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Robust State–Feedback Controllers for Systems Under Mixed Time/Frequency Domain Constraints

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Abstract

A successful controller design paradigm must take into account both model uncertainty and design specifications. Model uncertainty can be addressed using $\mathcal{H}_\infty$ or l_1 robust control theory, depending upon its characterization. However, these frameworks cannot accommodate the realistic case where the design specifications include both time and frequency–domain constraints. In this paper we propose an approach that takes explicitly into account both mixed time/frequency–domain constraints and model uncertainty. This is achieved by minimizing an upper bound of a set–induced operator norm, subject to additional frequency–domain specifications such as bounds on the $\mathcal{H}_2$ and $\mathcal{H}_\infty$–norm of relevant transfer functions. The main result of the paper shows that, by using the parametrization proposed in [8], the problem can be cast into a finite–dimensional, differentiable convex optimization problem, that can be solved by using gradient–based methods. This theory is illustrated with an application to the lateral dynamics of a B767 aircraft.

1. Introduction

A large number of control problems require designing a controller capable of achieving acceptable performance under system uncertainty and design specifications, usually including both, time and frequency–domain constraints. However, despite its practical importance, this problem remains to a large extent still open, even in the simpler case where the system under consideration is linear. During the last decade a large research effort has led to procedures for designing robust controllers capable of achieving desirable properties under various classes of model uncertainty. In particular, a powerful framework has been developed, addressing the issues of robust stability and robust performance in the presence of norm–bound perturbations by minimizing an $\mathcal{H}_\infty$–bound [23]. The $\mathcal{H}_\infty$–framework, combined with μ–analysis [5] (in order to exploit the structure of the uncertainty) has been successfully applied to a number of hard practical control problems (see for instance [15]). However, in spite of this success, it is clear that plain $\mathcal{H}_\infty$–control can only address a subset of the common performance requirements since, being a frequency–domain method, it cannot address time–domain specifications. Some approaches that incorporate time–domain constraints into the $\mathcal{H}_\infty$–formalism have been recently developed [14][11] [17]. However, these approaches require solving large, non–differentiable optimization problems and typically result in a very large controller order, necessitating some type of model reduction [17].

A different approach to robust control has been pursued in [21][4], where robustness and disturbance rejection are approached using the l_1–optimal control theory introduced by Vidyasagar [21] and developed by Pearson and coworkers [4]. These methods are attractive since they allow for an explicit solution to the robust performance problem. However, they cannot accommodate some common classes of frequency–domain specifications (such as $\mathcal{H}_2$ or $\mathcal{H}_\infty$–bounds).

Finally, a third approach to controlling time–domain constrained systems exploits the concept of positively invariant sets [2] [18] [16] [20]. Although this approach leads to simple design algorithms and has recently been extended to encompass some robustness considerations, it cannot handle frequency–domain specifications.

In this paper we propose an approach to design static state–feedback controllers satisfying mixed time/frequency–domain specifications. Satisfaction of time–domain constraints is achieved by minimizing an upper bound of a set–induced operator norm, while robust stability is guaranteed by imposing a bound on the $\mathcal{H}_\infty$–norm on a relevant closed–loop transfer function. The main result of the paper shows that, by using the parametrization proposed in [8], the problem can be cast into a finite–dimensional, differentiable convex optimization problem, that can be solved by using gradient–based methods. Moreover, additional specifications such as $\mathcal{H}_2$–bounds are easily incorporated into the formalism.

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2. Preliminaries

Consider the following linear, shift–invariant, discrete–time system

$$\Sigma : \begin{cases} x(k+1) &= Ax(k) + B_1 w(k) + B_2 u(k) \\ \zeta_\infty(k) &= C_1 x(k) + D_1 u(k) \\ y(k) &= x(k) \\ u(k) &= Ky(k) \end{cases} \quad (1)$$

where (A, B_2) is controllable, D_1 has full column rank, $x \in R^n$ represents the states, $u \in R^m$ represents the control action, $\zeta_\infty \in R^q$ represents variables subject to performance specifications and $w \in R^r$ represents an exogenous disturbance. Given a state–feedback matrix $K \in R^{n \times n}$, the closed–loop system can be expressed as follows:

$$\Sigma_{cl} : \begin{cases} x_{cl}(k+1) &= A_{cl} x_{cl}(k) + B_1 w(k) \\ \zeta_\infty(k) &= C_{cl1} x_{cl}(k) \\ &= (C_1 + D_1 K) x_{cl}(k) \end{cases} \quad (2)$$

where $A_{cl} = A + B_2 K$. Let $T_\infty(z)$ denote the closed–loop transfer function from $w(z)$ to $\zeta_\infty(z)$. In face of equation (2) we can state the design objectives of the *constrained robust–control problem* as follows:

P1. Given the system Σ and two convex, compact, balanced sets [9] containing the origin in their interior, $\mathcal{G} \subset R^n$ and $\mathcal{W} \subset R^m$, find a stabilising state–feedback gain matrix K such that:

$$\|T_\infty(z)\|_\infty \leq \gamma \quad (3)$$
$$x(k) \in \mathcal{G}, \ \forall k \quad (4)$$
$$u(k) \in \mathcal{W}, \ \forall k \quad (5)$$

Next, we recall a result concerning constrained control problems [16]:

Definition 1 ([9])
The Minkowsky Functional p of a balanced convex set $\mathcal{G}$ containing the origin in its interior is defined by

$$p(x) = \inf_{\tau > 0} \left\{ \tau : \frac{x}{\tau} \in \mathcal{G} \right\} \quad (6)$$

A well known result in functional analysis (see for instance [9]) establishes that p defines a seminorm in R^n. Furthermore, when $\mathcal{G}$ is compact, this seminorm becomes a norm. This result is exploited in the following lemma.

Lemma 1 ([16])
Consider the system:

$$x(k+1) = Ax(k) \quad (7)$$

and let $\|.\|_{\mathcal{G}}$ denote the operator norm induced in $R^{n \times n}$ by $\mathcal{G}$ (i.e. $\|A\|_{\mathcal{G}} \triangleq \sup_{\|x\|_{\mathcal{G}}=1} \|Ax\|_{\mathcal{G}}$). Then, given $\mathcal{G}$ and an initial condition $x_o \in \mathcal{G}$, the trajectory $x(k) \in \mathcal{G}$ for all k iff $\|A\|_{\mathcal{G}} \leq 1$.

Moreover, it is shown in [18] [19] that minimising $\|A\|_{\mathcal{G}}$ maximises robustness against parametric model uncertainty and minimises the effects of the disturbance w. From Lemma 1 it follows that the robust constrained control problem can be cast into the following constrained optimization format:

$$\min_K \|A + B_2 K\|_{\mathcal{G}} \quad (8)$$

subject to:

$$\begin{aligned} &\|T_\infty(z)\|_\infty \leq \gamma \\ &\|K\|_{\mathcal{G},\mathcal{W}} \leq 1 \end{aligned} \quad (9)$$

where $\|K\|_{\mathcal{G},\mathcal{W}} \triangleq \sup_{\|x\|_{\mathcal{G}} \leq 1} \|Kx\|_{\mathcal{W}}$. However, this constrained optimisation problem is not convex (since it can be easily shown that the constraint $\|T_\infty(z)\|_\infty$ is not convex in K). Thus, the existence of a global maximum is not guaranteed. In order to solve this difficulty, we recall the following results. Consider the following convex sets[2]:

$$\begin{aligned} \Theta &:= \{X \in R^{n \times n} : X = X^T > 0\} \\ \Upsilon &:= \{(X, W) \in \Theta \times R^{m \times n}\} \\ \Omega &:= \{(\tau, X) \in R \times \Theta : \tau > 0, \frac{1}{\tau^2} I - X \leq 0\} \\ \Psi &:= \{(\tau, X, W) \in \Omega \times R^{m \times n}\} \end{aligned}$$

then, the following results hold:

Lemma 2 ([8])
Consider the stable system Σ_{cl}. Assume that (A_{cl}, B_1) is controllable and (C_{cl1}, A_{cl}) is observable. Then the following statements are equivalent:

1. $\|T_\infty(z)\|_\infty < \gamma$
2. *ARI: There exists a symmetric positive definite matrix Y such that*

$$\begin{aligned} R(X) &:= A_{cl} Y A_{cl}^T - Y + B_1 B_1^T \\ &\quad + A_{cl} Y C_{cl1}^T M^{-1} C_{cl1} Y A_{cl}^T < 0, \quad (10) \\ M(X) &:= \gamma^2 I - C_{cl1} Y C_{cl1}^T \quad (11) \end{aligned}$$

   Moreover, Y can be chosen to be the same as the one in item 3 below.

3. *There exists a symmetric positive definite matrix Y such that*

$$\begin{pmatrix} A_{cl} \\ C_{cl1} \end{pmatrix} Y \begin{pmatrix} A_{cl}^T & C_{cl1}^T \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \begin{pmatrix} B_1^T & 0 \end{pmatrix} - \begin{pmatrix} Y & 0 \\ 0 & \gamma^2 I \end{pmatrix} < 0 \quad (12)$$

   Moreover, Y can be chosen to be the same as the one in item 2 above.

[2] Convexity of Ω is shown in the Appendix

Lemma 3 ([8])
Consider the same system as in Lemma 2 and let $K = WX^{-1}$ with $K \in R^{m\times n}$ and $X \in R^{n\times n}$, then the matrix mapping $Q(X,W) : \Upsilon \to R^{n\times n}$

$$Q(W,X) := \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \begin{pmatrix} B_1^T & 0 \end{pmatrix} - \begin{pmatrix} X & 0 \\ 0 & \gamma^2 I \end{pmatrix} + \begin{pmatrix} A + B_2WX^{-1} \\ C_1 + D_1WX^{-1} \end{pmatrix} X \begin{pmatrix} A + B_2WX^{-1} \\ C_1 + D_1WX^{-1} \end{pmatrix}^T \tag{13}$$

is convex on Υ. Furthermore, there exists a static state-feedback $K = WX^{-1}$ such that $\|T_\infty(z)\|_\infty < \gamma$ if and only if there is a $(X,W) \in \Upsilon$ such that $Q(X,W) < 0$. Here convexity is defined in terms of the usual ordering of symmetric matrices, i.e.:

$$\begin{aligned} & Q[\alpha(W_1,X_1) + (1-\alpha)(W_2,X_2)] \\ \leq\ & \alpha Q(W_1,X_1) + (1-\alpha)Q(W_2,X_2). \end{aligned} \tag{14}$$

for two pairs of matrices $(W_1,X_1) \in \Upsilon$ and $(W_2,X_2) \in \Upsilon$, and $\alpha \in [0,1]$.

Note that $Q(X,W)$ represents the left side argument of inequality (10) with a state-feedback $K = WX^{-1}$ in place and $Y = X \in \Theta$. It should be noted that a pair $(X,W) \in \Upsilon$ that satisfies $Q(X,W) < 0$ implies that $\|T_\infty(z)\|_\infty < \gamma$. However, a controller $K = WX^{-1}$ that satisfies the $\mathcal{H}_\infty$-bound $\|T_\infty(z)\|_\infty < \gamma$ does not necessarily imply $Q(X,W) < 0$. Hence $Q(X,W) < 0$ is only a sufficient condition for $\|T_\infty(z)\|_\infty < \gamma$, not a necessary one. Note that Lemma 3 only refers to the existence of a static state-feedback matrix (that satisfies the $\mathcal{H}_\infty$-bound) in terms of $Q(X,W)$, the parametrization of this controller, however, is not unique.

3. Convex Upper Bounds for $\|A + BK\|_\mathcal{G}$ and $\|K\|_{\mathcal{G},\mathcal{W}}$

In this section we use the results of section 2 to formulate the robust constrained control problem in a convex optimisation setting. We begin by providing a convex upper bound for $\|K\|_F$, the Frobenius norm of the state-feedback gain.

Theorem 1
Consider the Froebenius norm of the state-feedback gain matrix $\|K\|_F = \|WX^{-1}\|_F$, then

$$\begin{aligned} J_B(\tau,X,W) &= \frac{1}{2}Tr(\tau^2X^{-1}) + \frac{1}{2}Tr(W^TW) \\ & \quad s.t.\ \tau^2 X \geq I \end{aligned} \tag{15}$$

represents a differentiable upper bound for $\|WX^{-1}\|_F$ such that

$$\|WX^{-1}\|_F \leq J_B(\tau,X,W). \tag{16}$$

Furthermore, $J_{B2}(\tau,X,W)$ is convex on Ψ.

Proof: The following chain of inequalities proofs that $J_B(\tau,X,W)$ indeed is an upper bound for $\|WX^{-1}\|_F$.

$$\begin{aligned} \|WX^{-1}\|_F &= \sqrt{Tr(WX^{-1}X^{-1}W^T)} && (17) \\ &\leq \sqrt{Tr(\tau^2X^{-1}W^TW)} && (18) \\ & \quad s.\ t.\ \frac{1}{\tau^2}I - X \leq 0 \\ &\leq \sqrt{\tau^2\lambda_{max}(X^{-1})Tr(W^TW)} && (19) \\ &\leq \frac{1}{2}\tau^2\lambda_{max}(X^{-1}) + \frac{1}{2}Tr(W^TW) && (20) \\ &\leq \frac{1}{2}Tr(\tau^2X^{-1}) + \frac{1}{2}Tr(W^TW) && (21) \end{aligned}$$

assuming $\frac{1}{\tau^2}I - X \leq 0$ holds. Equation (18) follows from (17) by the scaling of $\|K\|_F$ with $\tau^2 X \geq I$. (19) follows from (18) by the application of Lemma 4 in the Appendix. (20) follows from (19) by application of the arithmetic–geometric mean inequality with $\alpha = \frac{1}{2}$ and the facts that $\lambda_{max}(\tau^2X^{-1}) \geq 0$ and $Tr(W^TW) \geq 0$ (see the Appendix). The last inequality finally follows from $\lambda_{max}(Z) \leq Tr(Z)$ for any $Z \in \Theta$. Convexity of $Tr(W^TW)$ is shown in [6] (p. 556, problem 33) and the remaining convexity proofs are provided in the Appendix (see Theorems 4 and 5). As the sum of convex mappings is convex, overall convexity follows. It can be easily verified that $J_B(\tau,X,W)$ is continuous and differentiable in all the variables involved.

With this result an upper bound for $\|A_{cl}\|_F$ can be derived using the triangular inequality as follows

$$\|A + B_2K\|_F \leq \|A\|_F + \|B_2K\|_F \tag{22}$$

where $\|A\|_F$ is a constant and $\|B_2K\|_F$ can be expressed in terms of $J_B(\tau,X,B_2W)$ which in turn results in a convex upper bound for $\|A_{cl}\|_F$. Convexity of $J_B(\tau,X,B_2W)$ is easily shown using Theorem 1 and the fact that $Tr(W^TB_2^TB_2W)$ is convex in W (see e.g. [7]). By using this upper bound on $\|A_{cl}\|_F$ and the results of Lemmas 2 and 3 and Theorem 1 we are now in the position to restate problem **P1** as a convex optimisation problem. Since all finite–dimensional matrix norms are equivalent [6], it follows that there exist constants c_1 and c_2, depending only on the geometry of the sets $\mathcal{G}$ and $\mathcal{W}$, such that $\|.\|_\mathcal{G} \leq c_1\|.\|_F$ and $\|.\|_{\mathcal{G},\mathcal{W}} \leq c_2\|.\|_F$. Hence, a suboptimal solution to problem **P1** can be obtained by solving the following auxiliary minimization problem.

P2: Robust constrained control with an $\mathcal{H}_\infty$-bound:

$$\begin{aligned} & \min_{(\tau,X,W)\in\Psi} J_B(\tau,X,B_2W) \\ & s.t.\ Q(X,W) < 0 \\ & \quad J_B(\tau,X,W) \leq \frac{\gamma_u}{c_2} \end{aligned} \tag{23}$$

where γ_u is the maximum control effort allowed. In general, as $Tr(B_2KK^TB_2^T) \leq \lambda_{max}(B_2B_2^T)Tr(KK^T)$, the minimisation problem (23) will also reduce the control effort in terms of $J_B(\tau,X,W)$.

4. A Gradient–Based Formulation

Ellipsoid or Cutting-Plane methods are applicable to this type of problem (for a review of the advantages and disadvantages of these methods see [3] and references therein). However, in many cases, descent–methods are preferred, since they have faster convergence rates. In this section we give a convex characterization of the constraints in terms of differentiable functions, and we use it to cast the original problem into an unconstrained optimisation form, amenable to solution by descent–type algorithms.

Theorem 2
Let $(X, W) \in \Upsilon$, then the scalar measure for the $\mathcal{H}_\infty$-bound is defined as:

$$J_I(X, W, t_f) = Tr\{e^{Q(X,W)t_f}\} \tag{24}$$

where the scaling factor t_f is introduced for algorithmic reasons (see section 5). $J_I(X, W, t_f)$ has the following properties:

1. *$J_I(X, W, t_f)$ is non-negative, and for a given t_f, it is continuous, differentiable and convex on Υ.*
2. $$\lim_{t_f \to \infty} \min_{X,W} J_I(X, W, t_f) = 0 \quad \Leftrightarrow \quad Q(X, W) < 0 \tag{25}$$

Proof: Convexity follows from (14), Weyl's Theorem and Lemma 6 (see Appendix). The latter property of $J_I(X, W, t_f)$ follows from the fact that $Q(X, W) < 0$ is equivalent to stability of $Q(X, W)$ as $Q(X, W)$ is hermitian. As $Tr\{e^{Q(X,W)t_f}\}$ is the sum of the exponential of the eigenvalues of $Q(X, W)t_f$, (25) follows directly.

Remark 2: Expression for the first order gradients of $J_I(X, W, t_f)$ can be found using the matrix series expansion of the involved matrix exponential and Kleinman's Lemma (see [22], [13]).

Remark 3: Using this technique, additional constraints such as $\frac{1}{\tau^2}I - X \leq 0$ can be converted to convex scalar functions as well. Also, at this point we want to emphasize that suboptimal design objectives such as $J_B(\tau, X, W) \leq \frac{\gamma_u}{c_2}$ or $J_B(\tau, X, B_2 W) \leq f_u$ can be accommodated in the same way by forming scalar penalty functions $Tr\{e^{[J_B(\tau,X,W) - \frac{\gamma_u}{c_2}]t_f}\}$ or $Tr\{e^{[J_B(\tau,X,B_2W) - f_u]t_f}\}$. In general, as long as the constraint is in the form of a hermitian matrix inequality or a scalar inequality this method will result in a penalty function with the same properties as in Theorem 2, maintaining the convexity properties of the original constraint.

5. Proposed Algorithm

We now form a cost function that combines all the performance costs and the constraint penalty functions into one overall cost function $J_O(X, W, t_f)$.

$$J_O(X, W, t_f) = c_P \; J_P(\tau, X, W) + J_C(\tau, X, W, t_f) \tag{26}$$

where $J_P(\tau, X, W)$ represents the performance objective $J_B(\tau, X, B_2 W)$, c_P is a weighting factor, and $J_C(\tau, X, W, t_f)$ is the sum of all penalty function terms corresponding to inequality constraints including the $\mathcal{H}_\infty$-penalty function $J_I(X, W, t_f)$. The proposed algorithm starts at a small $t_f = t_{f_s}$ so that initial guesses W, X and τ that do not satisfy the constraints will not result in numerical overflow problems. In a feasibility stage we optimize on $J_C(\tau, X, W, t_f)$ only ($c_P = 0$), trying to find a feasible solution $\bar{W}, \bar{X}$ and $\bar{\tau}$ satisfying the relevant constraints. Once this minimisation has converged for $t_f = t_{f_s}$, t_f is increased and the optimization is repeated. This process terminates when a set $\bar{W}, \bar{X}$ and $\bar{\tau}$ is found. Now t_f is increased to a large value t_{fL} such that $J_C(\bar{\tau}, \bar{X}, \bar{W}, t_{fL}) \ll c_P \; J_P(\bar{\tau}, \bar{X}, \bar{W})$ (note, that in the limit as $t_f \to \infty$ all exponential terms in $J_C(W, X, \tau, t_f)$ will go to zero if the according constraints are satisfied). In fact, for large but finite t_f, $J_C(\tau, X, W, t_f)$ practically acts as a barrier function in the overall optimization process. For this t_{fL} now we optimize on the overall cost function $J_O(X, W, t_f)$. If during this optimization $c_P \; J_P(\tau, X, W)$ approaches values close to $J_C(\tau, X, W, t_f)$ we can always increase t_{fL} or c_P so that $J_P(\tau, X, W)$ remains the dominating cost in the overall optimization.
Alternatively we can also define a penalty function approach in which we optimise the overall cost function from the beginning. Starting with a small t_{f_s} (26) is minimized. Once this optimization has converged, we increase t_f and repeat the minimization. This iteration will terminate as soon as all the constraints are satisfied.

6. Example

We illustrate our approach on a discrete 4^{th}–order system (see [12]), representing the lateral dynamics of a BOEING 767 aircraft. The state–space matrices are given as follows:

$$A = \begin{pmatrix} 0.9966 & 0.0227 & -0.0084 & -0.1120 \\ -0.0037 & 0.7952 & 0.1633 & 0.0005 \\ -0.0063 & -0.6008 & 0.7661 & 0.0003 \\ -0.0007 & -0.0645 & 0.1779 & 1.0000 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0.1885 \\ -0.0003 \\ -0.0007 \\ 0.2000 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -0.0029 \\ -0.0762 \\ -0.6529 \\ -0.0683 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 0.0100 & 0 & 0.0100 & 0 \end{pmatrix}, \quad D_1 = 0.0100.$$

The open–loop system is stable; the open–loop $\mathcal{H}_\infty$-norm $\|T_\infty(z)\|_\infty$ is 7.4826 and the minimally achievable norm $\|T_\infty(z)\|_\infty$ is approximately 0.0069. The Froebenius norm of the open–loop system–matrix A is $\|A\|_F = 1.9102$. Figure 1 shows $\|A_{cl}\|_F$ versus the specified $\mathcal{H}_\infty$-bound γ for several values of γ obtained by solving the following optimisation problem:

$$\min_{(\tau,X,W)\in\Psi} \quad J_B(\tau, X, B_2 W) + \|A\|_F$$
$$s.\,t. \;\; Q(X, W) < 0.$$

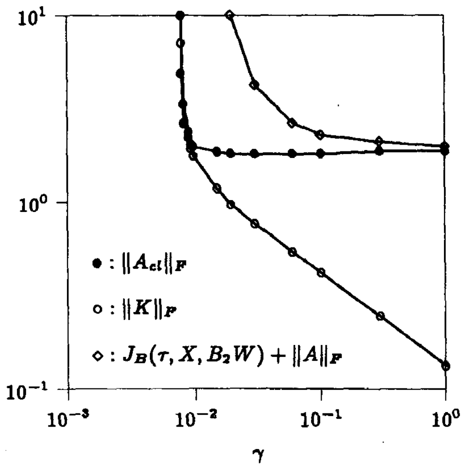

Figure 1: $\|A_{cl}\|_F$ versus specified $\mathcal{H}_\infty$–bound γ

The uppermost curve is the actual cost that we optimize on. From the theory as well as the plot it follows that $J_B(\tau, X, B_2W) + \|A\|_F$ is an upper bound for $\|A_{cl}\|_F$. Interesting is the fact that $\|A_{cl}\|_F$ decreases monotonically up to $\gamma = 0.1$. For $\gamma > 0.1$ $\|A_{cl}\|_F$ increases and converges to the open–loop norm $\|A\|_F$. This fact is due to the stability of the open–loop system. Note, that for $\gamma \geq 7.4826$, $W = 0$ will satisfy the required $\mathcal{H}_\infty$–bound and "stabilize" the system and $\|A_{cl}\|_F = \|A\|_F$ in this case. Hence, for large γ, $\|A_{cl}\|_F$ and $J_B(\tau, X, B_2W) + \|A\|_F$ converge to the same value, namely $\|A\|_F$. $\|K\|_F$ on the other hand shows the typical performance/stability robustness tradeoff.

7. Conclusions

During the last decade, a large research effort has been devoted to the problem of designing robust controllers, capable of guaranteeing stability in the face of plant uncertainty. As a result, a powerful $\mathcal{H}_\infty$–framework has been developed, addressing the issue of robust stability in the presence of norm–bounded plant perturbations. In general, suboptimal controllers are preferred, since optimal $\mathcal{H}_\infty$–controllers may exhibit some undesirable properties, such as very large gains. Since suboptimal controllers are not unique, the extra degrees of freedom available can then be used to optimize some performance measure. This leads naturally to a robust performance problem: design a controller guaranteeing a desired level of performance in the face of plant uncertainty. However, in spite of a large research effort, this problem has not completely been solved.

Alternatively, the extra degrees of freedom can be used to solve a problem of the form *nominal performance with robust stability*. In this case the controller yields a desired performance level for the nominal system while guaranteeing stability for all possible plant perturbations. The problem that we address in this paper, finding a feedback controller such that both time–domain and $\mathcal{H}_\infty$–constraint are satisfied, falls under this class.

In the first part of the paper we show that, by using a technique similar to [8], this problem can be cast into a finite–dimensional convex constrained optimization form. In the second part of the paper we show that this optimization problem can be transformed into a unconstrained differentiable optimization problem, amenable to solution by gradient–based methods.

The proposed design method results in low–order controllers (as opposed to procedures based upon the Youla parametrization [3]) that do not exhibit the large gain often associated with optimal $\mathcal{H}_\infty$controllers.

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Appendix

Theorem 3 (Weyl's Theorem, [6], p.181)
Let $G,\ H \in R^{n\times n}$ be Hermitian matrices, and let the eigenvalues of G, H and $G+H$ be arranged in the following order
$\lambda_1(G) \le \lambda_2(G) \le ... \le \lambda_n(G) = \lambda_{max}(G)$
$\lambda_1(H) \le \lambda_2(H) \le ... \le \lambda_n(H) = \lambda_{max}(H)$
$\lambda_1(G+H) \le \lambda_2(G+H) \le ... \le \lambda_n(G+H) = \lambda_{max}(G+H)$,
then

$$\lambda_i(G+H) \le \lambda_i(G) + \lambda_{max}(H) \tag{27}$$

for all $i = 1, 2, ...n$.
In particular we have

$$\lambda_{max}(G+H) \le \lambda_{max}(G) + \lambda_{max}(H). \tag{28}$$

and, for $H \le 0$,

$$\begin{aligned} \lambda_i(G+H) &\le \lambda_i(G) \\ \lambda_{max}(G+H) &\le \lambda_{max}(G). \end{aligned}$$

Lemma 4 ([22], p.630)
Let $G,\ H \in R^{n\times n}$ be Hermitian matrices such that $G \ge 0$ and $H \ge 0$, then

$$Tr(GH) \le \lambda_{max}(G) Tr(H). \tag{29}$$

Lemma 5 ([1])
(Arithmetic–Geometric Mean Inequality)
Let x and y be two non-negative scalars, then

$$x^\alpha y^{(1-\alpha)} \le \alpha x + (1-\alpha) y \tag{30}$$

for every $\alpha \in (0,1)$.

Lemma 6 ([7])
Let Z_1 and Z_2 be Hermitian matrices and $\alpha \in (0,1)$, then

$$\begin{aligned} Tr\{e^{[\alpha Z_1 + (1-\alpha) Z_2] t_f}\} &\le [Tr(e^{Z_1 t_f})]^\alpha [Tr(e^{Z_2 t_f})]^{(1-\alpha)} \\ &\le \alpha Tr(e^{Z_1 t_f}) + (1-\alpha) Tr(e^{Z_2 t_f}). \end{aligned}$$

Theorem 4
The function

$$J(\tau, X) = Tr(\tau^2 X^{-1}) \tag{31}$$

is convex in $R \times \Theta$

Proof: The proof is essentially equivalent to the proof of Lemma 4.4 in [8] and is omitted here.

Theorem 5
The set $\Omega = \{(\tau, X) \in R \times \Theta : \tau > 0, \frac{1}{\tau^2} I - X \le 0\}$ is convex.

Proof: Consider the mapping $f : R \times \Theta \to R^{n\times n}$ given by

$$f(\tau, X) = \frac{1}{\tau^2} I - X. \tag{32}$$

$f(\tau, X)$ is affine in X and $\frac{1}{\tau^2}$ is convex for all $\tau > 0$. Convexity of Ω follows immediately.