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An Exact Solution to General Four-Block Discrete-Time Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Problems via Convex Optimization

H. Rotstein and M. Sznaier

***Abstract*—The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem can be motivated as a nominal LQG optimal control problem subject to robust stability constraints, expressed in the form of an $\mathcal{H}_\infty$ norm bound. While at the present time there exist efficient methods to solve a modified problem consisting on minimizing an *upper bound* of the $\mathcal{H}_2$ cost subject to the $\mathcal{H}_\infty$ constraint, the original problem remains, to a large extent, still open.**

This paper contains a solution to a general four-block mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem, based upon constructing a family of approximating problems. Each one of these problems consists of a finite-dimensional convex optimization and an unconstrained standard $\mathcal{H}_\infty$ problem. The set of solutions is such that in the limit the performance of the optimal controller is recovered, allowing one to establish the existence of an optimal solution. Although the optimal controller is not necessarily finite-dimensional, it is shown that a performance arbitrarily close to the optimal can be achieved with rational (and thus physically implementable) controllers. Moreover, the computation of a controller yielding a performance ϵ-away from optimal requires the solution of a single optimization problem, a task that can be accomplished in polynomial time.

***Index Terms*—$\mathcal{H}_2$, $\mathcal{H}_\infty$, multiobjective.**

I. INTRODUCTION

Consider the system illustrated in Fig. 1, where the signals $w_\infty \in R^{p_1}$ (an l^2 signal) and $w_2 \in R^{p_2}$ (white noise) represent exogenous disturbances, $u \in R^{p_u}$ represents the control action, $\zeta_\infty \in R^{m_1}$ and $\zeta_2 \in R^{m_2}$ represent regulated outputs, and where $y \in R^{m_y}$ represents the measurements. This paper is concerned with the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem of finding an internally stabilizing controller $u(z) = K(z)y(z)$ such that the rms value of the performance output ζ_2 due to w_2 is minimized, subject to the specification $\|T_{\zeta_\infty w_\infty}(z)\|_\infty \leq \gamma$. This problem was originally introduced in [2] and has received considerable attention since. A large portion of this work (see for instance [2], [5], [19], [17], [7] and references therein) addresses the related problem of minimizing an *upper bound* of the $\mathcal{H}_2$ norm, subject to the $\mathcal{H}_\infty$ constraint. This modified problem is based upon the intuitively plausible idea that minimizing this upper bound should also reduce the actual objective function. Unfortunately, numerical results [1] suggest that for some examples the solution to the "modified" problem may yield an $\mathcal{H}_2$ norm larger than the one achieved by the "central" solution to the pure $\mathcal{H}_\infty$ problem. These examples illustrate the need to develop tools for solving the *exact* $\mathcal{H}_2/\mathcal{H}_\infty$ problem.

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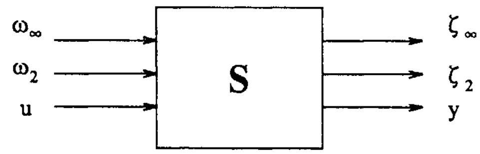


Fig. 1. The generalized plant.

In the state-feedback case, some partial results in this direction were presented in [10]. By fixing instead the order of the controller, [14] used Lagrange multipliers to find necessary conditions for optimality. Unfortunately, this approach is prone to numerical difficulties. An alternative approach is to use the Youla parameterization to recast the $\mathcal{H}_2/\mathcal{H}_\infty$ problem as an *infinite*-dimensional convex optimization [3]. Truncation then yields a finite-dimensional problem which is, at least in principle, tractable [4]. At the moment it is not clear whether one can actually solve the resulting optimization for any sensible problem and how the choice of controller order affects the achievable performance.

The approach pursued in this paper evolves from [16]. The extension to a general multi-input/multi-output (MIMO), four-block problem is not straightforward but can be achieved by using some of the ideas in [12] and [15]. As in [16], it will be shown that a *sub-optimal* solution to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem can be obtained by solving a finite-dimensional convex optimization problem followed by an unconstrained $\mathcal{H}_\infty$ minimization. Additional results include the existence of an optimal solution, the convergence in the $\mathcal{H}_2$ topology, and the fact that the optimal performance achieved over $\mathcal{H}_\infty$ and the smaller (and physically more meaningful) space $\mathcal{A}_o$ is the same.

II. PRELIMINARIES

A. Notation

$\mathcal{L}^\infty$ denotes the Lebesgue space of complex-valued matrix functions which are essentially bounded on the unit circle, equipped with the norm $\|G(z)\|_\infty \doteq \operatorname{ess\,sup}_{|z|=1} \bar{\sigma}(G(z))$, where $\bar{\sigma}$ denotes the largest singular value. By $\mathcal{H}_\infty(\mathcal{H}_\infty^\sim)$ we denote the subspace of functions in $\mathcal{L}^\infty$ with a bounded analytic continuation outside (inside) the unit disk. $\mathcal{RH}_\infty$ denotes the subspace of real rational transfer matrices of $\mathcal{H}_\infty$ and $\mathcal{A}_o$ denotes the subset of $\mathcal{H}_\infty$ functions continuous in the unit circle. The norm on $\mathcal{H}_\infty$ is defined by $\|G(z)\|_\infty \doteq \operatorname{ess\,sup}_{|z|>1} \bar{\sigma}(G(z))$. By $\mathcal{H}_2$ we denote the space of complex-valued matrix functions $G(z)$ with analytic continuation outside the unit disk and square integrable there, equipped with the usual $\mathcal{H}_2$ norm

$$\|G\|_2^2 \doteq \sup_{\gamma>1} \frac{1}{2\pi} \oint_{|z|=\gamma} |G(z)|_F^2 \frac{dz}{z}$$

where $\|\cdot\|_F$ denotes the Frobenious norm.

Also of interest is the Banach space $\mathcal{H}_{\infty,\delta}$ of transfer matrices in $\mathcal{H}_\infty$ which have analytic continuation outside the disk of radius δ, $0 < \delta < 1$ equipped with the norm $\|G(z)\|_{\infty,\delta} \doteq \sup_{|z|>\delta} \bar{\sigma}(G(z))$. Similarly, the space $\mathcal{H}_{2,\delta}$ is defined as the Banach space of transfer matrices having analytic continuation outside $|z| = \delta$ and square integrable there, equipped with the norm

$$\|G\|_{2,\delta}^2 \doteq \sup_{\gamma>\delta} \frac{1}{2\pi} \oint_{|z|=\gamma} |G(z)|_F^2 \frac{dz}{z}.$$

Given $G(z) \in \mathcal{H}_\infty$ one can write the formal series $G(z) = \sum_{i=0}^{\infty} G_i z^{-i}$. The series converges pointwise for each $|z| > 1$ and uniformly outside any disk with radius larger than one. The projection operator $\mathcal{P}_n : \mathcal{H}_\infty \to \mathcal{RH}_\infty$ is defined by

$$\mathcal{P}_n(G(z)) \doteq \sum_{i=0}^{n-1} G_i z^{-i}. \tag{1}$$

It is a standard result that $G \in \mathcal{H}_2$ if and only if $\sum_{i=0}^{\infty} \|G_i\|_F^2 < \infty$. In this case, by Parseval's theorem $\|G\|_2 = \sum_{i=0}^{\infty} \|G_i\|_F^2$, and the series converges also in the $\mathcal{H}_2$ norm.

B. The Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Problem

Assume that the generalized discrete-time plant P is finite-dimensional and linear time invariant. Let $T(z)$ and $S(z)$ denote the closed-loop transfer matrices from w_∞ to ζ_∞ and from w_2 to ζ_2, respectively, obtained when connecting a stabilizing controller from y to u. Using the Youla parameterization, the set of all such transfer matrices can be parameterized by [18]

$$\begin{aligned} T(z) &= T_{11}(z) - T_{12}(z)Q(z)T_{21}(z) \\ S(z) &= S_{11}(z) - S_{12}(z)Q(z)S_{21}(z) \end{aligned} \tag{2}$$

where T_{ij}, S_{ij} are stable transfer matrices, and $Q(z) \in \mathcal{H}_\infty$ is the "free parameter" in the parameterization. In order to stress the dependence on Q, the notation $T(Q),\ S(Q)$ is sometimes used in the sequel. The parameterization allows for precisely stating the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem as follows.

Problem 1 (Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Problem): Find the optimal value of the performance measure

$$\mu \doteq \inf_{Q \in \mathcal{H}_\infty} \{\|S_{11} - S_{12}QS_{21}\|_2 \quad \text{such that } \|T_{11} - T_{12}QT_{21}\|_\infty \le 1\} \tag{3}$$

and, given $\epsilon > 0$, a controller Q such that $\|S(Q)\|_2 \le \mu + \epsilon$ and $\|T(Q)\|_\infty \le 1$.

Lemma 1: Let $S_{12},\ S_{21}$ have generically full column and row rank, respectively, and assume that a solution to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimal control problem exists. Then this solution is unique.

Proof: Let Q_1 and Q_2 solve Problem 1, and assume by contradiction that $Q_1 \neq Q_2$. By the strict convexity of the $\mathcal{H}_2$ norm $S_{11} - S_{12}Q_1S_{21} = S_{11} - S_{12}Q_2S_{21}$. Since by assumption S_{12} has full column rank and S_{21} has full row rank, necessarily $Q_1 = Q_2$. □

In general, Problem 1 admits a minimizing solution in $\mathcal{H}_\infty$ but not in $\mathcal{A}_o$ [8], implying that the optimal controller cannot be approximated by a rational transfer function. Moreover, the optimal closed-loop system is in general not exponentially stable. From an engineering standpoint, these undesirable properties motivate the following problem.

Problem 2 (Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control Problem in $\mathcal{A}_o$): Find the optimal value of the performance measure

$$\mu_R \doteq \inf_{Q \in \mathcal{A}_o} \{\|S_{11} - S_{12}QS_{21}\|_2 \quad \text{such that } \|T_{11} - T_{12}QT_{21}\|_\infty \le 1\} \tag{4}$$

and, given $\epsilon > 0$, find a controller $Q \in \mathcal{A}_o$ such that $\|S(Q_R)\|_2 \le \mu_R + \epsilon$ and $\|T(Q_R)\|_\infty \le 1$.

In the sequel we solve these problems by constructing an optimizing sequence of controllers $\{Q_i\}$ such that the corresponding $T(Q_i)$ satisfies $\|T(Q_i)\|_\infty \le 1$ and such that $\|S(Q_i)\|_2 \to \mu$. We begin by reviewing some mathematical background required for establishing convergence. All the material is taken from [9] and included here for ease of reference.

C. Preliminaries on Complex Analysis

Let $\{f_n\}$ denote a sequence of complex-valued functions, each of whose domain contains an open subset U of the complex plane. The sequence $\{f_n\}$ *converges normally* in U to f if $\{f_n\}$ is pointwise convergent to f in U and this convergence is uniform on each compact subset of U. The relevance of normal convergence is highlighted by the following theorems.

Theorem 1: Suppose that each function in a sequence $\{f_n\}$ is analytic in an open set U and that the sequence converges normally in U to the limit function f. Then f is analytic in U.

A family $\mathcal{F}$ of functions analytic in U is said to be *normal* if each sequence $\{f_n\}$ from $\mathcal{F}$ contains at least one normally convergent subsequence. The following result is a corollary of the classical Montel's theorem.

Theorem 2: Let $\mathcal{F} = \{f_n\}$ be a family of functions analytic in an open set U. If $\|f_n\|_\infty \le 1$ for each $f_n \in \mathcal{F}$, then $\mathcal{F}$ is a normal family in U.

III. Problem Solution

A. Problem Transformation

It is a standard result that the parameterization of all stabilizing controllers can be selected so that $T_{12},\ T_{21}$ are inner and co-inner, respectively, and there exist $T_{12\perp},\ T_{21\perp}$ such that $[T_{12}\ \ T_{12\perp}]$ and $\begin{bmatrix} T_{21} \\ T_{21\perp}\end{bmatrix}$ are unitary. As a consequence, the equality

$$\|T_{11} - T_{12}QT_{21}\|_\infty = \left\| \begin{matrix} G_{11} - Q^\sim & G_{12} \\ G_{21} & G_{22} \end{matrix} \right\|_\infty$$

holds, where $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22}\end{bmatrix}$ is a stable transfer matrix. In the sequel it is assumed that G has a state-space realization of the form

$$G = \begin{pmatrix} A_e & B_a & B_b \\ C_a & D_{aa} & D_{ab} \\ C_b & D_{ba} & D_{bb} \end{pmatrix} \tag{5}$$

and, for simplicity the notation

$$\begin{aligned} B_e &= [B_a \quad B_b], & C_e &= \begin{bmatrix} C_a \\ C_b \end{bmatrix} \\ D_{er} &= [D_{aa} \quad D_{ab}], & D_{ec} &= \begin{bmatrix} D_{aa} \\ D_{ba} \end{bmatrix} \end{aligned} \tag{6}$$

is used. With these definitions Problem 1 may be reformulated as follows.

Problem 3: Compute $Q \in \mathcal{H}_\infty$ such that $\left\| \begin{matrix} G_{11} - Q^\sim & G_{12} \\ G_{21} & G_{22} \end{matrix} \right\|_\infty \le 1$ and $\|S\|_2$ is minimized.

B. Computation of a Solution over $\mathcal{H}_\infty$

In this section, a sequence of finite-dimensional convex optimization problems is introduced. The nth problem has $\mathcal{O}(n)$ variables, and its optimal cost μ^n satisfies $\mu^n \le \mu$. The sequence of problems approximates Problem 1 in the sense that $\mu^n \to \mu$, and the partial solutions converge to the optimal solution as $n \to \infty$. The formulation of the approximating problems requires some results from [12] which are reviewed next. Consider the Riccati equations

$$\begin{aligned} \hat{X} &= A_e \hat{X} A_e^T + B_e B_e^T + \left(A_e \hat{X} C_a^T + B_e D_{er}^T\right) \\ &\quad \times \left(I - D_{er} D_{er}^T - C_a \hat{X} C_a^T\right)^{-1} \left(C_a \hat{X} A_e^T + D_{er} B_e^T\right) \\ \hat{Y} &= A_e^T \hat{Y} A_e + C_e^T C_e + \left(A_e^T \hat{Y} B_a + C_e^T D_{ec}\right) \\ &\quad \times \left(I - D_{ec}^T D_{ec} - B_a^T \hat{Y} B_a\right)^{-1} \left(B_a^T \hat{Y} A_e + D_{ec}^T C_a\right). \end{aligned} \tag{7}$$

From [12], there exists a Q satisfying the *strict* $\mathcal{H}_\infty$ constraint if and only if there exist positive-definite solutions $\hat{X}$ and $\hat{Y}$ to these

Riccati equations such that $\rho(\hat{X}\hat{Y}) < 1$. This is assumed in what follows. For ease of notation, let $x \doteq \hat{X}^{1/2}$, $y \doteq \hat{Y}^{1/2}$.

Theorem 3: Let G have a state-space realization as in (5), and let $Q^n_{\mathrm{FIR}}(z) = \sum_{i=0}^{n-1} Q_i z^{-i}$. Then there exists a $Q^n_{\mathrm{tail}}(z) \in \mathcal{H}_\infty$ such that

$$\left\| \begin{matrix} G_{11} - \sum_{i=0}^{n-1} Q_i^T z^i - z^n Q^n_{\mathrm{tail}}(z)^\sim & G_{12} \\ G_{21} & G_{22} \end{matrix} \right\|_\infty \le 1$$

if and only if $\bar{\sigma}(W(\mathbf{Q_n})) \le 1$, where we have (8), as shown at the bottom of the page.

Proof: For a proof see [12, Th. 8]. □

Using the projection operator defined in (1), consider the optimization problem

Problem 4:

$$\mu^n = \min_{[Q_0^n \quad Q_1^n \quad \cdots \quad Q_{n-1}^n]} \left\| \mathcal{P}_n \left[S_{11}(z) - S_{12}(z) \left(\sum_{i=0}^{n-1} Q_i^n z^{-i} \right) S_{21}(z) \right] \right\|_2$$
$$\text{s.t.} \quad \bar{\sigma}(W(\mathbf{Q_n})) \le 1.$$

The number n of coefficients in $Q^n_{\mathrm{FIR}}(z) = \sum_{i=0}^{n-1} Q_i^n z^{-i}$ is called the "horizon" in the sequel.

Lemma 2: Problem 4 is convex and $\mu^n \le \mu^{n+1} \le \mu$.

Proof: Convexity is obvious. To prove the bounds, note that if $Q^{n+1}_{\mathrm{FIR}}(z) = \sum_{i=0}^{n} Q_i^{n+1} z^{-i}$ is feasible for Problem 4 with horizon $n+1$, then $\mathcal{P}_n(Q^{n+1}_{\mathrm{FIR}})$ is feasible for Problem 4 with horizon n. Also

$$\begin{aligned} &\left\| \mathcal{P}_n \left[S_{11} - S_{12} \mathcal{P}_n \left(Q^{n+1}_{\mathrm{FIR}} \right) S_{21} \right] \right\|_2 \\ &\le \left\| \mathcal{P}_{n+1} \left(S_{11} - S_{12} Q^{n+1}_{\mathrm{FIR}} S_{21} \right) \right\|_2 \\ &\le \left\| S_{11} - S_{12} Q S_{21} \right\|_2. \end{aligned}$$

A similar argument shows that $\mu^n \le \mu$ for every n. □

As a consequence of Lemma 2, $\mu^n \to \mu^{\lim}$. The equality $\mu^{\lim} = \mu$ is established next.

Theorem 4: Given a solution Q^n_{FIR} to Problem 4, select Q^n_{tail} as in Theorem 3 and define

$$Q^n(z) \doteq Q^n_{\mathrm{FIR}}(z) + z^{-n} Q^n_{\mathrm{tail}}(z). \tag{9}$$

Assume that a feasible solution to Problem 1 exists, and S_{12}, S_{21} are generically full column and row rank, respectively. Then $\mu^n \uparrow \mu$ and the sequence $\{Q^n(z)\}$ converges normally to a solution of Problem 1.

Proof: See Appendix A. □

Since the sequence $\{Q^n\}$ converges normally, so does the sequence of truncated closed-loop transfer matrices $S_n \doteq \mathcal{P}_n[S_{11} - S_{12} Q^n S_{21}]$. Moreover, it can also be easily shown that the sequence S_n is a Cauchy sequence in the $\mathcal{H}_2$ topology and hence converges in the $\mathcal{H}_2$-norm. However, since normal convergence does not imply uniform convergence, one cannot conclude that Q^n will provide an approximate solution to the problem, even if n is taken very large.

IV. Computation of a Solution over $\overline{\mathcal{RH}}_\infty$

In this section, an optimization problem involving only a *finite* number of elements of the impulse response of S is presented. This problem can be used to compute a rational ϵ-suboptimal solution to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem. To establish this fact, it is first shown that Problem 2 can be solved by considering a sequence of modified problems. Then, it is shown that the optimal cost achievable with controllers in $\mathcal{RH}_\infty$ can be made arbitrarily close to the optimum over $\mathcal{H}_\infty$ (i.e., $\mu_R = \mu$).

A. A Change of Variables

Consider a real rational transfer matrix $F(z)$, and define the mapping

$$F_\delta(z) \doteq F(\delta z), \quad 0 < \delta < 1 \tag{10}$$

which amounts to a change of variables $z \to \delta \cdot z$. Note that $F(z)$ has a pole at z_o if and only if $F_\delta(z)$ has a pole at z_o/δ, and hence F_δ is analytic outside the unit disk if and only if F is analytic outside the disk of radius δ.

Lemma 3: Let $K(z)$ be a controller such that $K_\delta(z)$ internally stabilizes $P_\delta(z)$. Then $K(z)$ internally stabilizes $P(z)$, $\|T_\delta\|_\infty \ge \|T\|_\infty$, $\|S_\delta\|_2 \ge \|S\|_2$, and $\|T_\delta(z)\|_\infty = \|T(z)\|_{\infty,\delta}$.

Proof: Internal stability follows from the previous observation. The $\mathcal{H}_\infty$ bound follows from the Maximum Modulus theorem, while the $\mathcal{H}_2$ bound can be obtained from the expansion $S_\delta(z) = \sum_{i=0}^\infty \left(S_i/\delta^i\right) z^{-i}$. The equality $\|T_\delta(z)\|_\infty = \|T(z)\|_{\infty,\delta}$ holds by definition. □

In the sequel, δ is selected close enough to one so that T_{ij} and S_{ij} are analytic outside the disk of radius δ.

B. A Modified $\mathcal{H}_2/\mathcal{H}_\infty$ Problem

Consider the following modified $\mathcal{H}_2/\mathcal{H}_\infty$ problem.

Problem 5 (Problem $\mathcal{H}_2/\mathcal{H}_{\infty,\delta}$): Find

$$\mu_\delta \doteq \min_{Q \in \overline{\mathcal{RH}}_{\infty,\delta}} \{ \|S_{11} - S_{12} Q S_{21}\|_2 \quad \text{such that } \|T_{11} - T_{12} Q T_{21}\|_{\infty,\delta} \le 1 \} \tag{11}$$

and the corresponding controller Q_δ, where $^-$ denotes closure.

Note that the set $\{Q \in \overline{\mathcal{RH}}_\infty : \|T_{11} - T_{12} Q T_{21}\|_{\infty,\delta} \le 1\}$ is compact in the $\mathcal{H}_\infty$ topology and thus Q_δ is well defined. Comparing the solution to this optimization problem for increasing δ with the solution to Problem 1 gives the following result.

Theorem 5: Given $\epsilon > 0$ there exists δ, $0 < \delta < 1$ such that $\mu_\delta \le \mu + \epsilon$.

Proof: See Appendix B. □

$$W(\mathbf{Q_n}) = \begin{bmatrix} yA_e^n x & yA_e^{n-1}B_a & \cdots & yA_eB_a & yB_a & yA_e^{n-1}B_b & yA_e^{n-2}B_b & \cdots & yA_eB_b & yB_b \\ C_aA_e^{n-1}x & C_aA_e^{n-2}B_a & \cdots & C_aB_a & D_{aa} & C_aA_e^{n-2}B_b & C_aA_e^{n-3}B_b & \cdots & C_aB_b & D_{ab} \\ C_aA_e^{n-2}x & C_aA_e^{n-3}B_a & \cdots & D_{aa} & 0 & C_aA_e^{n-3}B_b & C_aA_e^{n-4}B_b & \cdots & D_{ab} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_ax & D_{aa} & 0 & \cdots & 0 & D_{ab} & 0 & 0 & \cdots & 0 \\ C_bA_e^{n-1}x & C_bA_e^{n-2}B_a & \cdots & C_bB_a & D_{ba} & C_bA_e^{n-2}B_b & C_bA_e^{n-3}B_b & \cdots & C_bB_b & -Q_0^t \\ C_bA_e^{n-2}x & C_bA_e^{n-3}B_a & \cdots & D_{ba} & 0 & C_bA_e^{n-3}B_b & C_bA_e^{n-4}B_b & \cdots & -Q_0^t & -Q_1^t \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_bx & D_{ba} & 0 & \cdots & 0 & -Q_0^t & -Q_1^t & -Q_2^t & \cdots & -Q_{(n-1)}^t \end{bmatrix} \tag{8}$$

Corollary 1:

1) $\mu_\delta \geq \mu_R$.

2) The optimal cost of Problems 1 and 2 are equal, i.e., $\mu = \mu_R$.

Proof: Since $Q_\delta \in \overline{\mathcal{RH}}_{\infty,\delta}$, it can be approximated arbitrarily and uniformly by a transfer matrix in $\mathcal{RH}_\infty$ (e.g., by the partial Cesaro sums of its Taylor expansion). Let $\hat{Q}_\delta$ be a real rational approximation to Q_δ. From Lemma 3, $\hat{Q}_\delta$ is feasible for Problem 2 so that 1) follows. To show 2), note that $\mu_\delta \geq \mu_R \geq \mu$ and, from Theorem 5, a δ can be found so that $\mu_\delta \leq \mu + \epsilon$ for any given $\epsilon > 0$. □

Thus, although the solution of the $\mathcal{H}_2/\mathcal{H}_\infty$ problem is not generically in $\mathcal{A}_o$, the infimum achievable with controllers in $\overline{\mathcal{RH}}_\infty$ is equal to the optimal cost over $\mathcal{H}_\infty$.

Finally, we show convergence of the closed-loop systems and of the controllers in the $\mathcal{H}_2$ topology.

Lemma 4: Consider a sequence $0 < \delta_i \uparrow 1$. Then, the sequence of corresponding closed loops $S_i \doteq S_{11} - S_{12}Q_{\delta_i}S_{21}$ converges in the $\mathcal{H}_2$ topology. Moreover, if S_{12} and S_{21} have generically full column and row rank, respectively, on the unit circle, then the sequence of controllers converges in the $\mathcal{H}_2$ topology, i.e., $\|Q_{\delta_i} - Q^{\lim}\|_2 \to 0$.

Proof: Since $\|S_i\|_2 \downarrow \mu$, it follows that Q_{δ_i} is an optimizing sequence. Thus, from [8, Lemma 3] it follows that the sequence S_i is Cauchy in the $\mathcal{H}_2$ topology. To complete the proof note that if S_{12} (S_{21}) has full column (row) rank on the unit disk, it follows (from continuity) that $S_{12,\delta}$ $(S_{21,\delta})$ has full column (row) rank for all δ larger than some $\tilde{\delta}$. Hence, Q_δ and Q^*, the optimal solution to Problem 1, are unique. The fact that $\|Q_{\delta_i} - Q^*\|_2 \to 0$ as $\delta \to 1$ follows now from [8, Lemma 3] and the equality $\mu = \mu_R$. □

C. Computing an Approximate Solution

From the Proof of Theorem 5, if a suboptimality level $\epsilon > 0$ is given, then for a δ which can be computed in terms of the data, the solution Q_δ to Problem 5 satisfies $\mu_\delta \leq \mu + \epsilon$. Moreover, Q_δ can be approximated arbitrarily close by

$$Q_\delta^n(z) = \sum_{i=0}^{n-1} Q_i z^{-i} + z^{-n} Q_{\text{tail}}^n(z) \tag{12}$$

where Q_{tail}^n is defined in Theorem 3 and where $(Q_0 \ Q_1 \ \cdots \ Q_{n-1})$ solves the following *finite-dimensional* convex optimization problem:

$$\begin{aligned}\mu_\delta^n = & \min_{[Q_0 \quad Q_1 \quad \cdots \quad Q_{n-1}]} \\ & \left\| \mathcal{P}_n \left(S_{11}(z) - S_{12}(z) \sum_{i=0}^{n-1} Q_i z^{-i} S_{21}(z) \right) \right\|_2 \\ \text{s.t.} \quad & \bar{\sigma}(W_\delta(\mathbf{Q_n})) \leq 1\end{aligned}$$

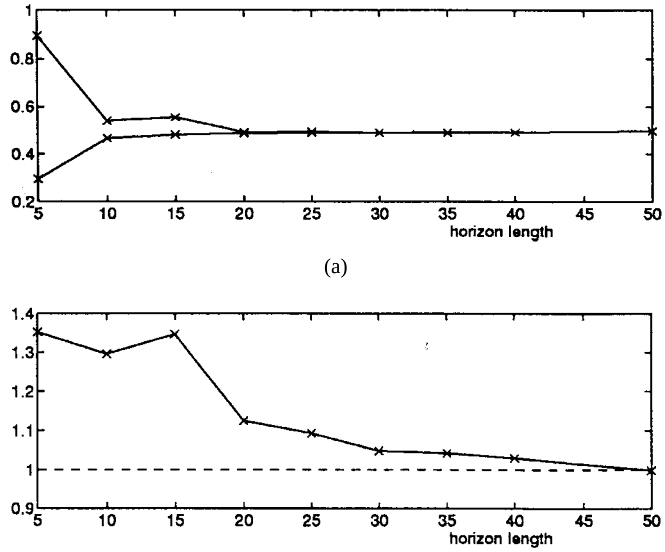
where n is larger than some precomputable bound N_δ. To see this, solve Problem 5 for a fixed $\delta < 1$. Then $\|T_{11} - T_{12}Q^n T_{21}\|_{\infty,\delta} \leq 1$. Moreover, by using the change of variables introduced in Section IV-A so that $T_{12,\delta}$ and $T_{21,\delta}$ are inner and co-inner respectively over $|z| = \delta$, we have that $\|Q^n\|_{\infty,\delta} \leq 1 + \|T_{11}\|_{\infty,\delta}$. Hence

$$\begin{aligned}\|S_{11} - S_{12}Q^n S_{21}\|_{\infty,\delta} &\leq \|S_{11}\|_{\infty,\delta} \\ &+ \|S_{12}\|_{\infty,\delta}(1 + \|T_{11}\|_{\infty,\delta})\|S_{21}\|_{\infty,\delta} < M\end{aligned}$$

for some constant M. Expanding $S_{11} - S_{12}Q^n S_{21} = \sum_{i=0}^{\infty} S_i z^{-i}$, it follows that $\bar{\sigma}(S_i) \leq \delta^i \cdot M$, which yields the following bound for the truncation error:

$$\|(I - \mathcal{P}_n)(S_{11} - S_{12}Q^n S_{21})\|_2 \leq \text{constant} \times \frac{M\delta^n}{\sqrt{1 - \delta^2}} \tag{13}$$

where the constant depends only on the dimensions of S_{11}. By taking n sufficiently large, say $n \geq N_\delta$, μ_δ^n approximates μ_δ as closely as desired. Note, though, that N_δ is usually very large and hence may not be useful for computations. This difficulty can be circumvented by combining the upper bound introduced in this section with the lower bound introduced in Section III-B to obtain sequences of suboptimal and superoptimal solutions.



(a)

(b)

Fig. 2. (a) Performance μ_n and $\|S_{11} - S_{12}Q^n S_{21}\|_2$ as a function of the horizon length. (b) $\|T_{11} - T_{12}Q^n T_{21}\|_\infty$ as a function of the horizon length.

V. Numerical Example

In this section we present a numerical example to illustrate the results discussed above. Let P be as in (S) with

$$A = \begin{bmatrix} 1.1314 & 1.1815 & -.1791 \\ -.9064 & .2005 & .1689 \\ -.5154 & -.3643 & .7966 \end{bmatrix}$$

$$D = \begin{bmatrix} -.0621 & -.0507 & -.0339 & -.0369 \\ -.0060 & .0297 & -.1171 & .0050 \\ -.0197 & .0897 & -.0834 & -.1230 \\ -.1227 & .0144 & .1279 & .0687 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} .0142 & .1967 \\ -.0043 & .0906 \\ .0519 & -.0999 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} -.0715 \\ -.1253 \\ .0104 \end{bmatrix} \quad B_3 = \begin{bmatrix} -.0631 \\ -.2842 \\ -.1383 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} .1612 & -.0574 & -.2380 \\ .2318 & .1363 & -.0082 \end{bmatrix}$$

$$\begin{bmatrix} C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} .1173 & .0853 & -.0379 \\ -.0815 & .1149 & -.1224 \end{bmatrix}.$$

Minimizing $\|T\|_\infty$ and using the central solution gives $\|T^*\|_\infty = .872$, $\|S\|_2 = 1.069$, while the minimization of the $\mathcal{H}_2$-norm gives $\|T\|_\infty = 2.166$, $\|S^*\|_2 = .372$. The value of μ^n defined in Theorem 3 for increasing values of n is shown in Fig. 2(a), together with the actual norm of $\|S_{11} - S_{12}Q^n S_{21}\|_2$. For short horizons, μ^n is a poor estimate of the two-norm and hence the two values are far apart; however, for n as small as 20, the two values become almost indistinguishable. Note also that μ^n increases monotonically. Fig. 2(b) shows the value of $\|T_{11} - T_{12}\mathcal{P}_n\{Q^n\}T_{21}\|_\infty$ also for increasing horizon lengths. Again, the norm is significantly larger than unity for small values of n and decreases until satisfying the $\mathcal{H}_\infty$

TABLE I
COMPARISON OF DIFFERENT δ'S

δ	$\|T\|_2$	$\|S\|_\infty$	N
0.85	0.4839	0.983	54
0.90	0.4777	0.983	85
0.95	0.4767	0.989	185

constraint with less than 1% tolerance for $n = 50$. At this point the algorithm was stopped, with $\mu_{50} = .49$. The corresponding 52-order controller was approximated by considering a balanced truncation of the Cesaro sums $Q_c^n(z) = \sum_{i=0}^{n-1}(1 - i/n)Q_i^n z^{-i}$ [6], yielding an 11th-order admissible controller achieving virtually identical $\mathcal{H}_2$-performance. Further model reduction yielded the following third-order controller:

$$k_r(z) = \frac{5.0473z^3 + 9.1395z^2 + 4.6276z + 1.2564}{z^3 + .9548z^2 + .4584z + .1811}. \tag{14}$$

The corresponding values of $\|T\|_2$ and $\|S\|_\infty$ are $\|T\|_2 = 0.4905$ and $\|S\|_\infty = 0.989$. Finally, Table I shows the performance achieved for different values of δ.

VI. CONCLUSION

In this paper a solution to a general mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem has been presented. As opposed to most of the literature on the subject, the $\mathcal{H}_2$ norm, rather than an upper bound, is minimized. The main idea is to construct a family of optimization problems and then show that the set of solutions thus generated converges to a solution of the original problem. At each step, the optimization problems are convex and have a structure which allows for finding a solution in polynomial time [11], leading to computationally tractable problems. While these computations are not inexpensive, they are cheaper than those required by other currently available methods. In addition, our approach provides additional new insight into some properties of the optimal solutions. This includes the fact that, although an optimal solution is not in general "well-behaved" since it is not continuous on the border of the region of stability (and thus the resulting closed-loop system is not exponentially stable), the optimal performance can be approached arbitrarily close by a real-rational controller. Moreover, from a practical standpoint, our approach allows for finding exponentially stable suboptimal solutions with a prescribed degree of stability, by selecting $\delta < 1$ in Problem 5, or an ϵ suboptimal solution. An extension of these ideas to more general objective functions can be found in [13]

APPENDIX A: PROOF OF THEOREM 4

It suffices to show that a feasible solution to Problem 1 achieving an $\mathcal{H}_2$ cost of $\mu^{\lim}$ exists. Consider the sequence of functions $\{Q^n\}$. Since $\|T_{11} - T_{12}Q^nT_{21}\|_\infty \leq 1$ and since T_{12} and T_{21} are inner and co-inner, respectively, it follows that

$$\|Q^n\|_\infty = \|T_{12}Q^nT_{21}\|_\infty \leq 1 + \|T_{11}\|_\infty. \tag{A1}$$

From Theorem 2 this implies that $\{Q^n\}$ is a normal family. Let $Q^{\lim}$ denote the limit function of some normally convergent subsequence. It is first claimed that $\|T_{11} - T_{12}Q^{\lim}T_{21}\|_\infty \leq 1$. To see this, consider $\gamma > 1$ and $\epsilon > 0$. Since $Q^n(z)$ converges normally to $Q^{\lim}$ in $|z| > 1$, it follows that there exists N such that for each $n \geq N$, $\bar{\sigma}(Q^n(z) - Q^{\lim}(z)) \leq \epsilon$ for any $|z| \geq \gamma$. Then

$$\bar{\sigma}(T_{11}(z) - T_{12}(z)Q^{\lim}(z)T_{21}(z)) \leq 1 + \epsilon$$

and, since ϵ is arbitrary

$$\bar{\sigma}(T_{11}(z) - T_{12}(z)Q^{\lim}(z)T_{21}(z)) \leq 1$$

for each $|z| \geq \gamma$. Thus

$$\begin{aligned}\|T_{11}(z) - T_{12}(z)Q^{\lim}(z)T_{21}(z)\|_\infty \\ = \sup_{|z|>1} \bar{\sigma}(T_{11}(z) - T_{12}(z)Q^{\lim}(z)T_{21}(z)) \leq 1.\end{aligned} \tag{A2}$$

To show that $\|S_{11} - S_{12}Q^{\lim}S_{21}\|_2 = \mu^{\lim}$, consider an arbitrary $\epsilon > 0$ and take $\gamma > 1$. Since $Q^{\lim}$ is also an $\mathcal{H}_2$ function, there exists an $N_1(\gamma)$ such that for every $n \geq N_1$

$$\begin{aligned}&\|S_{11} - S_{12}Q^{\lim}S_{21}\|_{2,\gamma} \\ &\quad \leq \|\mathcal{P}_n(S_{11} - S_{12}Q^{\lim}S_{21})\|_{2,\gamma} + \epsilon/2.\end{aligned} \tag{A3}$$

Normal convergence of Q^n implies that there exists an $N_2(\gamma)$ such that $\|Q^n - Q^{\lim}\|_{2,\gamma} \leq \epsilon/2$ (this follows from the fact that a fixed multiple of the $\mathcal{H}_\infty$ norm over-bounds the $\mathcal{H}_2$ norm). Then, for every $n \geq \max\{N_1, N_2\}$ we have that

$$\begin{aligned}\|S_{11} - S_{12}Q^{\lim}S_{21}\|_{2,\gamma} &\leq \|\mathcal{P}_n(S_{11} - S_{12}Q^nS_{21})\|_{2,\gamma} \\ &\quad + \|Q^{\lim} - Q^n\|_{2,\gamma} + \epsilon/2 \\ &\leq \mu^{\lim} + \epsilon/2 + \epsilon/2.\end{aligned} \tag{A4}$$

Since ϵ is arbitrary, it follows that

$$\|S_{11} - S_{12}Q^{\lim}S_{21}\|_{2,\gamma} \leq \mu^{\lim} \quad \gamma > 1.$$

Hence, from the definition of the $\mathcal{H}_2$ norm it follows that

$$\|S_{11} - S_{12}Q^{\lim}S_{21}\|_2 \leq \mu^{\lim}.$$

Suppose now that the whole sequence $\{Q^n\}$ is not converging. Then (see, e.g., [9, p. 298]) there exists another subsequence, say $\{Q^{m_k}\}$, which converges normally to a different limit function, say $\hat{Q}^{\lim}$. But then $\hat{Q}^{\lim}$ solves Problem 1 and hence by Lemma 1 $Q^{\lim} = \hat{Q}^{\lim}$. This contradicts the assumption that the whole sequence is not converging.

APPENDIX B: PROOF OF THEOREM 5

From continuity arguments it follows that there exists $0 < \bar{\delta} < 1$ such that for some constants $\kappa_{ij},\ \bar{\kappa}_{ij},\ \underline{\kappa}$, the following inequalities hold for each $\delta > \bar{\delta}$:

$$\begin{aligned}\|T_{ij} - T_{ij,\delta}\|_\infty &\leq \kappa_{ij}(1 - \delta) \\ \|T_{ij} - T_{ij,\delta}\|_2 &\leq \bar{\kappa}_{ij}(1 - \delta) \\ \|S_{12,\delta}\|_\infty &\leq \underline{\kappa}\|S_{12}\|_\infty.\end{aligned}$$

Recall that μ and μ_δ are defined by

$$\begin{aligned}\mu \doteq \inf_{Q \in \mathcal{H}_\infty} &\{\|S_{11} - S_{12}QS_{21}\|_2 \\ &\text{such that } \|T_{11} - T_{12}QT_{21}\|_\infty \leq 1\}\end{aligned} \tag{B1}$$

$$\begin{aligned}\mu_\delta \doteq \inf_{Q \in \mathcal{H}_\infty} &\{\|S_{11} - S_{12}QS_{21}\|_2 \\ &\text{such that } \|T_{11,\delta} - T_{12,\delta}Q_\delta T_{21,\delta}\|_\infty \leq 1\}.\end{aligned} \tag{B2}$$

Let $\epsilon > 0$ be given. Since by Theorem 4 an optimal solution to (B1) exists, and by assumption a transfer matrix satisfying the strict $\mathcal{H}_\infty$ inequality also exists, then for some $Q \in \mathcal{H}_\infty$

$$\begin{aligned}\|T_{11} - T_{12}QT_{21}\|_\infty &\leq \gamma < 1 \\ \|S_{11} - S_{12}QS_{21}\|_2 &\leq \mu + \epsilon/2.\end{aligned}$$

It follows from the first inequality that $\|Q\|_\infty < 1 + \|T_{11}\|_\infty$. It is claimed that $Q_{1/\delta}(z) \doteq Q(z/\delta)$ is a suboptimal solution to (B2). To see this, note that by the triangular inequality

$$\begin{aligned}&\|T_{11,\delta} - T_{12,\delta}QT_{21,\delta}\|_\infty \\ &\quad \leq \|T_{11} - T_{12}QT_{21}\|_\infty + \|T_{11} - T_{11,\delta}\|_\infty \\ &\qquad + \|T_{12}Q(T_{21} - T_{21,\delta})\|_\infty + \|(T_{12} - T_{12,\delta})QT_{21}\|_\infty \\ &\quad \leq \gamma + [\kappa_{11} + (\kappa_{21} + \kappa_{12})(1 + \|T_{11}\|_\infty)](1 - \delta).\end{aligned}$$

Using Lemma 3 and a similar argument we have that

$$\|S_{11} - S_{12}Q_{1/\delta}S_{21}\|_2 \le \mu + \epsilon/2 + [\bar{\kappa}_{11} + (\bar{\kappa}_{21}\underline{\kappa}\|S_{12}\|_\infty + \bar{\kappa}_{12}\|S_{21}\|_\infty)(1 + \|T_{11}\|_\infty)](1-\delta).$$

Selecting δ such that

$$[\kappa_{11} + (\kappa_{21} + \kappa_{12})(1 + \|T_{11}\|_\infty)](1-\delta) < 1 - \gamma$$

and

$$[\bar{\kappa}_{11} + (\bar{\kappa}_{21}\underline{\kappa}\|S_{12}\|_\infty + \bar{\kappa}_{12}\|S_{21}\|_\infty)(1 + \|T_{11}\|_\infty)](1-\delta) \le \epsilon/2$$

we have that $\|T_{11,\delta} - T_{12,\delta}QT_{21,\delta}\|_\infty < 1$ and $\|S_{11} - S_{12}Q_{1/\delta}S_{21}\|_2 \le \mu$. Thus $Q_{1/\delta}$ is feasible for (B2) and achieves a performance of at most $\mu + \epsilon$. This establishes the claim and the theorem, since necessarily $\mu_\delta \le \mu + \epsilon$.

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Optimal Random Perturbations for Stochastic Approximation Using a Simultaneous Perturbation Gradient Approximation

Payman Sadegh and James C. Spall

***Abstract*— The simultaneous perturbation stochastic approximation (SPSA) algorithm has recently attracted considerable attention for challenging optimization problems where it is difficult or impossible to obtain a direct gradient of the objective (say, loss) function. The approach is based on a highly efficient simultaneous perturbation approximation to the gradient based on loss function measurements. SPSA is based on picking a simultaneous perturbation (random) vector in a Monte Carlo fashion as part of generating the approximation to the gradient. This paper derives the optimal distribution for the Monte Carlo process. The objective is to minimize the mean square error of the estimate. The authors also consider maximization of the likelihood that the estimate be confined within a bounded symmetric region of the true parameter. The optimal distribution for the components of the simultaneous perturbation vector is found to be a symmetric Bernoulli in both cases. The authors end the paper with a numerical study related to the area of experiment design.**

***Index Terms*— Experiment design, optimal probability distribution, optimization, SPSA, stochastic approximation.**

I. Introduction

Consider the problem of determining the value of a p-dimensional parameter vector to minimize a loss function $L(\theta)$, where only measurements of the loss function are available (i.e., no gradient information is directly available). The simultaneous perturbation stochastic approximation (SPSA) algorithm has recently attracted considerable attention for challenging optimization problems of this type in application areas such as adaptive control, pattern recognition, discrete-event systems, neural network training, and model parameter estimation; see, e.g., [1]–[6].

SPSA was introduced in [7] and more thoroughly analyzed in [8]. The essential feature of SPSA—which accounts for its power and relative ease of use in challenging multivariate optimization problems—is the underlying gradient approximation that requires only two loss function measurements, regardless of the number of parameters being optimized. Note the contrast of two function measurements with the $2p$ measurements required in classical finite difference-based approaches (i.e., the Kiefer–Wolfowitz SA algorithm). Under reasonably general conditions, it was shown in [8] that the p-

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