while guaranteeing stability for all possible plant perturbations. In [1] and [2] an upper bound for the corresponding 2–norm is minimized while a H_∞–bound is satisfied. The dual problem has been solved in [3], in [16] it has been shown that the conditions derived in [2] and [3] are necessary and sufficient. These approaches are restricted to systems with "identical disturbance inputs" or "identical criterion outputs" and result in a set of coupled Riccati equations, that is, in general, difficult to solve. For the same class of systems the NPRS problem has been solved (see [4], [9] and [10]) for the static and dynamic state–feedback case and the dynamic full–order output–feedback case. There the problem is cast into a convex constrained optimization over a bounded set of matrices using non–differentiable constrained optimization techniques . This approach provides an efficient way of solving the problem when the controller is not restricted in structure.

Modern control applications in aeronautics and astronautics often rely on modeling techniques using finite–element analysis and thus involve high–order plant models. Additional requirements such as fixed order and/or fixed structure have to be included for a practical implementable controller. One approach in this direction can be found in [7] and [8]. In addition to a set of rather restrictive system assumptions (rank conditions as well as assumptions on system zeros) this approach requires a initial stabilizing controller guess that satisfies the H_∞–bound. No such assumptions are made in this paper.

In this paper we address the NPRS problem using a time–domain based penalty function approach. We formulate a time–domain cost function that explicitly incorporates the H_∞–bound. This cost function is continuous and differentiable in all the involved parameters. Also, unlike the approaches in [1] or [4] we optimize over the actual H_2–norms rather than their upper bounds. As in previous approaches dealing with fixed order/fixed structure controllers, this optimization problem is non–convex. However, we believe that the approach proposed here has a number of advantages over previously proposed methods. In particular, i) it does not require the use of homotopy based methods, ii) it incorporates multiple plant conditions and hence multiple operating–points of the plant as well as static and dynamic controllers (fixed order, fixed structure, strictly proper or proper), iii) the overall cost function is well defined even if the initial controller guess is not stabilizing, and finally, iv) the system assumptions are the least restrictive in comparison to the above approaches.

[1, 3] This work was supported in part by NASA under grants NAG-2-629 and NAG-1-1210.
[2] This work was supported in part by NSF under grant ECS-9211169.

2. Preliminaries

In general we consider n_p plant conditions of a system in the following form.

$$\Sigma^i_{2/\infty,op}: \left\{ \begin{array}{lcl} \dot{\bar{x}}^i & = & \bar{A}^i\bar{x}^i + \bar{B}^i_1 w^i_2 + \bar{B}^i_2 w^i_\infty + \bar{B}^i_3 u \\ z^i_2 & = & \bar{C}^i_1\bar{x}^i + \bar{D}^i_{11} w^i_2 + \bar{D}^i_{12} w^i_\infty + \bar{D}^i_{13} u \\ z^i_\infty & = & \bar{C}^i_2\bar{x}^i + \bar{D}^i_{21} w^i_2 + \bar{D}^i_{22} w^i_\infty + \bar{D}^i_{23} u \\ y^i & = & \bar{C}^i_3\bar{x}^i + \bar{D}^i_{31} w^i_2 + \bar{D}^i_{32} w^i_\infty \end{array} \right. \quad (1)$$

for $i = 1, 2, ..., n_p$. Each plant condition is assumed to be of n^{th} order. $(\bar{A}^i, \bar{B}^i_j)$ are controllable pairs and $(\bar{A}^i, \bar{C}^i_j)$, $j = 1, 2, ...n_p$ represent observable pairs. w_2 are assumed to be white noise signals with unit spectral density, thus representing the limiting case of disturbances with bounded spectrum (see e.g. [3]). w^i_∞ represent "disturbances" due to system uncertainties for each plant condition. It is assumed that, for each operating point, all uncertainties are lumped into one $\Delta^i(s)$-block, namely $w^i_\infty(s) = \Delta^i(s) z^i_\infty(s)$ with $\Delta^i(s)$ stable and $\|\Delta^i(s)\|_\infty \leq \frac{1}{\gamma^i}$ (we address the case of "unstructured uncertainties" here). Direct feedthrough terms from u to y^i can be incorporated as long as the corresponding feedthrough matrices are the same for every plant condition and the final controller design is well posed. In general we assume the controller $C(s)$ to be of the following form.

$$C(s): \left\{ \begin{array}{lcl} \dot{x}_c & = & A_c x_c + B_c y^i, \\ u & = & C_c x_c + D_c y^i \end{array} \right. \quad (2)$$

subject to $\bar{D}^i_{11} + \bar{D}^i_{13} D_c \bar{D}^i_{31} = 0$ for all $i = 1, 2, ..n_p$. This constraint is necessary for a finite 2-norm from w^i_2 to z^i_2. For technical reasons we impose the additional constraints $D_c \bar{D}^i_{32} = 0$ for all $i = 1, 2, ..n_p$ to simplify the gradient expressions. The order of the controller is prespecified and needs not be equal to the plant order. All matrices are assumed to be of compatible dimensions. A parametric representation C_o of a controller $C(s)$ is given by

$$C_0 = \begin{pmatrix} D_c & C_c \\ B_c & A_c \end{pmatrix}.$$

Given a controller C_o, the closed-loop plant conditions can be represented as follows.

$$\Sigma^i_{2/\infty,cl}(C_o): \left\{ \begin{array}{lcl} \dot{x}^i & = & A^i x^i + B^i_1 w^i_2 + B^i_2 w^i_\infty \\ z^i_2 & = & C^i_1 x^i \qquad\qquad + D^i_{12} w^i_\infty \\ z^i_\infty & = & C^i_2 x^i + D^i_{21} w^i_2 + D^i_{22} w^i_\infty \end{array} \right. \quad (3)$$

Let $\Sigma^i_{2,cl}(C_o)$ denote the subsystem of $\Sigma^i_{2/\infty,cl}(C_o)$ from w^i_2 to z^i_2 with $w^i_\infty = 0$,

$$\Sigma^i_{2,cl}(C_o): \left\{ \begin{array}{lcl} \dot{x}^i_2 & = & A^i x^i_2 + B^i_1 w^i_2 \\ z^i_2 & = & C^i_1 x^i_2 \end{array} \right. \quad (4)$$

and $\Sigma^i_{\infty,cl}(C_o)$ the subsystem from w^i_∞ to z^i_∞ with $w^i_2 = 0$

$$\Sigma^i_{\infty,cl}(C_o): \left\{ \begin{array}{lcl} \dot{x}^i_\infty & = & A^i x^i_\infty + B^i_2 w^i_\infty \\ z^i_\infty & = & C^i_2 x^i_\infty + D^i_{22} w^i_\infty. \end{array} \right. \quad (5)$$

With the constraint $D_c \bar{D}^i_{32} = 0$ it is easily verified that $D^i_{22} = \bar{D}^i_{22}$ for all $i = 1, 2, ..n_p$ and hence the direct feedthrough matrices from w^i_2 to z^i_2 are not dependent on C_0. Let $T^i_2(C_o, s)$ and $T^i_\infty(C_o, s)$ represent the transfer functions corresponding to the closed-loop systems $\Sigma^i_{2,cl}(C_o)$ and $\Sigma^i_{\infty,cl}(C_o)$ respectively. Now let us define the problem addressed in this paper.

Definition 1 *Assume n_p plant conditions as in (1) with the corresponding observability and controllability conditions. Then the mixed H_2/H_∞-design strategy is defined as follows. Find a stabilizing controller C_o such that the performance criterion $J_2(C_o, t_f)$ is minimized where*

$$J_2(C^*_o) = \min_{C_o} \lim_{t_f \to \infty} J_2(C_o, t_f) \quad (6)$$

$$J_2(C_o, t_f) = \sum_{i=1}^{n_p} \alpha^i J^i_2(C_o, t_f) \quad (7)$$

$$J^i_2(C_o, t_f) = \mathcal{E}[z^{iT}_2(t_f) z^i_2(t_f)] = \|T^i_2(C_o, s)\|^2_2 \quad (8)$$

subject to the constraints

$$\|T^i_\infty(C_o, s)\|_\infty < \gamma^i \quad (9)$$

for all $i = 1, 2, ..n_p$. γ^i are n_p parameters chosen by the designer, α^i are n_p weighting factors and $\mathcal{E}$ represents the expectation operator.

This is a constrained optimization problem where the H_∞-bound constraints can be expressed in terms of H_∞-Matrix Algebraic Riccati Equations $ARE^i(C_o, X^i) = 0$ or H_∞-Matrix Algebraic Riccati Inequalities $ARI^i(C_o, X^i) < 0$.

Lemma 1 *Consider a linear stable time-invariant system Σ*

$$\Sigma: \left\{ \begin{array}{lcl} \dot{x} & = & Ax + Bw \\ z & = & Cx + Dw \end{array} \right. \quad (10)$$

with transfer function $T(s)$ and $\gamma > \bar{\sigma}(D)$. Let $R = (\gamma^2 I - D^T D)$ and $S = (\gamma^2 I - DD^T)$. Then the following statements are equivalent (see e.g. [5]).

1. *$\|T(s)\|_\infty < \gamma$.*
2. *ARE: Assume (A, B) to be controllable and (C, A) to be observable, then the matrix equation*

$$[A^T + C^T D R^{-1} B^T] Z + Z [A^T + C^T D R^{-1} B^T]^T + Z B R^{-1} B^T Z + \gamma^2 C^T S^{-1} C = 0 \quad (11)$$

   has a unique real symmetric positive-definite solution Z such that $A + BR^{-1}[D^T C + B^T Z]$ is asymptotically stable.

3. *ARI: There is a symmetric positive-definite matrix X such that ([4], [5])*

$$[A^T + C^T D R^{-1} B^T] X + X [A^T + C^T D R^{-1} B^T]^T + X B R^{-1} B^T X + \gamma^2 C^T S^{-1} C < 0. \quad (12)$$

Rather than seeking explicit apriori solutions for the corresponding ARE's or ARI's to parametrize a controller satisfying the H_∞-constraints we define a time-domain function $J^i_\infty[ARI^i(C_o, X^i), t]$ representing the i^{th} H_∞-constraint. In this formulation the controller C_o and the n_p matrices X^i become the optimization variables in a gradient-based minimization algorithm that attempts to achieve $ARI^i(C_o, X^i) < 0$ for all $i = 1, 2, ..n_p$.

2. Definition of a Penalty/Barrier Function for the H_∞-Constraints

Definition 2 *Consider n_p systems $\Sigma^i_{\infty,cl}(C_o)$ as in equation (5) and define the set $\mathcal{X}$ of symmetric positive-definite matrices*
$\mathcal{X} = \{X^i : X^i = X^{iT}, X^i > 0, i = 1, 2, .., n_p\}$.

We introduce the penalty function

$$J_\infty(C_o, \mathcal{X}, t_f) = \sum_{i=1}^{n_p} J^i_\infty(C_o, X^i, t_f), \quad (13)$$

$$J^i_\infty(C_o, X^i, t_f) = Trace\ \{ e^{ARI^i(C_o,X^i)t_f} \} \quad (14)$$

with

$$\begin{aligned} ARI^i(C_o, X^i) &= [A^{iT} + C_2^{iT} D^i_{22} R^{i-1} B_2^{iT}]X^i \\ &+ X^i[A^{iT} + C_2^{iT} D^i_{22} R^{i-1} B_2^{iT}]^T \\ &+ X^i B_2^i R^{i-1} B_2^{iT} X^i \\ &+ \gamma^{i2} C_2^{iT} S^{i-1} C_2^i \end{aligned} \quad (15)$$

where $R^i = (\gamma^{i2} I - D_{22}^{iT} D^i_{22})$, $S^i = (\gamma^{i2} I - D^i_{22} D_{22}^{iT})$, and the corresponding minimization problem:

$$J_\infty(C_o^*, \mathcal{X}^*) = \min_{C_o,\mathcal{X}} \lim_{t_f\to\infty} J_\infty(C_o, \mathcal{X}, t_f). \quad (16)$$

The key properties associated with the penalty function (13) and the optimization problem (16) are expressed in the following theorem.

Theorem 1 *Consider n_p closed-loop plant conditions $\Sigma^i_{\infty,cl}(\hat{C}_o)$ as in equation (5) for a given controller $\hat{C}_o$, the set of symmetric positive-definite matrices $\mathcal{X}$ as defined above and assume $\gamma^i > \bar{\sigma}(D^i_{22})$ for all $i = 1, 2, .., n_p$. Then the following is true for all i, $i = 1, 2, .., n_p$.*

$$\|T^i_\infty(\hat{C}_o, s)\|_\infty < \gamma^i \quad \Leftrightarrow \quad \min_{\mathcal{X}} \lim_{t_f\to\infty} J_\infty(\hat{C}_o, \mathcal{X}, t_f) = 0$$

$$\|T^i_\infty(\hat{C}_o, s)\|_\infty > \gamma^i \quad \Leftrightarrow \quad \min_{\mathcal{X}} \lim_{t_f\to\infty} J_\infty(\hat{C}_o, \mathcal{X}, t_f) \to \infty.$$

Proof: Assume that $\hat{C}_o$ satisfies $\|T^i_\infty(\hat{C}_o, s)\|_\infty < \gamma^i$ for all $i = 1, 2, .., n_p$, then there is a set $\mathcal{X}$ such that $ARI^i(\hat{C}_o, X^i) < 0$ for all $i = 1, 2, .., n_p$. $ARI^i(\hat{C}_o, X^i)$ being Hermitian matrices, negative-definiteness is equivalent to stability of $ARI^i(\hat{C}_o, X^i)$ and hence the matrix exponential - in the limit as $t_f \to \infty$ - will be zero. Conversely, if $\min_{\mathcal{X}} \lim_{t_f\to\infty} J_\infty(\hat{C}_o, \mathcal{X}, t_f)$ is zero, then there is a set $\mathcal{X}$ of symmetric positive-definite matrices X^i such that $ARI^i(\hat{C}_o, X^i)$ are negative-definite which in turn implies that all H_∞-constraints are satisfied. The second statement follows immediately.□

Now consider the case where the chosen controller results in $\|T^i_\infty(\hat{C}_o, s)\|_\infty \le \gamma^i$ for some i. Then, for this i, $ARI^i(\hat{C}_o, X^i)$ may be non-positive. Numerically this case is not relevant, however, we can treat this case by modifying $J_\infty(C_o, \mathcal{X}, t_f)$ in equation (13) to the following form.

$$\hat{J}_\infty(C_o, \mathcal{X}, t_f, \varepsilon) = \sum_{i=1}^{n_p} \hat{J}^i_\infty(C_o, X^i, t_f, \varepsilon)$$

$$\hat{J}^i_\infty(C_o, X^i, t_f, \varepsilon) = Trace\ \{ e^{[ARI^i(C_o,X^i)+\varepsilon I]t_f} \}$$

for a small positive ε. Now, in the limit, as $t_f \to \infty$, $\hat{J}_\infty(C_o, \mathcal{X}, t_f, \varepsilon)$ will be finite if all H_∞-constraints are satisfied such that for each $i = 1, 2, .., n_p$, $ARI^i(\hat{C}_o, X^i) < -\varepsilon I$. Then the cost $\min_{\mathcal{X}} \lim_{t_f\to\infty} \hat{J}_\infty(C_o, \mathcal{X}, t_f, \varepsilon)$ is zero. We summarize these observations in the following Lemma.

Lemma 2 *Under the assumptions of Theorem 1 the following statements are equivalent:*

- $ARI^i(\hat{C}_o, X^i) < -\varepsilon I$
- $\|T^i_\infty(\hat{C}_o, s)\|_\infty \le \gamma^i - \delta(\varepsilon)$
- $\min_{\mathcal{X}} \lim_{t_f\to\infty} \hat{J}_\infty(\hat{C}_o, \mathcal{X}, t_f, \varepsilon) = 0$

for some small $\varepsilon \ge 0$ and $\delta(\varepsilon) \ge 0$.

$J^i_\infty(C_o, \mathcal{X}, t_f)$ can be interpreted as the trace of the transition matrix of a system $\dot{e}(t) = ARI^i(C_o, X^i)e(t)$ so that the H_∞-constraints can be viewed as the problem of simultaneously stabilizing n_p plants. This justifies the usage of the term "time-domain" penalty function. $J_\infty(C_o, \mathcal{X}, t_f)$ is continuous and differentiable with respect to C_o and all X^i. Explicit gradient expressions can be found in the Appendix and in [12]. This property invites the use of gradient-based optimization algorithms.

3. A Cost Function for the Mixed H_2/H_∞-Design

Now we define the new unconstrained cost function for the mixed H_2/H_∞-design problem.

Definition 3 *Under the assumptions in Definition 1 we define the unconstrained mixed H_2/H_∞-cost function as follows.*

$$J_{2/\infty}(C_o, \mathcal{X}, t_f, \varepsilon) = c_2 J_2(C_o, t_f) + \hat{J}_\infty(C_o, \mathcal{X}, t_f, \varepsilon) \quad (17)$$

where c_2 is a scaling factor. In the limit, as $t_f \to \infty$, the corresponding optimization problem

$$J^*_{2/\infty}(C_o^*, \mathcal{X}^*, \varepsilon) = \min_{C_o,\mathcal{X}} \lim_{t_f\to\infty} J_{2/\infty}(C_o, \mathcal{X}, t_f, \varepsilon) \quad (18)$$

solves the design strategy in Definition 1 if the controller can satisfy all the H_∞-bounds.

With the assumptions of controllability on (A^i, B_1^i) and observability on (C_1^i, A^i), $J^*_{2/\infty}(C_o^*, \mathcal{X}^*, \varepsilon)$ is finite if and only if the controller is internally stabilizing and all the H_∞-bounds are satisfied with a set $\hat{\mathcal{X}}$ such that $ARI^i(\hat{C}_o, \hat{X}^i) < -\epsilon I$ for all plant conditions ($i = 1, 2, \ldots n_p$). If the H_∞-constraints are satisfied but $\hat{J}_\infty(C_o, \mathcal{X}, t_f, \epsilon) > 0$ we can reduce ϵ to ensure $\min_{\mathcal{X}} \hat{J}_\infty(C_o, \mathcal{X}, t_f) = 0$ in the limit as $t_f \to \infty$. Hence, if all the H_∞-constraints are satisfied, then we can find an ϵ such that for $t_f \to \infty$ the objective $J_2(C_o, t_f)$ will dominate in the overall optimization. In this case the desired design objective as in Definition 1 will be recovered in the limit for $t_f \to \infty$. If, on the other hand, the controller is destabilizing or some of the H_∞-constraints are violated, then the combined cost function will remain unbounded as $t_f \to \infty$. Gradient expressions for $J_2(C_o, t_f)$ can be found in [6].

$J_{2/\infty}(C_o, \mathcal{X}, t_f, \epsilon)$ is neither convex in C_o nor in $\mathcal{X}$. Convexity can be achieved in the cases of static and dynamic state-feedback for the pure H_∞-problem only (see e.g. [4]). For the general case of controllers with fixed structure and/or order such convexity results are not yet known.

4. The Algorithm

Principally there are two possible ways to approach the optimization problem. The first approach involves the minimization of $J_{2/\infty}(C_o, \mathcal{X}, t_f, \epsilon)$ as a whole using a penalty function approach. Depending on the eigenvalues of the corresponding ARI's and whether or not the initial controller guess is stabilizing (see [6] on gradients for $J_2(C_o, t_f)$), we select an initial $t_{f,0}$ for which we solve the minimization problem (18). Once a solution of (18) has converged for $t_{f,i-1}$, we increase the finite time to $t_{f,i}$. This process terminates when the largest implementable $t_{f,i}$ has been reached or when the algorithm has converged to a steady-state value in terms of $t_{f,i}$; that is, the controller parameters do not change significantly (as a function of t_f) and all H_∞-constraints have been satisfied.

On the other hand we can first find a stabilizing, H_2-optimal controller using the method in [6]. If this controller satisfies all H_∞-constraints, the program terminates at this point. If there is a "conflict" between the H_2- and the H_∞-objectives, namely, if some of the H_∞-constraints are violated, the algorithm proceeds with the computation of a controller that satisfies all H_∞-constraints, disregarding the performance objective. This problem is solved by optimizing on $\hat{J}_\infty(C_o, \mathcal{X}, t_f, \epsilon)$ for increasing t_f, using the previously determined H_2-optimal controller as a stabilizing initial guess. After this controller has been found, we solve the optimization problem $\min_{C_o} \min_{\mathcal{X}} J_{2/\infty}(C_o, \mathcal{X}, t_{f,o}, \epsilon)$ for a large $t_{f,o}$. For sufficiently large $t_{f,o}$, $\hat{J}_\infty(C_o, \mathcal{X}, t_{f,o}, \epsilon)$ will act as a barrier function. That is, controllers C_o that violate one or more of the H_∞-constraints will be rejected during the line search. However, when applying this type of "bootstrap" algorithm we have to make sure that the controller that we get out of the pure H_∞-optimization phase is capable of recovering the optimal H_2-cost. This is necessary as the overall optimization problem is not convex and we may end up in a local minimum from which we cannot recover the optimal performance. The scalar c_2 may be used to scale the overall cost function properly.

This finite-time approach allows us to control numerical overflow problems arising from an initially destabilizing controller guess or a "bad" guess for the initial set $\mathcal{X}$ (and hence large eigenvalues of the corresponding ARI's) by choosing a small $t_{f,o}$.

The numerical implementation utilizes the optimization toolbox of MATLAB as well as the software package SANDY. For the example below we applied the barrier function approach.

5. Example

We illustrate our approach on two 4^{th}-order longitudinal dynamic models of a F15 aircraft. The first plant condition represents a subsonic flight condition while the second operating condition is supersonic. The state-space matrices according to (1) are given as follows.

$$\bar{A}^1 = \begin{pmatrix} 0.0082 & -25.7084 & 0 & -32.1709 \\ -0.0002 & -1.2763 & 1.0000 & 0 \\ 0.0007 & 1.0218 & -2.4052 & 0 \\ 0 & 0 & 1.0000 & 0 \end{pmatrix},$$

$$\bar{B}_1^1 = \begin{pmatrix} 0.0082 & 0.0462 \\ 0.0002 & 0.0023 \\ -0.0007 & -0.0018 \\ 0 & 0 \end{pmatrix},$$

$$\bar{B}_2^1 = \begin{pmatrix} -0.5585 & 0 & 0 \\ 0 & -0.2793 & 0 \\ 0 & 0.9991 & 20.9938 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\bar{B}_3^1 = \begin{pmatrix} -6.8094 \\ -0.1497 \\ -14.0611 \\ 0 \end{pmatrix},$$

$$\bar{C}_1^1 = \begin{pmatrix} 0.0707 & 0 & 0 & 0 \\ 0 & 0 & 0.3162 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{D}_{13}^1 = \begin{pmatrix} 0 \\ 0 \\ 20 \end{pmatrix},$$

$$\bar{C}_2^1 = \begin{pmatrix} 0.0147 & 0 & 0 & 0 \\ 0 & 0.0147 & 0 & 0 \\ 0 & -0.1688 & 0 & 0 \end{pmatrix}, \quad \bar{C}_3^1 = I.$$

$$\bar{A}^2 = \begin{pmatrix} -0.0117 & -95.9107 & 0 & -32.1129 \\ -0.0001 & -1.8794 & 1.0000 & 0 \\ 0.0006 & -3.6163 & -3.4448 & 0 \\ 0 & 0 & 1.0000 & 0 \end{pmatrix},$$

$$\bar{B}_1^2 = \begin{pmatrix} 0.0117 & 0.0661 \\ 0.0001 & 0.0013 \\ -0.0006 & 0.0025 \\ 0 & 0 \end{pmatrix},$$

$$\bar{B}_2^2 = \begin{pmatrix} -0.7985 & 0 & 0 \\ 0 & -0.3993 & 0 \\ 0 & 2.0457 & 78.4635 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\bar{B}_3^2 = \begin{pmatrix} -25.4041 \\ -0.2204 \\ -53.4246 \\ 0 \end{pmatrix},$$

$\bar{C}_1^2 = \bar{C}_1^1$, $\bar{D}_{13}^2 = \bar{D}_{13}^1$, $\bar{C}_2^2 = \bar{C}_2^1$, $\bar{C}_3^2 = I$.
All other matrices are assumed to be zero.
Some preliminary analysis of the design conditions is given below:

P1: Open-loop: stable, $\|T_2^1(s)\|_2 = 0.1068$,
$\|T_\infty^1(s)\|_\infty = 23348.3$;
Minimally achievable H_2-norm: 0.0032,
Minimally achievable H_∞-norm with this approach and a first order proper controller: ≈ 0.056.
P2: Open-loop: stable, $\|T_2^2(s)\|_2 = 0.031$,
$\|T_\infty^2(s)\|_\infty = 8013.3$;
Minimally achievable H_2-norm: 0.00223,
Minimally achievable H_∞-norm with this approach and a first order proper controller: ≈ 0.096.

Simulation results are presented in the Figures 1 and 2. Figure 1 represents the mixed design for the 1^{st} plant condition only. The 2^{nd} operating point is not taken into consideration for this design-curve. The trade-off between performance and robustness is typical, performance improvement implies invariably a deterioration in stability robustness and vice versa. For all design points the achieved $\|T_\infty^1(s)\|_\infty$ is at the specified γ-bound. The best compromise between performance and robustness is at the point $\|T_\infty^1(s)\|_\infty \approx 0.1$ with a corresponding $\|T_2^1(s)\|_2 \approx 0.018$. Dramatic reduction in either robustness or performance can be achieved at the other design points.

Figure 2 shows the simulation results when both plant conditions are accounted for in the design optimization. The weighting factors α^i were chosen to be $\alpha^1 = 1$ and $\alpha^2 = 1$. Hence both H_2-norms are weighted equally. The same value γ_{spec} over which the H_2-norms is plotted, was applied to both plant conditions. Hence this is only a two-dimensional example out of a generally four-dimensional surface. In a mixed design for multiple plants, γ_{spec} will provide an actual constraint for only some of the n_p operating conditions leaving the other plants unconstrained in terms of the robustness constraints. In our example the resulting $\|T_\infty^1(s)\|_\infty$ was always below the specified γ_{spec} while $\|T_\infty^2(s)\|_\infty$ was on the specified robustness boundary for all design points.

6. Conclusion

In this paper we have presented a time-domain approach to the mixed H_2/H_∞-design problem that does not depend on homotopy methods for the controller design. We have defined a Penalty/Barrier function that represents the H_∞-objectives in an overall cost function. The overall cost function is continuous and differentiable. Explicit gradient expressions have been derived for the objective function. The algorithmic treatment in a gradient-based finite-time setting allows us to have initially destabilizing controllers as well as controllers with fixed structure and fixed order.

References

[1] D.S. Bernstein, W.H. Haddad, "LQG Control With an H_∞ Performance Bound: A Riccati Equation Approach," *IEEE Trans. AC-34*, No. 3, 1989, pp. 293-305.

[2] D.S. Bernstein, W.H. Haddad, "Robust Stability and Performance via Fixed-Order Dynamic Compensation with Guaranteed Cost Bound," *Mathematics of Control, Signals, and Systems 3*, 1990, pp. 139-163.

[3] J. Doyle, K. Zhou and B. Bodenheimer, "Optimal Control with Mixed H_2 and H_∞ Performance Objectives," ACC 1989, pp. 2065-2070.

[4] P.P. Khargonekar, M.A. Rotea, "Mixed H_2/H_∞ Control: A Convex Optimization Approach," *IEEE Trans. AC-36*, No. 7, 1991, pp. 824-837.

[5] P.P. Khargonekar, K. Zhou, "An Algebraic Riccati Equation Approach to H^∞ Optimization," *Systems and Control Letters 11*, 1988, pp. 85-91.

[6] U. Ly, "A Design Algorithm for Robust Low-Order Controllers," Ph.D. Dissertation, Department of Aeronautics and Astronautics, Stanford University, 1982.

[7] D.B. Ridgely, L. Valavani, M. Dahleh, G. Stein, "Solution to the General Mixed H^2/H^∞ Control Problem - Necessary Conditions for Optimality," ACC 1992, pp. 1348-1352.

[8] D.B. Ridgely, C.P. Mracek, L. Valavani, "Numerical Solution of the General Mixed H^2/H^∞ Optimization Problem," ACC 1992, pp. 1353-1357.

[9] M.A. Rotea, P.P. Khargonekar, "H^2-Control with an H_∞-Constraint: The State Feedback Case," *Automatica*, Vol. 27, No. 2, 1991, pp. 307-316.

[10] M.A. Rotea, P.P. Khargonekar, "Simultaneous H^2/H^∞ Optimal Control with State Feedback," ACC 1990, pp. 2380-2384.

[11] E. Schömig, M. Sznaier, U. Ly, " Minimum Control Effort State-Feedback H_∞-Control", ACC 1993, pp. 595-599.

[12] E. Schömig, "Mixed H_2/H_∞-Control with General Output-Feedback Using Parameter Optimization Methods," Ph.D. Dissertation, Electrical Engineering Department, University of Washington, 1993.

[13] M. Steinbuch, O.H. Bosgra, "Necessary Conditions for Static and Fixed Order Dynamic Mixed H_2/H_∞ Optimal Control," ACC 1991, pp. 1137-1142.

[14] M. Steinbuch, O.H. Bosgra, "Robust Performance in H_2/H_∞ Optimal Control," CDC 1991, pp. 549-550.

[15] A. Weinmann, "Uncertain Models and Robust Control," Springer-Verlag, Wien New York, 1991.

[16] H.H. Yeh, S.S. Banda, B.C. Chang, "Necessary and Sufficient Conditions for Mixed H_2/H_∞ Optimal Control," *IEEE Trans. AC-37*, No. 3, 1992, pp. 355-358.

Appendix: Computation of the Gradient

Assume a closed-loop system $\Sigma^i_{\infty,cl}(C_o)$ as in (5) and let

$$\hat{A}^i = A^i + B_2^i \bar{R}^{i^{-1}} D_{22}^{iT} C_2^i$$
$$\hat{B}^i = B_2^i \bar{R}^{i^{-1}} B_2^{iT}$$
$$\hat{C}^i = C_2^{iT}[D_{22}^i \bar{R}^{i^{-1}} D_{22}^{iT} + I] C_2^i$$

with $\bar{R}^i = (\gamma^2 I - D_{22}^{iT} D_{22}^i)$. Note, that $\hat{A}^i$, $\hat{B}^i$ and $\hat{C}^i$ are functions of C_o. For notational simplicity this dependency is omitted here. The direct feedthrough term D_{22}^i from w_∞^i to z_∞^i, however, is not a function of C_o due to the assumption $D_c \bar{D}_{32}^i = 0$ for all i. Using a Cholesky factorization of $X^i = Q^{iT} Q^i$, where Q^i are upper triangular matrices, the corresponding $ARI^i(C_o, X^i) = ARI^i(C_o, Q^{iT}Q^i)$ for the i^{th} plant condition is of the form

$$\begin{aligned} ARI^i(C_o, Q^i) &= \hat{A}^{iT} Q^{iT} Q^i + Q^{iT} Q^i \hat{A}^i \\ &\quad + Q^{iT} Q^i \hat{B}^i Q^{iT} Q^i + \hat{C}^i. \end{aligned} \quad (19)$$

Rather than optimizing over $\mathcal{X}$, the set of symmetric positive–definite matrices, we choose Q^i as the optimization parameter for the i^{th} plant condition. Hence the assumption of $X^i \geq 0$ is explicitly accounted for. Note, that the case $X^i \geq 0$, that is, singularity of X^i, is not significant as we optimize over the set of stabilizing controllers. Using the fact that $Trace(TU) = Trace(UT)$ for compatible matrices U and T and a power series expansion of $e^{ARI^i[C_o,(Q^i+\epsilon\Delta Q^i)]t_f}$, it can be shown that

$$\begin{aligned} & J_\infty^i[C_o, (Q^i + \epsilon\Delta Q^i), t_f] - J_\infty^i(C_o, Q^i, t_f) \\ &\quad = 2\epsilon t_f Trace\{\mathcal{F}(C_o, Q^i)\Delta Q^i\} \end{aligned} \quad (20)$$

where

$$\begin{aligned} \mathcal{F}(C_o, Q^i) &= \{(\hat{A}^i + \hat{B}^i Q^{iT} Q^i) e^{ARI^i(C_o,Q^i)t_f} \\ &\quad + e^{ARI^i(C_o,Q^i)t_f}(\hat{A}^i + \hat{B}^i Q^{iT} Q^i)^T\} Q^{iT}. \end{aligned} \quad (21)$$

In this derivation all higher–order terms in ϵ are discarded. Hence, applying Kleinman's lemma (see [11], [15]), the first derivative of $J_\infty^i(C_o, Q^i, t_f)$ with respect to Q^i is given by

$$\frac{\partial Trace\ \{ e^{ARI^i(C_o,Q^i)t_f} \}}{\partial Q^i} = 2 t_f Q^i [\mathcal{F}(C_o, Q^i)]^T. \quad (22)$$

Gradients with respect to the controller matrix C_o are derived using the same procedure as above. The overall gradient follows from the summation of the individual gradients for each plant condition. The expressions are more complicated and are omitted here (see [12]). Note, however, that only matrix–exponentials and matrix–multiplications are needed to compute the gradients. Expressions for the derivative of $J_2(C_o, t_f)$ with respect to C_o can be found in [6].

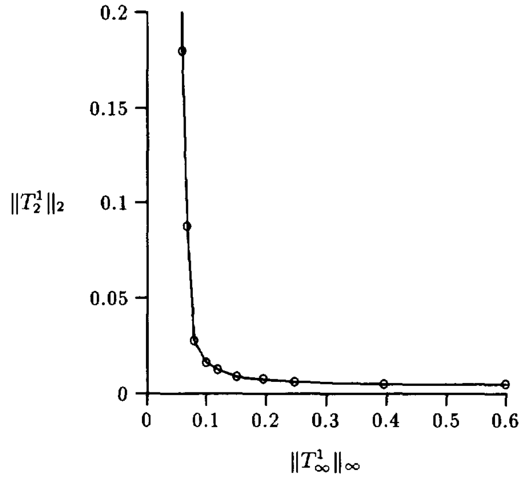

Figure 1: First plant condition

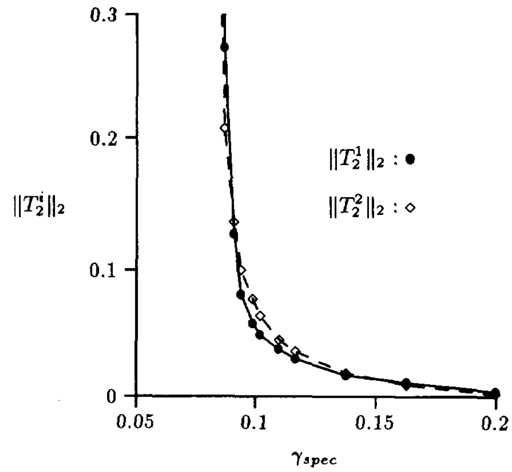

Figure 2: Both plant conditions