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A Pessimistic Approach to Frequency Domain Model (In)Validation

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***Abstract*— Model (In)Validation has been extensively studied during the past decade, leading to efficient algorithms to establish whether some a priori information, typically a given model, uncertainty and noise description, is consistent with some observed experimental data. These algorithms are *optimistic* in the sense that a given description is considered to be not invalidated by the experimental data when there exists at least one pair (noise, uncertainty) that together with the nominal model can reproduce the observed data. Thus, they can substantially underestimate the size of the model uncertainty, leading to poor performance or even instability when this information is used to design a robust controller. To solve this difficulty, in this paper we take a *pessimistic* approach, where we seek the smallest ball such that the resulting model is invalidated by the experimental data *for all* uncertainties outside this ball. Thus, a robust controller designed using this information is guaranteed to stabilize the unknown plant. The main result of the paper provides an LMI–based optimization procedure for computing an upper bound on the radius of this ball for the case of structured uncertainty entering the plant in an LFT form. Moreover, in the case of up to two uncertainty blocks this upper bound is indeed exact.**

I. INTRODUCTION AND MOTIVATION

Before a given system description, obtained either from first principles or an identification step, can be used to synthesize a controller, it must be validated using experimental data. In addition, this step can be used to obtain bounds on the modelling error that can be directly incorporated into a robust controller design algorithm such as μ–synthesis.

Model (in)validation of Linear Time Invariant (LTI) systems in a Robust Control setting has been extensively addressed in the past decade (see for instance [13], [10], [3], [2], [9], [11], [16] and references therein). The main result ([3], [2]) shows that in the case of LTI causal unstructured uncertainty and general Linear Fractional (LFT) dependence, model (in)validation reduces to a convex optimization problem that can be efficiently solved, by applying norm constrained interpolation theory. In the case of structured uncertainty, the problem has been shown in [15] to be NP–hard in the number of uncertainty blocks. However, computable weaker conditions (sufficient for the model to be invalidated) in the form of Linear Matrix Inequalities (LMIs) are available, by reducing the problem to the (in)validation of a scaled model subject to a scaled unstructured uncertainty as proposed in [3], [15], [11], or alternatively, by stating the invalidation problem as one of violation of robust performance by any admissible uncertainty (and solved as a structured singular value problem type) as in [13], [9], [6]. Moreover, as shown in [6], these conditions are indeed necessary and sufficient in the case of (arbitrarily) slow time–varying uncertainty.

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A potential difficulty when using these approaches to estimate the size of the uncertainty, as a first step to a robust controller design, stems from the fact that they are *optimistic* in the sense that a given description is considered to be not invalidated when there exists at least one pair (noise, uncertainty) that together with the nominal model can reproduce the observed experimental data. Thus, they can substantially underestimate the size of the model uncertainty, leading to poor performance or even instability when this information is used to design a robust controller. This effect is illustrated next with a very simple example:

Example 1: Consider the following candidate model represented in the standard z-domain as:

$$y(z) = \frac{\delta(z)}{z} u(z) + \eta(z) \tag{1}$$

with a priori information:

$$\mathcal{N}_o = \left\{ \eta(e^{j\omega}) : \|\eta\|_\infty \leq 0.5 \right\} \tag{2}$$

$$\mathbf{\Delta}_o = \left\{ \delta(e^{j\omega}) : \|\delta\|_\infty \leq \gamma \right\} \tag{3}$$

and assume the experimental input/output data is given by: $u(\omega_k) = 1$ and $y(\omega_k) = e^{-j\omega_k}$, respectively. It can be easily shown that $\gamma_{min} = 0.5$ is the minimum uncertainty size such that the model (1) is not invalidated by the experimental data. Thus, the set of (constant) output feedback $\mathcal{H}_\infty$ controllers $u(z) = Ky(z)$ that robustly stabilize the plant, that is

$$\left\| \frac{\delta(z)K}{z} \right\|_\infty \leq 1 \text{ for all } \delta \in \mathbf{\Delta}_o.$$

is given by $|K| < \frac{1}{\gamma_{min}} = 2$. Hence $K = 1$ is a suitable "robust" controller. Note however that $\tilde{\delta} = 1.5$ is also in the uncertainty consistency set (that is, it could also have generated the given input/output data). In this case, the corresponding "real" plant is given by $y(z) = 1.5z^{-1}u(z) + \eta(z)$, which is *destabilized* by the controller $K = 1$.

To avoid this difficulty, in this paper we propose a *pessimistic* approach to model (in)validation, where we seek the smallest ball guaranteed to include the uncertainty consistency set, e.g. such that the resulting model is invalidated by the experimental data *for all* uncertainties outside this ball. Thus, a robust controller designed using this information is guaranteed to stabilize the unknown plant. For instance, in the simple example above it is easily seen that the experimental information is invalidated for all $\|\delta\|_\infty > 1.5$, which implies that any controller $|K| < \frac{2}{3}$ will stabilize *all* plants in the consistency set, e.g. all plants compatible with the available a priori information and a posteriori experimental data.

The main result of the paper provides an LMI–based optimization procedure for computing an upper bound on the radius of a ball guaranteed to contain the uncertainty consistency, for the case of structured uncertainty entering the plant in an LFT form. Moreover, in the case of up to two uncertainty blocks this upper bound is indeed exact. This is accomplished by first recasting the problem into a robust performance violation form and then invoking the S–procedure to obtain sufficient conditions for the latter problem[1]. These results are illustrated with some simple examples.

II. PRELIMINARIES

A. Notation

We consider the model (in)validation problem for discrete time, causal, linear time-invariant (LTI) stable systems subject to LTI structured uncertainty. We will represent discrete LTI systems by their complex-valued transfer function matrices $H(z) = \sum_{k=0}^{\infty} H(k)z^k$. $\mathcal{L}_\infty$ denotes the Lebesgue space of complex valued matrix functions essentially bounded on the unit circle, equipped with the norm: $\|H(z)\|_\infty \doteq \operatorname{ess\,sup}_{|z|=1} \bar{\sigma}(H(z))$ where $\bar{\sigma}$ denotes the largest singular value. By $\mathcal{H}_\infty$ we refer to the subspace of functions in $\mathcal{L}_\infty$ with a bounded analytic continuation inside the unit disk, with norm defined as $\|H(z)\|_\infty \doteq \operatorname{ess\,sup}_{|z|<1} \bar{\sigma}(H(z))$.

$\mathcal{L}_\infty[0, 2\pi]$ denotes the space of bounded vector valued functions $x(e^{j\omega})$ equipped with the norm $\|x\|_\infty \doteq \operatorname{ess\,sup}_{\omega \in [0,2\pi)} \|x(e^{j\omega})\|_2$ where for a fixed frequency $\|x(e^{j\omega_o})\|_2$ stands for the standard 2-norm in $\mathbf{C}^n$. $\mathcal{BL}_\infty[0, 2\pi](\epsilon)$ is the ball in this space centered at the origin with radius ϵ.

Given a complex valued matrix M, M^* denotes its conjugate transpose. $M > 0$ ($M \geq 0$) indicates the matrix is positive (semi)definite. $\mathcal{F}_u(M, \Delta)$ denotes the upper linear fractional interconnection of matrices M and Δ, defined as:

$$\mathcal{F}_u(M, \Delta) \doteq M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12} + M_{22}.$$

B. Pessimistic Model Invalidation Setup

Consider the upper linear fractional (LFT) interconnection $\mathcal{F}_u(P, \Delta)$ shown in Figure 1 of a discrete–time, causal, stable, LTI candidate model $P \in \mathcal{RH}_\infty$ and dynamic structured uncertainty Δ. In the sequel we will assume that the block Δ is known to belong to the set:

$$\mathbf{\Delta}(\gamma) = \{\Delta \in \mathcal{H}_\infty : \Delta = \operatorname{diag}(\delta_1 I_1, \ldots, \delta_n I_n), \|\Delta\|_\infty \leq \gamma\} \tag{4}$$

[1]Thus these conditions are indeed necessary and sufficient for cases where the S–procedure is lossless.

that is a block diagonal structure consisting of (possibly repeated) scalar blocks[2]. In addition, we will assume that a (perhaps very coarse) bound is available on the size of the admissible uncertainty. For instance, if the plant P has been identified from experimental data using an interpolatory algorithm, such a bound can be obtained by computing the diameter of information (see for instance [12], Chapter 10). Finally, we will assume that the available experimental information consists of samples $y(e^{j\omega_k})$ of the frequency response of the plant to a known input $u(e^{j\omega_k})$, corrupted by bounded measurement noise $\eta(e^{j\omega_k})$. In summary, the *a priori* information and *a posteriori* experimental data are given by:

$$\begin{array}{rcl} \mathcal{S} &=& \{\mathcal{F}_u(P, \Delta), \Delta \in \mathbf{\Delta}(\gamma)\} \\ \mathbf{\Delta}(\gamma) &=& \{\Delta : \Delta = diag(\delta_1 I_1, \ldots, \delta_n I_n), \|\Delta\|_\infty \leq \gamma\} \\ \mathcal{N} &=& \{\eta : \eta \in \mathcal{BL}_\infty[0, 2\pi](\epsilon)\} \\ y(e^{j\omega_k}) &=& H(e^{j\omega_k})u(e^{j\omega_k}) + \eta(e^{j\omega_k}) \end{array} \tag{5}$$

where H denotes the frequency response of the (unknown) plant.

Motivated by Example 1, in this paper we will consider the following "pessimistic" model (in)validation problem for the setup described above:

Problem 1: Given the experimental information $\{u(e^{j\omega}), y(e^{j\omega})\}$, the nominal plant P and noise level ϵ, find the minimum α such that the model is invalid for all $\Delta \in \mathbf{\Delta}$, $\|\Delta\|_\infty > \alpha$.

As we show in the sequel, an upper bound on α can be found by solving a convex optimization problem, obtained through the use of the S–procedure. Thus, this bound is exact in cases where the procedure is lossless.

Remark 1: Problem 1 can be thought of as a specific instance of worst case model set identification problem where the aim is to find a nominal model and a bound on its uncertainty given some a priori information on the true plant, noise and a posteriori experimental data so that nominal plant together with the uncertainty bound covers the consistency set (see [12], Chapter 10 for details). In classical worst case identification the a priori information on the true plant is usually in the form of bounds on its gain and stability margin ([5]). Whereas, here the uncertainty (Δ) itself can be considered as the plant to be identified, given the a priori information on the true system in the form of a candidate model P, the structure of the uncertainty and how these two are interconnected. The identified worst-case uncertainty bound is such that any uncertainty exceeding this bound is inconsistent with the experimental data hence it covers the consistency set.

[2]Since our goal is to find bounds on the uncertainty such that the resulting model is guaranteed to be invalid, assuming this uncertainty structure does not entail any loss of generality: Uncertainty structures containing full matrix blocks can be converted to this form by augmenting the plant with suitable input/output signals. In this case the proposed procedure will provide bounds on each of the elements of the matrix.

III. MAIN RESULTS

A. Problem Transformation

The first step towards obtaining tractable conditions for pessimistic invalidation is to transform Problem 1 into a *robust performance violation* form. To this effect, begin by noting that, since only a finite number of input/output measurements are available through experiment, it is possible to assume that both $u(e^{j\omega})$ and $y(e^{j\omega})$ are the impulse responses of some known systems $S_u, S_y \in \mathcal{RH}_\infty$[3]. Under this assumption, the condition

$$y = \mathcal{F}_u(P, \Delta)(u) + \eta$$

can be rewritten as follows:

$$\tilde{\eta} = \mathcal{F}_u(M, \tilde{\Delta})\mathbf{1}$$

or equivalently

$$\begin{bmatrix} q(e^{j\omega}) \\ \tilde{\eta}(e^{j\omega}) \end{bmatrix} = \underbrace{\begin{bmatrix} M_{11}(e^{j\omega}) & M_{12}(e^{j\omega}) \\ M_{21}(e^{j\omega}) & M_{22}(e^{j\omega}) \end{bmatrix}}_{M(e^{j\omega})} \begin{bmatrix} p(e^{j\omega}) \\ \mathbf{1} \end{bmatrix} \tag{6}$$

$$p_j(e^{j\omega}) = \tilde{\delta}_j(e^{j\omega}) q_j(e^{j\omega}) \ \forall j = 1, \dots, n$$

where

$$\begin{aligned} M_{11}(e^{j\omega}) &= \gamma P_{11}(e^{j\omega}) \\ M_{12}(e^{j\omega}) &= P_{12}(e^{j\omega}) S_u(e^{j\omega}) \\ M_{21}(e^{j\omega}) &= -\tfrac{\gamma}{\epsilon} P_{21}(e^{j\omega}) \\ M_{22}(e^{j\omega}) &= \tfrac{1}{\epsilon}(S_y(e^{j\omega}) - P_{22}(e^{j\omega}) S_u(e^{j\omega})). \end{aligned}$$

This transformation is depicted in Figure 1 where $\tilde{\Delta}$ and $\tilde{\eta}$ are normalized so that they lie in the corresponding unit balls.

In this framework, having $\left\| \mathcal{F}_u(M, \tilde{\Delta})\mathbf{1} \right\|_\infty > 1$ is equivalent to the invalidation of the original model.

B. A Sufficient Condition for Pessimistic Invalidation

Theorem 1: Given the input-output pair $\{u(e^{j\omega}), y(e^{j\omega})\}$ and the sets of admissible noise and uncertainty, then:

$$\left\| \mathcal{F}_u(M, \tilde{\Delta})\mathbf{1}(e^{j\omega_o}) \right\|_2 > 1$$

for all $\Delta \in \mathbf{\Delta}(\gamma)$ with $\left| \tilde{\delta}_i(e^{j\omega_o}) \right| > \alpha_i(e^{j\omega_o})$ if there exist $x_j(e^{j\omega_o}) > 0$ for $j = 1, \dots, n$ such that the following LMI is feasible at ω_o

$$M^* \begin{bmatrix} -X_u(e^{j\omega}) & 0 & 0 & 0 \\ 0 & \alpha_i^2(e^{j\omega}) x_i(e^{j\omega}) & 0 & 0 \\ 0 & 0 & -X_l(e^{j\omega}) & 0 \\ 0 & 0 & 0 & I \end{bmatrix} M - \begin{bmatrix} -X_u(e^{j\omega}) & 0 & 0 & 0 \\ 0 & x_i(e^{j\omega}) & 0 & 0 \\ 0 & 0 & -X_l(e^{j\omega}) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} > 0 \tag{7}$$

where $X_u(e^{j\omega}) = diag[x_1(e^{j\omega}), \dots, x_{i-1}(e^{j\omega})]$ and $X_l(e^{j\omega}) = diag[x_{i+1}(e^{j\omega}), \dots, x_n(e^{j\omega})]$.

[3] S_u and S_y can be found for instance by solving a boundary Nevanlinna-Pick interpolation problem.

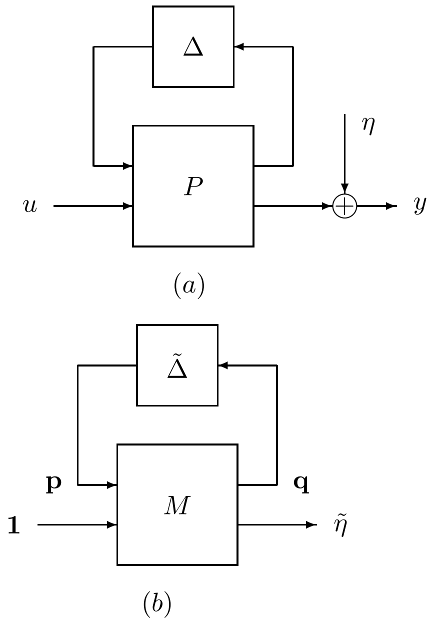

Fig. 1. (a) Pessimistic Model (In)Validation Set–up. (b) Equivalent "Robust" Performance Violation Formulation.

Proof: At ω_o pre and post multiplying (7) by $p^* = [p_1^*(e^{j\omega_o}) \ \dots \ p_i^*(e^{j\omega_o}) \ \dots \ p_n^*(e^{j\omega_o}) \ \ \mathbf{1}]$ and p, and using the relations in (6) gives:

$$\left\| \tilde{\eta}(e^{j\omega_o}) \right\|_2^2 > \sum_{\substack{j=1 \\ j \neq i}}^{n} x_j(e^{j\omega_o}) \left(\left| q_j(e^{j\omega_o}) \right|^2 - \left| p_j(e^{j\omega_o}) \right|^2 \right)$$
$$+ x_i(e^{j\omega_o}) (\left| p_i(e^{j\omega_o}) \right|^2 - \alpha_i^2(e^{j\omega_o}) \left| q_i(e^{j\omega_o}) \right|^2) + 1 \tag{8}$$

The first term on the right hand side of (8) is nonnegative since each $\tilde{\delta}_j$ is scaled to be a contraction. Since $\left| \tilde{\delta}_i(e^{j\omega_o}) \right| > \alpha_i(e^{j\omega_o})$ the second term is also nonnegative. Hence:

$$\left\| \mathcal{F}_u(M, \tilde{\Delta})\mathbf{1}(e^{j\omega_o}) \right\|_2 = \left\| \tilde{\eta}(e^{j\omega_o}) \right\|_2 > 1$$

that is the model is invalid for all $\Delta \in \mathbf{\Delta}(\gamma)$ with $\left| \tilde{\delta}_i(e^{j\omega_o}) \right| > \alpha_i(e^{j\omega_o})$. ■

Direct application of Theorem 1 leads to the following algorithm for finding an envelope (i.e. an upper bound) for the frequency response of each δ_i.

Algorithm 1: Given the a priori information P, $\mathcal{N}$, $\mathbf{\Delta}(\gamma)$ and experimental data $\{u(e^{j\omega_k}), y(e^{j\omega_k})\}$ at N different frequency:

0) Set $k = 0$.
1) Form the system M defined in (6). Set $i = 1$.
2) For each i, do:
   (i) Find the minimum value of $\alpha_i(e^{j\omega_k})$ such that (7) holds. [4]
   (ii) Scale M by $\begin{bmatrix} I & 0 & 0 \\ 0 & \alpha_i & 0 \\ 0 & 0 & I \end{bmatrix}$. Set $i = i+1$. If $i \leq n$ go to step 2.

[4] This step can be accomplished with a line search over $[0, 1]$.

(iii) The value of the envelope at ω_k is given by $\bar{E}(\omega_k) \doteq \gamma[\alpha_1(e^{j\omega_k}), \ldots, \alpha_n(e^{j\omega_k})]$.

3) If $k \leq N$, set $k = k+1$ and go to step 1.

4) The envelope can be obtained by interpolating the $\bar{E}_i(\omega_k)$ values over $[0, 2\pi)$.[5]

Remark 2: As additional experimental data becomes available, the envelope can be easily updated by taking the pointwise minimum of the old envelope and the new envelope.

In principle, one could suspect that the minimum α_i values will depend on the order in which they are calculated since i) the problem is not convex in all α_is and Xs, and ii) condition (7) is just sufficient in some cases. However, as shown next, this is not the case.

Theorem 2: Let α_k denote the values obtained using Algorithm 1 above. Then these values are independent of the order in which they are computed.

Proof: Given in the Appendix ■

C. Analysis of the conservatism of the condition

In this section we briefly comment on the conservatism of condition (7). Let $r \doteq (p^T \; u)^T$ and $s \doteq (q^T \; \eta^T)^T$ and define the following (quadratic) inequalities:

$$\begin{aligned} \sigma_i(r,s) &\doteq \alpha|s_i|^2 - |r_i|^2, \\ \sigma_j(r,s) &\doteq |r_j|^2 - \gamma|s_j|^2, \; j = 1,\ldots,n, \; j \neq i, \\ \sigma_{n+1}(r,s) &\doteq |s_{n+1}|^2 - |r_{n+1}|^2 \end{aligned} \tag{9}$$

Since u is a scalar, $\|\mathcal{F}_u(M, \tilde{\Delta})(e^{j\omega_o})\| > 1$ is equivalent to the condition:

$$\sigma_{n+1}(r) > 0 \; \forall r \text{ such that } \sigma_i(r) < 0, i = 1,\ldots,n. \tag{10}$$

It is well known (see for instance [8]) that a *sufficient* condition for (10) to hold is the existence of $n+1$ multipliers $x_i > 0$ such that $\sum_i x_i \sigma_i(r,s) > 0$. Replacing the explicit dependence of s on r and rewriting the resulting condition in matrix form leads to the LMI (7). In general, the step above is conservative, and the gap between necessity and sufficiency is known to grow linearly with the number of blocks [7]. However, in the case of up to three Hermitian quadratic forms in a complex linear space, the two conditions are equivalent (e.g. the S–procedure is lossless) [4], [8]. Hence the conditions in Theorem 1 are indeed necessary and sufficient for uncertain structures containing up to two blocks.

IV. EXAMPLE

In this section, we illustrate the proposed method with a simple example. Consider the following true LTI system $\mathcal{F}_u(P, \Delta)$, with:

$$P_{11}(z) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad P_{12}(z) = \begin{bmatrix} 1 \\ \frac{-4.9z-5.1}{3.625z-6.375} \end{bmatrix}$$

$$P_{21}(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad P_{22}(z) = \begin{bmatrix} \frac{-4.9z-5.1}{3.625z-6.375} \\ \frac{1.5(z+1)^2}{18.6z^2-48.8z+32.6} \end{bmatrix}$$

[5]Note that it is not possible to deduce an exact bound for frequency response at frequencies not included in the experiment.

and

$$\Delta(z) = diag(\delta_1(z), \delta_2(z)) \tag{11}$$

$$\delta_1(z) = \frac{-0.24z^2 + 0.2z}{0.53z^2 - 0.4z + 1}$$

$$\delta_2(z) = \frac{0.044368(z-5)(z^2 - 1.244z + 0.5181)}{(z-1.149)(z^2+2z+2)}.$$

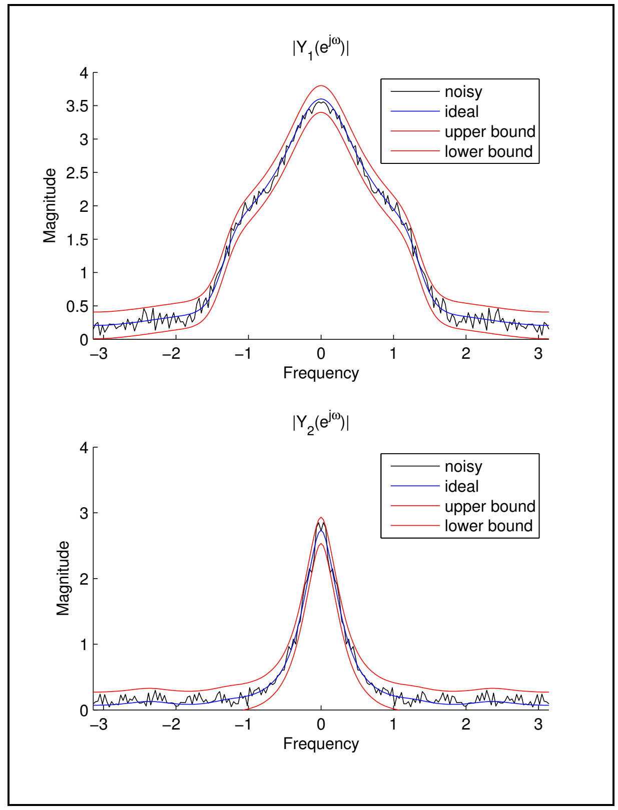

Fig. 2. Experimental data, noise free system, upper bound and lower bound of noisy system. The noise values are scaled by four for illustrative purposes.

Assume we are given P as a candidate model, together with the structure of Δ. Our experimental data consists of a set of $N = 200$ samples of the impulse response of $\mathcal{F}_u(P, \Delta)$ corrupted by additive noise in $\mathcal{N} = \mathcal{BL}_\infty[0, 2\pi](0.05)$. Noise samples are generated as complex numbers in this ball with uniform random phase and magnitude. We use $\gamma = 3$ as the coarse upper bound where $\|\Delta\|_\infty = 0.5976$ for the uncertainty in (11). The experimental output and the noise free plant are shown in Figure 2.

The goal is to find the "largest uncertainty" that could have generated this input/output pair given the a priori information. Figure 3 shows the frequency responses of this worst case uncertainty and the true uncertainty given in (11). Any δ with frequency response larger than the envelope at any frequency is invalidated by the experimental data. Moreover, for any δ whose frequency response is below the envelope it is possible to construct an admissible noise sequence to generate the experimental output.

It is important to note that, the worst case uncertainty does not necessarily correspond to the real uncertainty. It

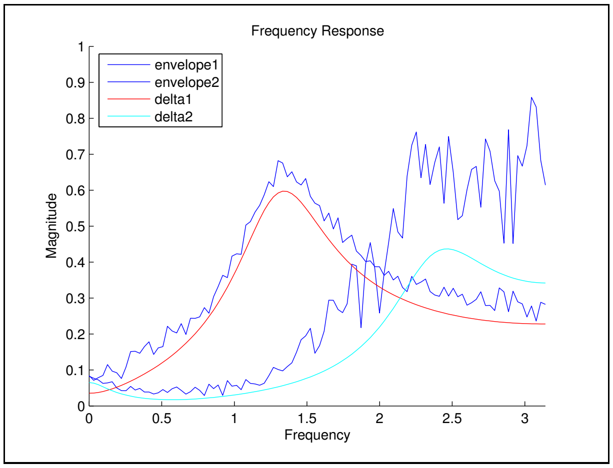

Fig. 3. Real uncertainties and the worst case uncertainties that could have generated the same input output pair.

is rather an upper bound of the real uncertainty since the real uncertainty could be anywhere in the consistency set, whereas the worst case uncertainty is on its boundary. As the consistency set shrinks by collecting multiple experimental data for a single frequency, the envelope will converge to the real uncertainty. In that sense, the presented algorithm can also be used as an identification scheme. However, our goal in this paper is not identification but to find a hard bound on the size of admissible uncertainty set given a single experiment.

V. CONCLUSIONS AND FUTURE WORK

Model (In)Validation is a critical step that must precede the use of models, obtained either from first principles or an identification step, to synthesize a controller. This problem has been extensively studied in the past decade, leading to several computationally efficient algorithms for invalidating a given (plant, uncertainty) description, based on noisy experimental data. A common feature of these algorithms is the fact that a given description is considered to be non–invalidated when a single pair (uncertainty, noise) is found such that, in conjunction with the assumed plant, can reproduce the experimental data. In this sense, these algorithms are overly *optimistic*, since they can substantially underestimate the actual size of the uncertainty, leading to poor performance, or even instability, when this description is used to synthesize a controller. This effect can be mitigated in part by pursuing a risk–adjusted approach such as the one pursued in [17], [14], that will reject as invalid, with probability close to 1, models validated only by a small set of (uncertainty, noise) pairs. Still, the resulting uncertainty bounds are optimistic, since the algorithm stops when a single validating pair is found.

To avoid these difficulties, in this paper we pursued a pessimistic approach, seeking to establish an outer bound on the uncertainty consistency set, that is, to find a set such that the model is guaranteed to be invalid *for all* uncertainties outside it. The main result of the paper shows that this problem can be reduced to a tractable convex optimization form by recasting it into a robust performance violation form. Since the latter is convexified by exploiting the S-procedure, the resulting conditions are generically only sufficient, that is, will tend to overbound the set. We conjecture that these conditions are also necessary and sufficient in the case of an arbitrary number of slowly time varying uncertainty blocks, but no formal proof is available at the moment. Efforts are currently under way pursuing research in this direction and towards reducing the conservatism in the LTI case by combining the approach described in this paper with risk adjusted methods.

APPENDIX
PROOF OF THEOREM 2

For simplicity, and without loss of generality, we will prove that the results of Algorithm 1 do not depend on the order in which α_1 and α_2 are calculated. However, the same reasoning applies to any pair (α_i, α_j). For a given frequency, start by assuming that α_1 is the minimum solution calculated in Step 2.i in algorithm 1 when $i = 1$. If α_2 is the minimum solution given α_1 (i.e in the second iteration when $i = 2$), it is indeed the solution to the following optimization problem:

$$\alpha_2 = \min_{\alpha\in[0,1],B} \alpha \;\; subject\ to$$

$$\begin{aligned} &M^* \begin{bmatrix} -\alpha_1^2 b_1 & 0 & 0 & 0 \\ 0 & \alpha^2 b_2 & 0 & 0 \\ 0 & 0 & -B_l & 0 \\ 0 & 0 & 0 & I \end{bmatrix} M \\ &- \begin{bmatrix} -b_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & -B_l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} > 0 \\ &B = diag[b_1, \dots, b_n] > 0 \end{aligned} \tag{12}$$

On the other hand, let $\overline{\alpha_2}$ be the solution when we initialize the algorithm with $i = 2$. That is:

$$\overline{\alpha_2} = \min_{\alpha\in[0,1],C} \alpha \;\; subject\ to$$

$$\begin{aligned} &M^* \begin{bmatrix} -c_1 & 0 & 0 & 0 \\ 0 & \alpha^2 c_2 & 0 & 0 \\ 0 & 0 & -C_l & 0 \\ 0 & 0 & 0 & I \end{bmatrix} M \\ &- \begin{bmatrix} -c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & -C_l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} > 0 \\ &C = diag[c_1, \dots, c_n] > 0. \end{aligned} \tag{13}$$

If $\overline{\alpha_2} < \alpha_2$, there exists no $X = diag[x_1, x_2, \dots, x_n] > 0$

such that

$$\begin{aligned} &M^* \begin{bmatrix} -\alpha_1^2 x_1 & 0 & 0 & 0 \\ 0 & \overline{\alpha_2}^2 x_2 & 0 & 0 \\ 0 & 0 & -X_l & 0 \\ 0 & 0 & 0 & I \end{bmatrix} M \\ &- \begin{bmatrix} -x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & -X_l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} > 0 \end{aligned} \tag{14}$$

since α_2 is the minimum of all $\alpha \in [0,1]$ that satisfies (12). By duality (see [1] p.29) this is equivalent to the existence of a positive semidefinite matrix $W = W^* \geq 0$, $W \neq 0$ such that the following holds for all $X = diag[x_1, x_2, \ldots, x_n] > 0$:

$$\begin{aligned} &\operatorname{trace} W \left(M^* \begin{bmatrix} -\alpha_1^2 x_1 & 0 & 0 & 0 \\ 0 & \overline{\alpha_2}^2 x_2 & 0 & 0 \\ 0 & 0 & -X_l & 0 \\ 0 & 0 & 0 & I \end{bmatrix} M \right. \\ &\left. - \begin{bmatrix} -x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & -X_l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \leq 0 \end{aligned} \tag{15}$$

In particular, (15) is true for $\overline{X} = diag[c_1/\alpha_1^2, c_2 \ldots, c_n] > 0$ where we choose c_i values the same as the arguments of (13) in the optimal solution. Plugging $\overline{X}$ into (15) and some simple manipulation leads:

$$\begin{aligned} &\operatorname{trace} W^{1/2} \left(M^* \begin{bmatrix} -c_1 & 0 & 0 & 0 \\ 0 & \overline{\alpha_2}^2 c_2 & 0 & 0 \\ 0 & 0 & -C_l & 0 \\ 0 & 0 & 0 & I \end{bmatrix} M \right. \\ &\left. - \begin{bmatrix} -c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & -C_l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) W^{1/2} \\ &+ \operatorname{trace} W^{1/2} \begin{bmatrix} (\frac{1}{\alpha_1^2} - 1)c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} W^{1/2} \leq 0. \end{aligned} \tag{16}$$

However, the first term in (16) is positive because of the positive definiteness constraint in (13) and the fact that $W = W^* \geq 0$, $W \neq 0$. Since $\alpha_1 \in [0,1]$, the second term is also nonnegative, leading to a contradiction. Hence $\overline{\alpha_2} \geq \alpha_2$. On the other hand, $\overline{\alpha_2}$ is always less than or equal to α_2 since the optimization problem (13) is less restrictive than (12). Thus, $\overline{\alpha_2} = \alpha_2$.

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