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Convex Necessary and Sufficient Conditions for Model (In)Validation under SLTV Structured Uncertainty

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Abstract—**This paper deals with the problem of model (in)validation of discrete–time, causal, LTI stable models subject to Slowly Linear Time Varying structured uncertainty, using frequency–domain data corrupted by additive noise. It is well known that in the case of structured LTI uncertainty the problem is NP hard in the number of uncertainty blocks. The main contribution of this paper shows that, on the other hand, if one considers arbitrarily slowly time varying uncertainty and noise in $\mathscr{L}_2$, then tractable, convex necessary and sufficient conditions for (in)validation can be obtained.**

I. Introduction

This paper deals with the problem of frequency–domain (in)validation of discrete–time, causal, Linear Time Invariant (LTI), stable models subject to Slowly Linear Time Varying (SLTV) structured diagonal uncertainty, that enters the model in Linear Fractional Transformation (LFT) form. In general terms, this problem can be formally stated as follows: Given frequency–domain data corrupted by additive noise, find whether the candidate model together with some combination of admissible uncertainty and noise could have generated this data. If the answer is negative, then the model is said to be invalidated and should be rejected; otherwise, is said to be not invalidated by the available experimental evidence.

Model (in)validation of LTI systems in a Robust Control setting has been extensively addressed in the past decade (see for instance [10], [7], [2], [1], [5], [9], [13] and references therein). The main result ([2], [1]) shows that in the case of LTI causal unstructured uncertainty and general LFT dependence, model (in)validation reduces to a convex optimization problem that can be efficiently solved, by applying norm constrained interpolation theory.

In the case of structured uncertainty, the problem still can be recast as a set of necessary and sufficient conditions, but in terms of bilinear matrix inequalities and has been shown in [12] to be NP–hard in the number of uncertainty blocks. However, computable weaker conditions (sufficient for the model to be invalidated) in the form of Linear Matrix Inequalities (LMIs) are available, by reducing the problem to the (in)validation of a scaled model subject to a scaled unstructured uncertainty as proposed by [2], [12], [9], or alternatively, by stating the invalidation problem as one of violation of robust performance by any admissible uncertainty (and solved as a structured singular value problem type) as in [10], [5].

This paper seeks to overcome the computational complexity of model (in)validation in the presence of structured uncertainty and in this sense, it is related to the approach in [5]. In fact, our starting point is a set of frequency–dependent LMI conditions with the same structure as the ones developed by [5]. However, by considering SLTV uncertainty operators with arbitrarily small variation rates (at the expense of relaxing the causality requirement, as is also the case of [5]), we obtain a necessary and sufficient condition for a model to be invalidated by experimental data. Note that since currently available analysis and design tools, e.g. μ–synthesis, provide tight conditions only for SLTV uncertainty[1], from a practical standpoint it is desirable also to allow for SLTV uncertainty in the model validation process. This avoids obtaining potentially more conservative LTI descriptions that, nevertheless, cannot be fully exploited for controller synthesis. Moreover, this set–up allows for directly dealing with an $\mathscr{L}_2$ characterization of the noise (as in the original formulation in [5]), rather than a set of pointwise in frequency euclidian norm constraints.

The paper is organized as follows. Section II presents the notation and conventions used through the paper and Section III states the model (in)validation problem under consideration. Section IV contains the main results; for ease of presentation the proof of the driver result of this paper is left to the Appendix. Finally, Section V illustrates the proposed method with a simulation example and Section VI presents some conclusions as well as directions for further research.

II. Preliminaries

Z, **R** and **C** denote the set of integer, real and complex numbers respectively. x denotes a complex–valued column vector, x^* its conjugate transpose row vector and $|x|$ its euclidean norm $\sqrt{x^*x}$. A^* denotes the conjugate transpose of matrix A, $A_{i,j}$ its (i,j) element, A_i its i-*th* column and $\overline{\sigma}(A)$ its maximum singular value. If $A = A^*$ then $A > 0$ ($A \leq 0$)

[1]These conditions are also tight in the case of of μ–simple LTI uncertainty structures, in which case the conditions provided in the present paper are also necessary and sufficient.

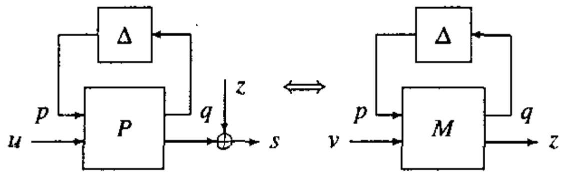

Fig. 1. Model (In)Validation Set-up

means that A is positive definite (negative semidefinite). $I, 0$ denote the identity and null matrices of compatible dimensions (when omitted).

Let ℓ_2^m denote the Hilbert space of vector–valued sequences $x = \{x_i\}_{i \in \mathbf{Z}}$, equipped with the norm $\|x\|_2^2 \doteq \sum_{i \in \mathbf{Z}} x_i^* x_i$ and $\mathscr{L}(\ell_2)$, the space of discrete–time, LTV, bounded operators M in ℓ_2, equipped with the norm $\|M\|_{\ell_2 \text{ ind}} \doteq \sup_{u \neq 0} \|Mu\|_2 / \|u\|_2$. Let $\mathscr{L}_2$ denote the Hilbert space of Lebesgue square integrable functions $x(\omega)$ equipped with the norm $\|x\|_2^2 \doteq \int_0^{2\pi} \operatorname{trace}\left(x(\omega)x(\omega)^*\right) \frac{d\omega}{2\pi}$; $\mathscr{L}_\infty$, the Lebesgue space of complex–valued matrix functions $X(z)$ essentially bounded on the unit circle, equipped with the norm $\|X\|_\infty \doteq \operatorname{ess}\sup_{|z|=1} \overline{\sigma}(X(z))$; $\mathscr{H}_\infty$, the subspace of functions in $\mathscr{L}_\infty$ with bounded analytic continuation inside the unit disk, equipped with the norm $\|X\|_\infty \doteq \operatorname{ess}\sup_{|z|<1} \overline{\sigma}(X(z))$ and $\mathscr{RL}_\infty$ ($\mathscr{H}_\infty$), the subspace of $\mathscr{L}_\infty$ ($\mathscr{H}_\infty$) of rational functions. Let $\mathscr{BX}(\gamma)$ be the open γ-ball in a normed space $\mathscr{X}$, $\mathscr{BX}(\gamma) = \{x \in \mathscr{X} : \|x\|_{\mathscr{X}} < \gamma\}$; $\overline{\mathscr{BX}}(\gamma)$, the closure of $\mathscr{BX}(\gamma)$ and $\mathscr{BX}$ ($\overline{\mathscr{BX}}$), the open (closed) unit ball in $\mathscr{X}$.

Finally, given a real–valued sequence x in ℓ_2^m, $x(e^{j\omega})$ denotes its Fourier transform $x(e^{j\omega}) \doteq \sum_{i \in \mathbf{Z}} x_i e^{j\omega i}$ and $X(z)$, the $\mathscr{Z}$-transform of a real–valued matrix sequence $\{X_i\}_{i \in \mathbf{Z}}$, $X(z) = \sum_{i \in \mathbf{Z}} X_i z^i$. We will use the same notation x, $\|x\|_2$ for elements in either ℓ_2^m or $\mathscr{L}_2$, in any case it will be clear from context which space we mean. λ denotes the unit delay operator and $M \star \Delta$, the upper linear fractional transformation $M \star \Delta = M_{21}\Delta(I - M_{11}\Delta)^{-1} M_{12} + M_{22}$.

III. Problem Statement

Consider the model (in)validation set–up, shown in Fig. 1 on the left, as an upper linear fractional interconnection $P \star \Delta$ between a discrete–time, causal, stable, LTI candidate model P:

$$\begin{aligned} q(e^{j\omega}) &= P_{11}(e^{j\omega})p(e^{j\omega}) + P_{12}(e^{j\omega})u(e^{j\omega}) \\ s(e^{j\omega}) &= P_{21}(e^{j\omega})p(e^{j\omega}) + P_{22}(e^{j\omega})u(e^{j\omega}) + z(e^{j\omega}) \end{aligned} \tag{1}$$

and a structured uncertainty block $\Delta = \operatorname{diag}(\Delta_1, \ldots, \Delta_n)$ with Δ_k full square:

$$p_k = \Delta_k(q_k), \quad k = 1, \ldots, n \tag{2}$$

which is assumed to belong to the set:

$$\mathbf{\Delta}_\nu^{SLTV}(\gamma) \doteq \left\{ \Delta \in \overline{\mathscr{BL}(\ell_2)}(\gamma) : \|\lambda\Delta - \Delta\lambda\|_{\ell_2 \text{ ind}} \leq \nu \right\},$$

with $\nu > 0$ but arbitrarily small, i.e. to the set of Linear Time Varying (LTV) operators bounded by γ, of arbitrarily slow variation[2] ν.

The block P consists of a nominal model of the actual system P_{22} and some description of how the uncertainty affects the model, given by blocks P_{11}, P_{12} and P_{21}. Furthermore, we assume that model P has a rational transfer function $P(z) \in \mathscr{RH}_\infty$ and that $\|P_{11}\|_\infty < \gamma^{-1}$ so that the interconnection $P \star \Delta$ is robustly ℓ_2 stable. Finally, (q, p) are partitioned according to the uncertainty structure as in (2) and (u, s, z) are scalar signals.

Given a known fixed frequency–domain input signal $u(e^{j\omega})$ and its output $s(e^{j\omega})$ possibly corrupted by additive noise $z(e^{j\omega})$ in the set

$$\mathscr{N} \doteq \overline{\mathscr{BL}_2}(\varepsilon),$$

the goal is to determine whether the candidate model P together with some admissible pair of uncertainty $\Delta \in \mathbf{\Delta}_\nu^{SLTV}(\gamma)$ and noise $z \in \mathscr{N}$ could have generated this input–output pair, i.e. whether:

$$s = (P \star \Delta)u + z, \quad \text{for some } (\Delta, z).$$

If the answer is affirmative, then the model is said to be not invalidated by the experimental evidence. On the contrary, if no such pair (Δ, z) exists, the model should be discarded.

Under the assumptions that both signals (u, s) are the impulse responses of some discrete–time, causal, stable, LTI, rational systems in $\mathscr{RH}_\infty$, equations (1) can be rewritten as follows:

$$\begin{aligned} q(e^{j\omega}) &= M_{11}(e^{j\omega})p(e^{j\omega}) + M_{12}(e^{j\omega})v(e^{j\omega}) \\ z(e^{j\omega}) &= M_{21}(e^{j\omega})p(e^{j\omega}) + M_{22}(e^{j\omega})v(e^{j\omega}), \end{aligned}$$

where v is an impulsive signal, i.e. $v(e^{j\omega}) = 1 \ \forall \omega \in [0, 2\pi)$, $u(e^{j\omega}) = S_u(e^{j\omega})v(e^{j\omega})$,

$$\begin{aligned} & M_{11}(e^{j\omega}) \doteq \gamma P_{11}(e^{j\omega}), \quad M_{12}(e^{j\omega}) \doteq P_{12}(e^{j\omega})S_u(e^{j\omega}), \\ & M_{21}(e^{j\omega}) \doteq -\frac{\gamma}{\varepsilon} P_{21}(e^{j\omega}), \\ & M_{22}(e^{j\omega}) \doteq \frac{1}{\varepsilon}\left(s(e^{j\omega}) - P_{22}(e^{j\omega})S_u(e^{j\omega})\right) \end{aligned} \tag{3}$$

and now (z, Δ) are assumed to be bounded in norm by 1, i.e. $z \in \overline{\mathscr{BL}_2}$, $\Delta \in \mathbf{\Delta}_\nu^{SLTV}$. In this framework, the model (in)validation problem can be precisely stated as follows.

Problem 1: Given the input-output pair $\{u(e^{j\omega}), s(e^{j\omega})\}$ and the admissible sets of noise $\mathscr{N}$ and uncertainty $\mathbf{\Delta}_\nu^{SLTV}(\gamma)$, determine whether there exists at least one pair $z \in \mathscr{N}$, $\Delta \in \mathbf{\Delta}_\nu^{SLTV}(\gamma)$ so that equations (1) and (2) hold; or equivalently, whether there exists at least one $\Delta \in \mathbf{\Delta}_\nu^{SLTV}$ so that:

$$\|z\|_2 = \|(M \star \Delta)v\|_2 \leq 1, \tag{4}$$

with system M defined as in (3) and v an impulsive input.

[2] Recall that if operator $\Delta \in \overline{\mathscr{BL}(\ell_2)}$ is time–invariant, it commutes with the delay operator, i.e. $\lambda\Delta = \Delta\lambda$ and therefore $\nu = 0$. On the other hand, $\nu = 2$ corresponds to the arbitrarily fast time–varying case, because $\|\lambda\Delta - \Delta\lambda\|_{\ell_2 \text{ ind}} \leq 2\|\Delta\|_{\ell_2 \text{ ind}}$.

IV. MAIN RESULTS

This section proposes a necessary and sufficient test that solves Problem 1, in terms of frequency–dependent Linear Matrix Inequalities. We begin by presenting the driver result of the paper.

Theorem 1: Consider a discrete–time, causal, stable, rational, LTI system $M(z) \in \mathscr{RH}_\infty$ and a structured uncertainty $\Delta \in \mathbf{\Delta}_\nu^{SLTV}$, $\Delta = \mathrm{diag}(\Delta_1, \dots, \Delta_n)$. Then, there exists some $\nu^* > 0$ such that $\|(M \star \Delta)v\|_2^2 > 1$, with v an impulsive input, for *any* $\Delta \in \mathbf{\Delta}_\nu^{SLTV}$ of variation $\nu \leq \nu^*$ if and only if there exists a hermitian matrix $X(\omega) > 0$ and a real transfer function $y(\omega) \geq 0$, such that $\forall\omega$ in $[0, 2\pi)$ the following inequalities hold:

$$M(e^{j\omega})^* \begin{bmatrix} X(\omega) & 0 \\ 0 & -1 \end{bmatrix} M(e^{j\omega}) - \begin{bmatrix} X(\omega) & 0 \\ 0 & -y(\omega) \end{bmatrix} \leq 0, \quad (5)$$

$X(\omega) = \mathrm{diag}(x_1(\omega)I_1, \dots, x_n(\omega)I_n)$ and

$$\int_0^{2\pi} y(\omega) \frac{d\omega}{2\pi} > 1. \quad (6)$$

Proof: Given in the appendix. ■

Direct application of this result leads to the following corollary, outlining a necessary and sufficient test for model (in)validation subject to structured SLTV uncertainties.

Corollary 1: Given a candidate model P, the experimental data $\{u(e^{j\omega}), s(e^{j\omega})\}$ and candidate noise and uncertainty sets $\{\mathscr{N}, \mathbf{\Delta}_\nu^{SLTV}(\gamma)\}$:

1) Form the system M defined in (3).
2) Evaluate at each frequency

$$\hat{y}(\omega) \doteq \sup\{y: \text{conditions (5) hold}\} \quad (7)$$

   and compute the integral $I(\hat{y}) \doteq \int_0^{2\pi} \hat{y}(\omega) \frac{d\omega}{2\pi}$.
3) Then there exists at least one $\Delta \in \mathbf{\Delta}_\nu^{SLTV}$ of arbitrarily small variation ν so that $\|(M \star \Delta)v\|_2 \leq 1$ with v an impulsive input (i.e. the model is not invalidated by the experimental data available so far) if and only if

$$I(\hat{y}) \leq 1. \quad (8)$$

Remark 1: Conditions (5) and (6), and therefore the proposed model (in)validation test, remain necessary and sufficient for LTI structures Δ with at most two different full blocks. The proof follows along the lines of that of Theorem 1. The sufficiency is straightforward for an arbitrary number of blocks; pre/postmultiplying (5) by $\begin{bmatrix} p^* & v^* \end{bmatrix}^*$ and its hermitian conjugate and using the facts that $\|\Delta\|_{\ell_2 \text{ ind}} \leq 1$ and it commutes with D immediately yields $|z(\omega)|^2 \geq y(\omega)$. The desired result follows then from (6). Necessity follows from the losslessness of the S-procedure in the case of at most three Hermitian quadratic forms in a complex linear space (see [3] and [6], Chapter 8, Section 8.1.2); if LMI (5) fails, it is always possible to construct a LTI $\hat{\Delta}$ so that $\|(M(e^{j\omega}) \star \hat{\Delta}(\omega))v(e^{j\omega})\|_2 \leq 1$, $v(e^{j\omega}) = 1$, $\forall\omega$.

On the other hand, if conditions (5) and (6) hold for a constant matrix X, then (as a consequence of operator D being constant and $D\Delta = \Delta D$) $\|(M \star \Delta)v\|_2^2 > 1$, for *any* $\Delta \in \mathbf{\Delta}^{LTV} \doteq \{\Delta \in \overline{\mathscr{BL}(\ell_2)}: \|\lambda\Delta - \Delta\lambda\|_{\ell_2 \text{ ind}} \leq 2\}$ and an inpulsive input v, i.e., model M subject to arbitrarily fast LTV uncertainty is invalidated by the experimental evidence.

Note that in principle applying the test above requires having experimental data at all frequencies. However, due to the continuity of $M(e^{j\omega})$, which in turns implies continuity of $X(\omega)$ and $y(\omega)$, the integral (6) can be approximated with arbitrary precision by a sum and thus the (in)validation test requires only a finite (albeit possibly large) number of experimental data points.

V. A SIMPLE EXAMPLE

In order to illustrate the proposed method, consider the following *true* LTI system $P \star \hat{\Delta}$, with:

$$P_{11}(z) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad P_{12}(z) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$P_{21}(z) = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \quad P_{22}(z) = \frac{3.5(z+1)^2}{18.6z^2 - 48.8z + 32.6}$$

and $\hat{\Delta}(z) = \mathrm{diag}(\hat{\Delta}_1(z), \hat{\Delta}_2(z), \hat{\Delta}_3(z))$, with:

$$\hat{\Delta}_1(z) = \frac{0.85(5.1 - 4.9z)}{(6.375 - 3.6250z)}$$

$$\hat{\Delta}_2(z) = \frac{0.65(5.001 - 4.9990z)}{(6.15 - 3.85z)}$$

$$\hat{\Delta}_3(z) = \frac{0.95(5.15 - 4.85z)}{(6.95 - 3.05z)}.$$

Assume we are given $P_{22}(z)$ as a candidate model for $P \star \hat{\Delta}$, together with a description of the uncertainty type[3] and how it enters the model in terms of the blocks (P_{11}, P_{12}, P_{21}) respectively. Our "experimental" data[4], $s(e^{j\omega})$, consists of a set of $N = 1000$ samples of the frequency response of $P \star \hat{\Delta}$, corrupted by complex additive noise in $\mathscr{N} \doteq \overline{\mathscr{BL}_2}(\varepsilon)$, with[5] $\varepsilon = 0.0894$. The plant, the model and the samples are shown in Fig. 2.

The goal is to check whether the given model subject to structured SLTV uncertainty is able to reproduce the experimental evidence within the assumed noise bound, i.e. whether there exist at least one $\Delta \in \mathbf{\Delta}_\nu^{SLTV}$ so that the *equivalent* closed–loop model as defined in (3) satisfies $\|(M \star \Delta)v\|_2 \leq 1$, where v denotes an impulsive input. If the answer is affirmative, it is also of interest to quantify the minimum size of the uncertainty γ, so that the model remains not invalidated by the data.

[3]Note that $\|\hat{\Delta}\|_\infty = 0.95$.

[4]In this example, we have generated the output noise samples as complex numbers with uniformly distributed random phase (between $[0, 2\pi)$) and (bounded) magnitude. We assume however that these frequency domain samples belong to some system in $\mathscr{RH}_\infty$ and therefore satisfy the conjugate symmetry property.

[5]This noise upper bound represents a 5% of the true frequency response energy.

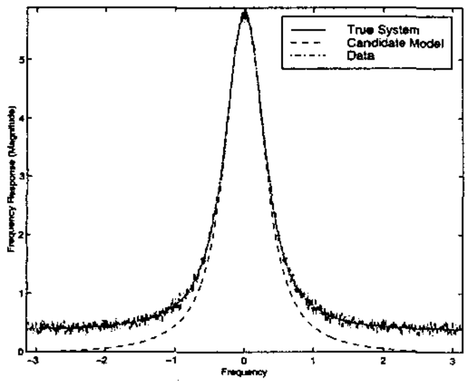

Fig. 2. Model, actual plant and samples.

To this end, we evaluated condition (7) at a grid of 1000 frequency points over the interval $[0, 2\pi)$. The second and fourth columns of Table I display the results of the proposed (in)validation test for increasing values of the uncertainty size γ in the interval $[0.2, 0.35]$. According to Lemma 1, the model remains invalidated by the available evidence, i.e. $\|s - (P \star \Delta)v\|_2 > \varepsilon$, for $\gamma \leq 0.2333$. Starting at $\gamma = 0.25$ the approximate value of $I(\hat{y}) \doteq \int_0^{2\pi} \hat{y}(\omega)\frac{d\omega}{2\pi}$ does not exceed 1 and we may conclude that there exists at least one admissible uncertainty in $\mathbf{\Delta}_v^{SLTV}(\gamma)$ so that the interconnection $P \star \Delta$ can explain the experimental data.

VI. Conclusions and Further Research

This paper presents a frequency–domain test for (in)validation of LTI models subject to SLTV structured diagonal uncertainties. By characterizing the noise in terms of its $\mathscr{L}_2$ norm and allowing an arbitrarily small variation rate of the uncertainty operator, at the expense of relaxing the causality requirement, this paper presents a set of frequency–dependent LMI based conditions, that are necessary and sufficient to solve the model (in)validation problem.

Efforts are currently under way to remove the noncausality limitation of the proposed method, by addressing the (in)validation problem in the time domain.

Appendix

Proof: [Sufficiency] Assume conditions (5) and (6) hold, i.e., for any $\varepsilon > 0$ and $\forall \omega \in [0, 2\pi)$, there exist $X(\omega) = X(\omega)^* > 0$ and a positive transfer function $y(\omega)$ so that:

$$M(e^{j\omega})^* \begin{bmatrix} X(\omega) & 0 \\ 0 & -I \end{bmatrix} M(e^{j\omega}) - \begin{bmatrix} X(\omega) & 0 \\ 0 & -y(\omega) \end{bmatrix} - \varepsilon I < 0 \quad (9)$$

and $\int_0^{2\pi} y(\omega)\frac{d\omega}{2\pi} > 1$. Following a reasoning similar to [8] (Lemma 2.3) or [6] (Chapter 6, Lemma 6.7), given $X(\omega)$ and the fact that $M(z)$ is rational, it is possible to construct a function $D(z) \in \mathscr{RH}_\infty$ that preserves the structure of X, such that $D^{-1}(z) \in \mathscr{RH}_\infty$ and $\forall \omega$, $X(\omega) = D(e^{j\omega})^* D(e^{j\omega})$. Since $D(z) \in \mathscr{RH}_\infty$, it admits the expansion $\sum_{i=0}^{\infty} D_i z^i$ where the sequence $\{D_i\}$ converges exponentially to zero. Denote by D the corresponding (LTI, causal) operator in $\mathscr{L}(\ell_2)$, $D \doteq \sum_{i=0}^{\infty} D_i \lambda^i$.

Pick $x(e^{j\omega}) = [p(e^{j\omega})^* \; v(e^{j\omega})^*]^*$, $v(e^{j\omega}) = 1$. Multiplying (9) from the left and from the right by $x(e^{j\omega})^*$ and $x(e^{j\omega})$ respectively, rearranging terms and integrating over $[0, 2\pi]$ yields:

$$\left[\|Dq\|_2^2 - \|Dp\|_2^2 - \varepsilon\left(\|p\|_2^2 + 1\right)\right] + \int_0^{2\pi} y(\omega)\frac{d\omega}{2\pi} < \|z\|_2^2. \quad (10)$$

Consider the term between brackets on the left hand side of the above equation. Following [6] (Chapter 6, page 91), we have that $\|D\Delta D^{-1}\|_{\ell_2\,\mathrm{ind}} \leq 1 + \nu\kappa(D)$ with $\kappa(D) \doteq \|D^{-1}\|_{\ell_2\,\mathrm{ind}} \sum_{i=0}^{\infty} i\overline{\sigma}(D_i) < \infty$. Letting $u = Dq$:

$$\begin{aligned} \|(D\Delta D^{-1})u\|_2^2 = \|Dp\|_2^2 &\leq [1 + \nu\kappa(D)]^2 \|Dq\|_2^2 \Leftrightarrow \\ 0 &\leq \|Dq\|_2^2 - \|Dp\|_2^2 + [\nu^2\kappa(D)^2 + 2\nu\kappa(D)]\|Dq\|_2^2 \\ &\leq \|Dq\|_2^2 - \|Dp\|_2^2 + [\nu^2\kappa(D)^2 + 2\nu\kappa(D)]\|D\|_{\ell_2\,\mathrm{ind}}^2 \|q\|_2^2. \end{aligned}$$

Denote $\alpha(\nu) \doteq [\nu^2\kappa(D)^2 + 2\nu\kappa(D)]\|D\|_{\ell_2\,\mathrm{ind}}^2$. Clearly $\alpha(\nu)$ approaches 0 as ν tends to 0. On the other hand, $\|q\|_2$ is uniformly bounded above by $\beta \doteq \|(I - M_{11}\Delta)^{-1}\|_{\ell_2\,\mathrm{ind}} \|M_{12}\|_\infty$ over the class $\mathbf{\Delta}_v^{SLTV}$ ([6], Appendix B) and by assumption $\int_0^{2\pi} y(\omega)\frac{d\omega}{2\pi} = 1 + \gamma$, $\gamma > 0$. Back to the term between brackets in (10):

$$\begin{aligned} &\left(\|Dq\|_2^2 - \|Dp\|_2^2 + \alpha(\nu)\|q\|_2^2\right) - \varepsilon\left(\|p\|_2^2 + 1\right) - \alpha(\nu)\|q\|_2^2 \geq \\ &-\varepsilon\left(\|q\|_2^2 + 1\right) - \alpha(\nu)\|q\|_2^2 \geq -\varepsilon(\beta^2 + 1) - \alpha(\nu)\beta^2 > -\gamma. \end{aligned}$$

Choosing $\varepsilon < \gamma/(2(1+\beta^2))$ and ν^* sufficiently small so that $\alpha(\nu^*) < \gamma/(2\beta^2)$ renders the left hand side of (10) always greater than 1 and yields the desired result:

$$1 < \|z\|_2^2 = \|(M \star \Delta)v\|_2^2, \quad v(e^{j\omega}) = 1,$$

for any $\Delta \in \mathbf{\Delta}_v^{SLTV}$ with $\nu \leq \nu^*$. ■

Before proceeding with the necessity part of the proof, we need the following preliminary result. See also [11].

Lemma 1: Let

$$M(e^{j\omega}) = \begin{cases} M_0, & \omega \in [\omega_0, \omega_0 + h] \\ 0, & \text{otherwise.} \end{cases}$$

If the following LMI:

$$M_0^* \begin{bmatrix} X & 0 \\ 0 & -I \end{bmatrix} M_0 - \begin{bmatrix} X & 0 \\ 0 & -1 \end{bmatrix} < 0 \quad (11)$$

γ	$I(\hat{y}) \approx$	γ	$I(\hat{y}) \approx$
0.2000	2.5887	0.2833	0.3274
0.2167	1.9169	0.3000	0.1703
0.2333	1.3577	0.3167	0.0769
0.2500	0.9093	0.3333	0.0281
0.2667	0.5683	0.3500	0.0077

TABLE I

Results of the Model (In)validation Test does not have a positive definite solution X, then there exist signals $r = [p^* \ v^*]^*$, $v(e^{j\omega}) = 1$ and $s = [q^* \ z^*]^*$ supported in $[\omega_0, \omega_0 + h]$ such that:

$$s = Mr, \quad \|q_k\|_2^2 \geq \|p_k\|_2^2 \ \ k = 1, \ldots, n, \quad \|z\|_2^2 \leq \|v\|_2^2. \quad (12)$$

Proof: Let P_k and Q_k be matrices of the form $[0 \cdots 0 \, I \, 0 \cdots 0]$, such that:

$$P_k r = \begin{cases} p_k & k = 1, \ldots, n \\ v & k = n+1 \end{cases} \qquad Q_k s = \begin{cases} q_k & k = 1, \ldots, n \\ z & k = n+1. \end{cases}$$

According to [4] (Lemma III.1), if (11) is not feasible then the following dual LMI has always a solution $W = W^* \geq 0$, $W \neq 0$:

$$\forall X, \ \text{trace}\left(M_0 W M_0^* \begin{bmatrix} X & 0 \\ 0 & -I \end{bmatrix} - W \begin{bmatrix} X & 0 \\ 0 & -I \end{bmatrix} \right) \geq 0$$
$$\Rightarrow \ \text{trace}(Q_k M_0 W M_0^* Q_k^* - P_k W P_k^*) \geq 0, \ k = 1, \ldots, n \quad (13)$$
$$P_{n+1} W P_{n+1}^* - Q_{n+1} M_0 W M_0^* Q_{n+1}^* \geq 0. \quad (14)$$

Let $m \doteq \text{rank}(W)$ and factor W as RR^*, where $R = [R_1 \cdots R_m]$ in $\mathbf{C}^{(n+1)\times m}$. Replacing the expression of W, equations (13) and (14) become:

$$\text{trace}\left(\sum_{i=1}^m Q_k M_0 R_i R_i^* M_0^* Q_k^* - \sum_{i=1}^m P_k R_i R_i^* P_k^* \right) \geq 0, \quad k = 1, \ldots, n \quad (15)$$
$$\sum_{i=1}^m P_{n+1} R_i R_i^* P_{n+1}^* - \sum_{i=1}^m Q_{n+1} M_0 R_i R_i^* M_0^* Q_{n+1}^* \geq 0.$$

Since $P_{n+1} W P_{n+1}^* \neq 0$ (otherwise robust stability would be violated[6]), we can always scale W so that $P_{n+1} W P_{n+1}^* = \sum_{i=1}^m |R_{n+1,i}|^2 = 1$. Moreover, we can always choose the elements $R_{n+1,i} \neq 0$, $i = 1, \ldots, m$ (e.g. by right multiplying R by a unitary matrix U).

Define the signal r over non-overlapping frequency intervals of length $h|R_{n+1,i}|^2$:

$$r(e^{j\omega}) = \begin{cases} \frac{R_i}{R_{n+1,i}}, & \omega \in [\omega_{i-1}, \omega_i], \quad i = 1, \ldots, m \\ 0, & \text{otherwise}, \end{cases}$$

with $\omega_i = \omega_{i-1} + h|R_{n+1,i}|^2$. Now by construction $v(e^{j\omega}) = 1$ in $[\omega_0, \omega_0 + h]$ and:

$$\int_0^{2\pi} r(e^{j\omega}) r(e^{j\omega})^* \frac{d\omega}{2\pi} = \sum_{i=1}^m \int_{\omega_{i-1}}^{\omega_i} \frac{R_i R_i^*}{|R_{n+1,i}|^2} \frac{d\omega}{2\pi} = \frac{h}{2\pi} \sum_{i=1}^m R_i R_i^*$$
$$\int_0^{2\pi} M(e^{j\omega}) r(e^{j\omega}) r(e^{j\omega})^* M(e^{j\omega})^* \frac{d\omega}{2\pi} = \frac{h}{2\pi} \sum_{i=1}^m M_0 R_i R_i^* M_0^*,$$

which together with (15) yields (12). ∎

Proof: [Necessity] Following [11], define at each frequency ω:

$$\hat{y}(\omega) \doteq \sup\{y \colon \text{conditions (5) hold}\}$$
$$y_k \doteq \max_{\omega \in [kh, (k+1)h]} \hat{y}(\omega)$$

for any partition over $[0, 2\pi]$. Note that since $y \leq M_{22}^* M_{22}$, $\hat{y}$ is well defined. Assume that condition (6) fails, i.e. $\int_0^{2\pi} \hat{y}(\omega) \frac{d\omega}{2\pi} \leq 1$. Consider first the case $\int_0^{2\pi} \hat{y}(\omega) \frac{d\omega}{2\pi} < 1$. Then, given an arbitrarily small $\varepsilon > 0$ there exists a $h_1(\varepsilon) > 0$ so that:

$$\int_0^{2\pi} \hat{y}(\omega) \frac{d\omega}{2\pi} \leq \sum_k y_k \frac{h}{2\pi} \leq 1 - \gamma(\varepsilon), \quad \forall h \leq h_1.$$

Pick $\gamma(\varepsilon) \doteq \varepsilon(2 - \frac{\varepsilon}{4}) > 0$.

Using the facts that the interconnection $M \star \Delta$ is uniformly robustly stable for the class $\mathbf{\Delta}_v^{SLTV}$ (see [6], Appendix B, Corollary B.5) and system M continuous on $[0, 2\pi)$, there exists a $h_2 > 0$ so that $\forall h \leq h_2$:

$$\|(M \star \Delta) - (\hat{M} \star \Delta)\|_{\ell_2 \, \text{ind}} \leq \frac{\varepsilon}{2}, \quad \forall \Delta \in \mathbf{\Delta}_v^{SLTV}, \quad (16)$$

where $\hat{M}(e^{j\omega}) \doteq M(e^{jkh})$ for $\omega \in [kh, (k+1)h]$.

Next, note that if $(X(\omega), y(\omega))$ solve LMI (5), then so do $X_\alpha(\omega) \doteq \alpha X(\omega)$ and $y_\alpha(\omega) \doteq \alpha y(\omega)$ for any $\alpha \in (0,1)$ [7]. Thus, it follows that $\hat{y}(\omega) \geq 0$. Define now the narrow-band system:

$$M_k \doteq M(e^{jkh}) \begin{bmatrix} I & 0 \\ 0 & \frac{1}{\sqrt{y_k + \varepsilon}} \end{bmatrix}, \quad \text{with} \quad h \leq \min(h_1, h_2);$$

[6]Consider the input signal r over non-overlapping frequency intervals of length $\frac{h}{m}$:

$$r(e^{j\omega}) = \begin{cases} R_i, & \omega \in [\omega_{i-1}, \omega_i], \quad i = 1, \ldots, m \\ 0, & \text{otherwise}, \end{cases}$$

with $\omega_i = \omega_{i-1} + \frac{h}{m}$. If $P_{n+1} W P_{n+1}$, then $\|v\|_2 = 0$ while (p, q) are not zero signals, violating robust stability against the class $\mathbf{\Delta}_v^{SLTV}$.

[7]This follows from noting that $\forall (p, q, v, z)$ and $\alpha \in (0, 1)$:

$$0 \geq |X(\omega)^{\frac{1}{2}} q(e^{j\omega})|^2 - |X(\omega)^{\frac{1}{2}} p(e^{j\omega})|^2 + y(\omega)|v(e^{j\omega})|^2 - |z(e^{j\omega})|^2 > |X(\omega)^{\frac{1}{2}} q(e^{j\omega})|^2 - |X(\omega)^{\frac{1}{2}} p(e^{j\omega})|^2 + y(\omega)|v(e^{j\omega})|^2 - \frac{1}{\alpha}|z(e^{j\omega})|^2.$$

by assumption the following LMI:

$$\begin{bmatrix} I & 0 \\ 0 & \frac{1}{\sqrt{y_k+\varepsilon}} \end{bmatrix} \left(M(e^{jkh})^* \begin{bmatrix} X(kh) & 0 \\ 0 & -I \end{bmatrix} M(e^{jkh}) - \begin{bmatrix} X(kh) & 0 \\ 0 & -(y_k+\varepsilon) \end{bmatrix} \right) \begin{bmatrix} I & 0 \\ 0 & \frac{1}{\sqrt{y_k+\varepsilon}} \end{bmatrix}$$
$$= M_k^* \begin{bmatrix} X(kh) & 0 \\ 0 & -I \end{bmatrix} M_k - \begin{bmatrix} X(kh) & 0 \\ 0 & -1 \end{bmatrix} < 0$$

is not feasible. Applying Lemma 1, there exist (piecewise constant) signals supported in $[kh,(k+1)h]$, $r^k = [(p^k)^*(v^k)^*]^*$, $v^k = 1$ and $s^k = [(q^k)^*(z^k)^*]^*$, so that:

$$s^k = M(e^{jkh})r^k, \quad \|q_i^k\|_2^2 \geq \|p_i^k\|_2^2 \qquad i = 1,\dots,n, \; . \\ \|z^k\|_2^2 \leq (y_k+\varepsilon)\|v^k\|_2^2. \tag{17}$$

Consider the following piecewise constant signals with support in $[0,2\pi)$:

$$\begin{matrix} \hat{p}(e^{j\omega}) \doteq p^k, & \hat{v}(e^{j\omega}) \doteq v^k, \\ \hat{q}(e^{j\omega}) \doteq q^k, & \hat{z}(e^{j\omega}) \doteq z^k, \end{matrix} \quad \omega \in [kh,(k+1)h],$$

the system $\hat{M}(e^{j\omega}) \doteq M(e^{jkh})$ for $\omega \in [kh,(k+1)h]$ and the perturbation $\hat{\Delta}$ ([6], [11]):

$$\hat{\Delta} = \mathrm{diag}(\hat{\Delta}_1,\dots,\hat{\Delta}_n), \quad \hat{\Delta}_i u \doteq \sum_k \frac{p_i^k \langle q_i^k, u\rangle}{\|q_i^k\|_2^2}.$$

By construction, $p^k = \hat{\Delta} q^k$ and $\hat{z} = (\hat{M} \star \hat{\Delta})\hat{v}$. It can also be shown that $\hat{\Delta} \in \mathbf{\Delta}_\nu^{SLTV}$, with $\nu \doteq 2\sin(\frac{h}{2})$. And according to Lemma 1 and (17), for an impulsive input $v(e^{j\omega}) = 1$:

$$\|(\hat{M} \star \hat{\Delta})v\|_2^2 \leq \sum_k (y_k+\varepsilon)\frac{h}{2\pi}.$$

Using (16), for this particular $\hat{\Delta}$ and this particular input v:

$$\|(M \star \hat{\Delta})v\|_2 \leq \|(M \star \hat{\Delta})v - (\hat{M} \star \hat{\Delta})v\|_2 + \|(\hat{M} \star \hat{\Delta})v\|_2$$
$$\leq \frac{\varepsilon}{2}\|v\|_2 + \left(\sum_k (y_k+\varepsilon)\frac{h}{2\pi}\right)^{\frac{1}{2}} \leq \frac{\varepsilon}{2} + (1-\gamma(\varepsilon)+\varepsilon)^{\frac{1}{2}} = 1. \tag{18}$$

Then there exists at least one $\hat{\Delta}$ of variation $\nu \doteq 2\sin(\frac{h}{2})$ with $h \leq \min(h_1,h_2)$, so that $\|(M \star \hat{\Delta})v\|_2^2 \leq 1$.

On the other hand, if $\int_0^{2\pi} \hat{y}(\omega)\frac{d\omega}{2\pi} \leq 1$ then given any $\varepsilon > 0$ there exists $h_1(\varepsilon) > 0$ so that:

$$\int_0^{2\pi} \hat{y}(\omega)\frac{d\omega}{2\pi} \leq \sum_k y_k \frac{h}{2\pi} \leq 1+\gamma(\varepsilon), \quad \forall h \leq h_1.$$

Pick $\gamma(\varepsilon) \doteq \varepsilon^2/4$. By following the same reasoning as before, we can construct a $\hat{\Delta}$ of variation $\nu \doteq 2\sin(\frac{h}{2})$ with $h \leq \min(h_1,h_2)$, so that $\|(M \star \hat{\Delta})v\|_2^2 \leq (1+\varepsilon)$. Since ε is arbitrarily small, we conclude that the model remains not invalidated against the class $\mathbf{\Delta}_\nu^{SLTV}$ of *arbitrarily small* variation ν, i.e.:

$$\lim_{\nu \to 0} \|(M \star \hat{\Delta})v\|_2^2 \leq 1.$$

■

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