Convex Necessary and Sufficient Conditions for Frequency Domain Model (In)Validation Under SLTV Structured Uncertainty

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***Abstract*—This paper deals with the problem of model (in)validation of discrete time, causal, linear time-invariant (LTI) stable models subject to slowly linear time-varying structured uncertainty, using frequency domain data corrupted by additive noise. It is well known that in the case of structured LTI uncertainty the problem is NP hard in the number of uncertainty blocks. The main contribution of this paper shows that, on the other hand, if one considers arbitrarily slowly time varying uncertainty and noise in $\mathcal{L}_2$, then tractable, convex necessary and sufficient conditions for (in)validation can be obtained. Additional results include a discussion of the case where the noise is characterized in terms of the $\mathcal{L}_\infty$ norm.**

***Index Terms*—Frequency domain model (in)validation, linear matrix inequalities, structured slowly linear time-varying (SLTV) uncertainty.**

I. Introduction

THIS paper deals with the problem of frequency domain (in)validation of discrete time, causal, linear time-invariant (LTI), stable models subject to slowly linear time-varying (SLTV) structured diagonal uncertainty, that enters the model in linear fractional transformation (LFT) form. In general terms, this problem can be formally stated as follows: Given frequency domain data corrupted by additive noise, find whether the candidate model together with some combination of admissible uncertainty and noise could have generated this data. If the answer is negative, then the model is said to be invalidated and should be rejected; otherwise, is said to be not invalidated by the available experimental evidence.

Model (in)validation of LTI systems in a robust control setting has been extensively addressed in the past decade (see, for instance, [1], [2], [6], [8], [10], [12], [15], and the references therein). The main result ([1], [2]) shows that in the case of LTI causal unstructured uncertainty and general LFT dependence, model (in)validation reduces to a convex optimization problem that can be efficiently solved, by applying norm constrained interpolation theory.

In the case of structured uncertainty, the problem still can be recast as a set of necessary and sufficient conditions, but in terms of bilinear matrix inequalities and has been shown in [14] to be NP hard in the number of uncertainty blocks. However, computable weaker conditions (sufficient for the model to be invalidated) in the form of linear matrix inequalities (LMIs) are available, by reducing the problem to the (in)validation of a scaled model subject to a scaled unstructured uncertainty as proposed by [2], [10], [14], or alternatively, by stating the invalidation problem as one of violation of robust performance by any admissible uncertainty (and solved as a structured singular value problem type) as in [6] and [12].

This paper seeks to overcome the computational complexity of model (in)validation in the presence of structured uncertainty and in this sense, it is related to the approach in [6]. In fact, our starting point is a set of frequency dependent LMI conditions with the same structure as the ones developed by [6]. However, by considering SLTV uncertainty operators with arbitrarily small variation rates (at the expense of relaxing the causality requirement, as is also the case of [6]), we obtain a necessary and sufficient condition for a model to be invalidated by experimental data. Note that since currently available analysis and design tools, e.g., μ-synthesis, provide tight conditions only for SLTV uncertainty,[1] from a practical standpoint it is desirable also to allow for SLTV uncertainty in the model validation process. This avoids obtaining potentially more conservative LTI descriptions that, nevertheless, cannot be fully exploited for controller synthesis. Moreover, this setup allows for directly dealing with an $\mathcal{L}_2$ characterization of the noise (as in the original formulation in [6]), rather than a set of pointwise in frequency Euclidean norm constraints.

The paper is organized as follows. Section II presents the notation and conventions used through this paper and Section III states the model (in)validation problem under consideration. Section IV contains the main results; for ease of presentation the proof of the driver result of this paper is left to the Appendix. Additional results in this section include a discussion of the case where the noise is characterized in terms of the $\mathcal{L}_\infty$ norm. Finally, Section V illustrates the proposed method with a simulation example and Section VI presents some conclusions as well as directions for further research.

II. Preliminaries

The following displays the notations and conventions used throughout this paper:

Z, **R**, **C** sets of integer, real, and complex numbers, respectively;

Manuscript received August 1, 2003; revised April 13, 2004. Recommended by Associate Editor P. A. Iglesias. This work was supported by the National Science Foundation under Grants ECS-0115946, ECS-0221562, and IIS-0312558.

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Digital Object Identifier 10.1109/TAC.2004.835387

[1]These conditions are also tight in the case of LTI uncertainty structures satisfying $2S + F \leq 3$, where S and F denote the number of repeated scalar and full blocks. In this case, the conditions provided in this paper are also necessary and sufficient.

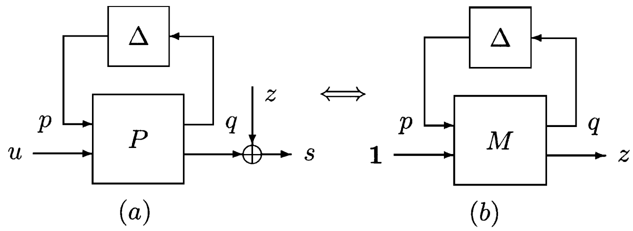

Fig. 1. (a) Model (in)validation setup. (b) Equivalent formulation.

x, x^*	complex valued column vector and its conjugate transpose row vector;
$\|x\|$	Euclidean norm of vector $x \in \mathbf{C}^m$: $\|x\| \doteq (x^*x)^{1/2}$;
$A_{i,j}$	(i,j) element of matrix A;
A_i	ith column of matrix A;
A^*	conjugate transpose of matrix A;
$\overline{\sigma}(A)$	maximum singular value of matrix A;
$\overline{\lambda}(A)$	if $A = A^*$, maximum eigenvalue of A;
$A \leq 0$	$A = A^*$ is negative semidefinite;
I,0	identity and null matrices of compatible dimensions, when omitted;
ℓ_2	Hilbert space of real valued vector sequences $\{x_i\}_{i\in\mathbf{Z}}$, equipped with the inner product: $\langle x, y\rangle \doteq \sum_{i\in\mathbf{Z}} x_i^* y_i$, and the norm: $\|x\|_2 \doteq \langle x, x\rangle^{1/2}$;
λ	unit delay operator;
P_N	truncation operator $P_N x$: $\{\ldots, 0, x_{-N+1}, \ldots, x_0, \ldots, x_{N-1}, 0, \ldots\}$;
$\mathcal{BX}(\gamma)$	open γ-ball in a normed space $\mathcal{X}$: $\mathcal{BX}(\gamma) = \{x \in \mathcal{X} : \|x\|_{\mathcal{X}} < \gamma\}$. $\mathcal{BX}$ stands for $\mathcal{BX}(1)$;
$\overline{\mathcal{BX}}(\gamma)$	closure of $\mathcal{BX}(\gamma)$;
$\mathcal{L}_\infty$	Lebesgue space of complex valued matrix functions $X(z)$ essentially bounded on the unit circle, equipped with the norm: $\|X\|_\infty \doteq \operatorname{ess\,sup}_{\|z\|=1} \overline{\sigma}(X(z))$;
$\mathcal{H}_\infty$	subspace of functions in $\mathcal{L}_\infty$ with bounded analytic continuation inside the unit disk, equipped with the norm: $\|X\|_\infty \doteq \operatorname{ess\,sup}_{\|z\|<1} \overline{\sigma}(X(z))$;
$\mathcal{RX}$	subspace of $\mathcal{X}$ of rational functions, $\mathcal{X}$ either $\mathcal{L}_\infty$ or $\mathcal{H}_\infty$;
$\mathcal{L}_2$	Hilbert space of Lebesgue square integrable vector functions $x(\omega)$ equipped with the norm $\|x\|_2 \doteq \int_0^{2\pi} \operatorname{trace}(x(\omega)x(\omega)^*)(dw/2\pi)$;
$\mathcal{L}(\ell_2)$	space of discrete time, LTV, bounded in ℓ_2 operators M, equipped with the norm: $\|M\|_{\ell_2 \mathrm{ind}} \doteq \sup_{u\neq 0}(\|Mu\|_2/\|u\|_2)$;
$x(e^{j\omega})$	Fourier transform of a real valued vector sequence in $\ell 2$: $x(e^{j\omega}) \doteq \sum_{i\in\mathbf{Z}} x_i e^{-j\omega i}$;
$X(z)$	$\mathcal{Z}$-transform of a real valued matrix sequence $\{X_i\}_{i\in\mathbf{Z}}$: but evaluated at $1/z$: $X(z) = \sum_{i\in\mathbf{Z}} X_i z^{-i}$;
$M \star \Delta$	upper linear fractional transformation $M \star \Delta = M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12} + M_{22}$.

III. Problem Statement

Consider the upper linear fractional (LFT) interconnection $P \star \Delta$ shown in Fig. 1(a) of a discrete time, causal, stable, LTI candidate model $P \in \mathcal{RH}_\infty$ and a structured uncertainty $\Delta \in \overline{\mathcal{BL}(\ell_2)}(\gamma)$, $\Delta = \operatorname{diag}(\Delta_1, \ldots, \Delta_n)$, Δ_k full square block.[2] The block P consists of a nominal model of the actual system P_{22}, and a description, given by the blocks P_{11}, P_{12}, and P_{21}, of how the uncertainty affects the model. In the sequel, we will assume that the block Δ is known to belong to the set

$$\mathcal{B}\boldsymbol{\Delta}_\nu^{\mathrm{SLTV}}(\gamma) \doteq \left\{\Delta \in \overline{\mathcal{BL}(\ell_2)}(\gamma) : \|\lambda\Delta - \Delta\lambda\|_{\ell_2 \mathrm{ind}} \leq \nu\right\} \tag{1}$$

with $\nu > 0$ but arbitrarily small, i.e., to the set of LTV operators bounded by γ, of arbitrarily slow variation[3] ν, as introduced in [9]. Finally, we will assume that the block P_{11} is such that the interconnection $P \star \Delta$ is robustly stable for all $\Delta \in \mathcal{B}\boldsymbol{\Delta}_\nu^{\mathrm{SLTV}}(\gamma)$.

The corresponding loop equations are given by

$$\begin{aligned} q &= P_{11}p + P_{12}u \\ s &= P_{21}p + P_{22}u + z \\ p_k &= \Delta_k q_k, \qquad k = 1, \ldots, n \end{aligned} \tag{2}$$

where the vector signals $u(e^{j\omega})$, $s(e^{j\omega})$ in $\mathcal{L}_2$ represent an arbitrary but known test input and its corresponding output respectively, corrupted by measurement noise $z(e^{j\omega})$ in a given convex set $\mathcal{N}$. In particular, following the initial formulation in [6], in most of this paper we will assume that

$$\mathcal{N} \doteq \overline{\mathcal{BL}_2}(\epsilon) \tag{3}$$

although we will also briefly consider other characterizations. The goal is to determine whether the candidate model P together with some admissible pair of uncertainty $\Delta \in \mathcal{B}\boldsymbol{\Delta}_\nu^{\mathrm{SLTV}}(\gamma)$ and noise $z \in \mathcal{N}$ could have generated this input–output pair, i.e., whether

$$s = (P \star \Delta)u + z, \qquad \text{for some}(\Delta, z). \tag{4}$$

If the answer is affirmative, then the model is said to be not invalidated by the experimental evidence. On the contrary, if no such pair (Δ, z) exists, the model should be discarded.

IV. Main Results

This section proposes necessary and sufficient conditions for frequency domain model (in)validation subject to SLTV structured uncertainty. These conditions are given in terms of frequency dependent LMIs. We begin by recasting the problem into the equivalent form shown in Fig. 1(b), that involves finding the minimum value, over Δ, of $\|(M \star \Delta)\mathbf{1}\|_2$.

A. Problem Transformation

Under the assumptions that $s(e^{j\omega}) \in \mathcal{L}_2$ and u is the impulse response of some known system $S_u \in \mathcal{RH}_\infty$, $u = S_u\mathbf{1}$, (2) can be rewritten as follows:

$$\begin{aligned} q(e^{j\omega}) &= M_{11}(e^{j\omega})p(e^{j\omega}) + M_{12}(e^{j\omega})\mathbf{1} \\ z(e^{j\omega}) &= M_{21}(e^{j\omega})p(e^{j\omega}) + M_{22}(e^{j\omega})\mathbf{1} \end{aligned}$$

[2]Without loss of generality, all results can be extended to cope with more general uncertainty structures.

[3]Recall that if operator $\Delta \in \overline{\mathcal{BL}(\ell_2)}$ is time invariant, it commutes with the delay operator, i.e., $\lambda\Delta = \Delta\lambda$ and therefore $\nu = 0$. On the other hand, $\nu = 2$ corresponds to the arbitrarily fast time varying case, since

$$\|\lambda\Delta - \Delta\lambda\|_{\ell_2 \mathrm{ind}} \leq 2\|\Delta\|_{\ell_2 \mathrm{ind}}.$$

where

$$\begin{aligned} M_{11}(e^{j\omega}) &\doteq \gamma P_{11}(e^{j\omega}) \\ M_{12}(e^{j\omega}) &\doteq P_{12}(e^{j\omega})S_u(e^{j\omega}) \\ M_{21}(e^{j\omega}) &\doteq -\frac{\gamma}{\epsilon}P_{21}(e^{j\omega}) \\ M_{22}(e^{j\omega}) &\doteq \frac{1}{\epsilon}\left(s(e^{j\omega}) - P_{22}(e^{j\omega})S_u(e^{j\omega})\right) \end{aligned} \tag{5}$$

and where (z, Δ) are suitable scaled so that $z \in \overline{\mathcal{BL}_2}$ and $\Delta \in \mathcal{B}\mathbf{\Delta}_\nu^{\mathrm{SLTV}} \doteq \mathcal{B}\mathbf{\Delta}_\nu^{\mathrm{SLTV}}(1)$. In this framework, the model (in)validation problem can be precisely stated as follows.

Problem 1: Given the input-output pair $\{u(e^{j\omega}), s(e^{j\omega})\}$ and the sets of admissible noise and uncertainty:

$$\begin{aligned} \mathcal{N} &\doteq \overline{\mathcal{BL}_2}(\epsilon) \\ \mathcal{B}\mathbf{\Delta}_\nu^{\mathrm{SLTV}} &\doteq \left\{\Delta \in \overline{\mathcal{BL}(\ell_2)} : \|\lambda\Delta - \Delta\lambda\|_{\ell_2 \mathrm{ind}} \leq \nu\right\} \end{aligned}$$

determine whether there exists at least one pair $z \in \mathcal{N}$, $\Delta \in \mathcal{B}\mathbf{\Delta}_\nu^{\mathrm{SLTV}}$ so that (2) hold or equivalently, whether there exists at least one $\Delta \in \mathcal{B}\mathbf{\Delta}_\nu^{\mathrm{SLTV}}$ so that

$$\|z\|_2 = \|(M \star \Delta)\mathbf{1}\|_2 \leq 1 \tag{6}$$

with system M defined as in (5).

B. Convex Necessary and Sufficient Condition

In this section, we show that the condition $\|(M \star \Delta)\mathbf{1}\|_2 > 1$ for all $\Delta \in \mathcal{B}\mathbf{\Delta}_\nu^{\mathrm{SLTV}}$ can be reduced to an LMI optimization. We begin by presenting the driver result of this paper.

Theorem 1: Consider a given stable, discrete-time, LTI system $M(z) \in \mathcal{RH}_\infty$ and an uncertainty structure

$$\mathbf{\Delta} = \left\{\Delta \in \mathcal{B}\mathbf{\Delta}_\nu^{\mathrm{SLTV}} : \Delta = \mathrm{diag}(\Delta_1, \ldots, \Delta_n)\right\}. \tag{7}$$

Assume that the interconnection $M \star \Delta$ is stable for all $\Delta \in \mathcal{B}\mathbf{\Delta}_\nu^{\mathrm{SLTV}}$. Then, there exists some $\nu^* > 0$ such that $\|(M \star \Delta)\mathbf{1}\|_2^2 > 1$ for *all* $\Delta \in \mathbf{\Delta}$ with $\nu \leq \nu^*$ if and only if there exist a Hermitian matrix $X(w) > 0$ and a real transfer function $y(\omega) \geq 0$, such that the following inequalities hold $\forall\omega$ in $[0, 2\pi)$:

$$M(e^{j\omega})^* \begin{bmatrix} X(\omega) & 0 \\ 0 & -I \end{bmatrix} M(e^{j\omega}) - \begin{bmatrix} X(\omega) & 0 \\ 0 & -y(\omega) \end{bmatrix} \leq 0 \tag{8}$$

$X(\omega) = \mathrm{diag}(x_1(\omega)I_1, \ldots, x_n(\omega)I_n)$ and

$$\int_0^{2\pi} y(\omega)\frac{d\omega}{2\pi} > 1. \tag{9}$$

Proof: Given in the Appendix. ■

Direct application of this result leads to the following corollary, outlining a necessary and sufficient test for model (in)validation subject to structured SLTV uncertainties.

Corollary 1: Given a candidate model P, experimental data $\{u(e^{j\omega}), s(e^{j\omega})\}$, and candidate noise and uncertainty sets $\{\mathcal{N}, \mathbf{\Delta}\}$:

1) form the system M defined in (5);

2) evaluate at each frequency

$$\hat{y}(\omega) \doteq \sup\{y : \text{conditions (8) hold}\} \tag{10}$$

and compute the integral $I(\hat{y}) \doteq \int_0^{2\pi} \hat{y}(\omega)(dw/2\pi)$;

3) then there exists at least one $\Delta \in \mathcal{B}\mathbf{\Delta}_\nu^{\mathrm{SLTV}}$ with arbitrarily small variation ν so that $\|(M \star \Delta)\mathbf{1}\|_2 \leq 1$ (that is the model is not invalidated by the experimental data available so far) if and only if

$$I(\hat{y}) \leq 1. \tag{11}$$

Remark 1: Conditions (8) and (9), and therefore the proposed model (in)validation test, remain necessary and sufficient for LTI structures Δ with at most two different full blocks. The proof follows along the lines of that of Theorem 1. The sufficiency is straightforward for an arbitrary number of blocks; pre/postmultiplying (8) by $[p^*(e^{j\omega})\ 1]^*$ and its Hermitian conjugate, and using the facts that $\|\Delta(e^{j\omega})\|_\infty \leq 1$ and it commutes with D immediately yields $\|z(\omega)\|^2 \geq y(\omega)$. The desired result follows then from (9). Necessity follows from the losslessness of the S-procedure in the case of at most three Hermitian quadratic forms in a complex linear space (see [4] and [7, Ch. 8, Sec. 8.1.2]; if LMI (8) fails $\forall\omega$, it is always possible to construct a LTI $\hat{\Delta}$ so that $\|(M(e^{j\omega}) \star \hat{\Delta}(e^{j\omega}))\mathbf{1}\|_2 \leq 1$.

Note that in principle applying the test above requires having experimental data at all frequencies. However, due to the continuity of $M(e^{j\omega})$, which in turns implies continuity of $X(\omega)$ and $y(\omega)$, the integral (9) can be approximated with arbitrary precision by a sum and thus the (in)validation test requires only a finite (albeit possibly large) number of experimental data points.

C. Alternative (In)Validation Procedure for Multiple-Input–Multiple-Output (MIMO) Systems

The algorithm outlined above provides a necessary and sufficient test for invalidating a MIMO system by applying a single (vector) input to the system. However, in some practical cases it may be desirable to consider one input at a time. For instance, if the model proves to be indeed invalidated by the data, such a procedure could indicate which subsystem has been incorrectly modeled. Clearly, applying one input at a time leads to LMIs of the form (8) and failure of any of these LMIs is a sufficient condition for invalidating the model. On the other hand, in principle, feasibility of these LMIs does not imply that the model is not invalidated by the data, since this does not guarantee that the same uncertainty Δ is used to validate each subsystem. Surprisingly, as we show in the sequel, it turns out that feasibility of these LMIs is both necessary and sufficient for the model not to be invalidated. Specifically, we have the following result.

Theorem 2: Consider a given stable, discrete time, LTI system $M(z) \in \mathcal{RH}_\infty$ and an uncertainty structure $\mathbf{\Delta}$ as in (7). Define

$$M^{(k)} = \begin{bmatrix} M_{11} & M_{12}^{(k)} \\ M_{21} & M_{22}^{(k)} \end{bmatrix}$$

where $M_{ij}^{(k)}$ denotes the k^{th} column of the operator M_{ij}. Then, there exists some $\nu^* > 0$ and n_u integers $i_1, i_2, \dots i_{n_u}$ such that

$$\max_k \left\{ \left\| \left(M^{(k)} \star \Delta \right) \lambda^{i_k} \mathbf{1} \right\|_2^2 \right\} > 1 \qquad \forall \Delta \in \mathbf{\Delta}, \quad \nu \le \nu^*$$

if and only if there exist n_u Hermitian matrices $X^{(k)}(w) > 0$ and real transfer functions $y^{(k)}(\omega) \ge 0$, such that the following inequalities hold $\forall \omega$ in $[0, 2\pi)$:

$$M(e^{j\omega})^{(k)*} \begin{bmatrix} X^k(\omega) & 0 \\ 0 & -I \end{bmatrix} M(e^{j\omega})^{(k)} - \begin{bmatrix} X^k(\omega) & 0 \\ 0 & -y^{(k)}(\omega) \end{bmatrix} \le 0 \quad (12)$$

$X^{(k)}(\omega) = \mathrm{diag}(x_1^{(k)}(\omega) I_1, \dots, x_n^{(k)}(\omega) I_n)$ and

$$\max_k \left\{ \int_0^{2\pi} y^{(k)}(\omega) \frac{d\omega}{2\pi} \right\} > 1. \quad (13)$$

Proof: The proof of sufficiency follows immediately from Theorem 1. The proof of necessity is given in the Appendix. ■

Corollary 2: A MIMO model can be (in)validated by sequentially applying a known signal to each input channel, provided that these signals are sufficiently spaced in time.

Proof: Follows by applying Theorem 2 to the system

$$\begin{aligned} M_{11} &\doteq P_{11} \quad M_{12} \doteq P_{12} \mathrm{diag}\left(S_u^{(1)}, \dots, S_u^{(n_u)} \right) \\ M_{21} &\doteq -\frac{\gamma}{\epsilon} P_{21} \\ M_{22} &\doteq \frac{1}{\epsilon} \left[\mathrm{diag}\left(s^{(1)}, \dots, s^{(n_u)} \right) - P_{22}\, \mathrm{diag}\left(S_u^{(1)}, \dots, S_u^{(n_u)} \right) \right] \end{aligned}$$

where $S_u^{(k)}$ and $s^{(k)}$ denote the spectra of the k^{th} input test signal and its corresponding output (i.e., when $u \doteq [0 \cdots 0 (u^k)^* 0 \cdots 0]^*$), respectively. ■

D. Arbitrarily Fast Time-Varying Uncertainty Case

Consider now the case of arbitrarily fast time varying uncertainty $\Delta \in \mathcal{B}\mathbf{\Delta}^{\text{LTV}}$

$$\mathcal{B}\mathbf{\Delta}^{\text{LTV}} \doteq \left\{ \Delta \in \overline{\mathcal{BL}(\ell_2)} : \| \lambda \Delta - \Delta \lambda \|_{\ell_2 \text{ind}} \le 2 \right\}$$

that is, the limit of the description (1) when $\nu \to 2$ (and $\gamma = 1$). It is well known that in the case of robust performance analysis the upper bound of μ obtained using frequency independent scales is a necessary and sufficient condition for robust stability [3], [11]. Motivated by this result and the close connection between Problem 1 and a modified μ-analysis problem pointed out in [6], [12], one may conjecture that a necessary and sufficient condition for invalidation in this case is the existence of a constant Hermitian matrix $X > 0$ and a $\mathcal{L}_2$ function $y(\omega) \ge 0$ so that conditions (8) and (9) hold. Indeed to show sufficiency pre/postmultiply (8) by $[p^* \ 1]$ and its Hermitian conjugate, and integrate, leading to

$$\|z\|_2^2 \ge \frac{1}{2\pi} \int_0^{2\pi} y(\omega) d\omega + \|\hat{q}\|_2^2 - \|\hat{p}\|_2^2$$

where we have defined $\hat{p} \doteq X^{1/2} p$ and $\hat{q} \doteq X^{1/2} q$. The fact that $\|(M \star \Delta)\mathbf{1}\|_2 > 1$ follows now by noting that since X is constant, it commutes with any $\Delta \in \mathcal{B}\mathbf{\Delta}^{\text{LTV}}$ and, thus, $\|\hat{q}\|_2^2 - \|\hat{p}\|_2^2 \ge 0$.

On the other hand, as we illustrate next with a simple counterexample, satisfaction of conditions (8) and (9) with constant scales X is *not necessary* for invalidation under arbitrarily fast varying uncertainty.

Counterexample 1: Consider a system with $P_{11} = P_{22} = 0$ and $P_{12} = P_{21} = 1$ so that $P \star \Delta = \Delta$. Assume that the experimental information is generated by the impulse response of some stable LTI system $\Delta = \hat{\Delta}$, with $\|\hat{\Delta}(e^{j\omega})\|_2 > 2$ and such that $\hat{\Delta}(e^{j\omega_o}) = 0$ for some ω_o. The corresponding transformed system (take $\gamma = 1$ and $\epsilon = 1$) is given by

$$\begin{aligned} M_{11}(z) &= 0 \quad M_{12}(z) = 1 \\ M_{21}(z) &= -1 \quad M_{22}(z) = \hat{\Delta}(z). \end{aligned}$$

Thus, the corresponding LMI reduces to

$$\begin{bmatrix} -x - 1 & \hat{\Delta}(e^{j\omega}) \\ \hat{\Delta}(e^{j\omega}) & x - \left| \hat{\Delta}(e^{j\omega}) \right|^2 + y(\omega) \end{bmatrix} \le 0. \quad (14)$$

Evaluating this LMI at ω_o yields $x + y(\omega_o) \le 0$ which implies that $x = 0$. It can be easily shown that, together with the fact that $y(\omega) \ge 0$, this implies $y(\omega) = 0 \ \forall \omega$. Hence, (9) fails. On the other hand, for any $\Delta \in \mathcal{B}\mathbf{\Delta}^{\text{LTV}}$, $\|\Delta\|_{\ell_2 \text{ind}} \le 1$ we have that

$$\begin{aligned} \|(M \star \Delta)\mathbf{1}\|_2 &= \left\| (\hat{\Delta} - \Delta)\mathbf{1} \right\|_2 \\ &\ge \left\| \hat{\Delta}(e^{j\omega}) \right\|_2 - \|\Delta\|_{\ell_2 \text{ind}} \|\mathbf{1}\|_2 > 1 \end{aligned}$$

and thus the model is indeed invalid.

E. $\overline{\mathcal{BL}_\infty}$ Noise Case and Connections With Earlier Results

In this section, we briefly analyze the situation where the measurement noise is characterized by a pointwise in frequency bound on its Euclidean norm rather than its energy. This case corresponds to the approximation proposed in [6] to handle model invalidation based on a finite set of frequency measurements. Motivated by the results of Section IV, one may rephrase conditions (8) and (9) as the existence of $y(\omega) \in \mathcal{L}_\infty$ and a Hermitian matrix $X(\omega) > 0$ such that the LMI in (8) holds and

$$\operatorname{ess} \sup_{\omega \in [0, 2\pi)} y(\omega) > 1. \quad (15)$$

Indeed, straightforward computations show that these conditions are equivalent to the sufficient[4] conditions obtained in [6] (Section V) for the model to be invalid under LTI, not necessarily causal, uncertainty. As we show in the sequel, the same conclusion holds if the uncertainty description is expanded to the class $\mathcal{B}\mathbf{\Delta}_\nu^{\text{SLTV}}$, $\forall \nu \le \nu^*$ and some ν^* sufficiently small.

Before proceeding, we require the following preliminary result.

Lemma 1: Consider a signal $q(e^{j\omega}) \in \mathcal{BL}_2$. Given an operator $\Delta \in \mathcal{B}\mathbf{\Delta}_\nu^{\text{SLTV}}$, let $p = \Delta q$ and let $f_h(e^{j\omega})$ be a continuous

[4]Necessary and sufficient for up to two full uncertainty blocks.

function on $[0, 2\pi]$, supported on the interval $[s, s+h]$, $h > 0$, i.e., $f_h(e^{j\omega}) \neq 0$ for $\omega \in [s, s+h]$, $f_h(e^{j\omega}) = 0$ otherwise.

Then, given $\epsilon > 0$ the following inequality holds:

$$\lim_{\nu \to 0} \int_s^{s+h} \left\|f_h(e^{j\omega})q(e^{j\omega})\right\|^2 - \left\|f^h(e^{j\omega})p(e^{j\omega})\right\|^2 \frac{d\omega}{2\pi} \geq -\epsilon. \tag{16}$$

Proof: See the Appendix. ■

Next, we present the main result of this section.

Theorem 3: Consider a given stable, discrete time, LTI system $M(z) \in \mathcal{RH}_\infty$ and the uncertainty structure $\mathbf{\Delta}$ defined in (7). Assume that the interconnection $M \star \Delta$ is stable for all $\Delta \in \mathcal{B}\mathbf{\Delta}_\nu^{\text{SLTV}}$.Then, if conditions (8) and (15) hold, there exists some $\nu^* > 0$ such that

$$\sup_{\omega \in [0, 2\pi)} \overline{\sigma}\left[(M \star \Delta)\mathbf{1}\right] > 1$$

for *all* $\Delta \in \mathbf{\Delta}$ with $\nu \leq \nu^*$.

Proof: If (15) holds, then there exists a closed interval of length h, $\Omega \doteq [s, s+h]$, where

$$\inf_\Omega y(\omega) = 1 + \gamma, \qquad \gamma > 0 \tag{17}$$

(recall that following [7] and [9], both $X(\omega)$ and $y(\omega)$ can be taken to be continuous on $[0, 2\pi)$). Also, if (8) holds, then for any $\epsilon > 0$ we have at each frequency ω

$$\left\|\tilde{q}(e^{j\omega})\right\|^2 - \left\|\tilde{p}(e^{j\omega})\right\|^2 - \epsilon\left(\left\|p(e^{j\omega})\right\|^2 + 1\right) + y(\omega) < \left\|z(e^{j\omega})\right\|^2 \tag{18}$$

where we assume that $v(e^{j\omega}) = 1$, $\forall \omega$, and that the signals $(\tilde{q}, \tilde{p}) \doteq (Dq, Dp)$ are related through the SLTV operator $\tilde{\Delta} \doteq D\Delta D^{-1}$ of variation rate $\tilde{\nu}$.

Let $f_h(e^{j\omega})$ be a continuous function on $[0, 2\pi]$, supported on Ω and such that

$$\|f_h\|_2^2 = \int_\Omega |f(\omega)|^2 \frac{d\omega}{2\pi} = 1.$$

Multiplying both sides of (18) by $|f_h(e^{j\omega})|^2$ and integrating over the interval Ω, we have

$$\left(\int_\Omega \|f_h(\omega)\tilde{q}(\omega)\|^2 - \|f_h(\omega)\tilde{p}(\omega)\|^2 \frac{d\omega}{2\pi}\right) + \int_\Omega y(\omega)\,|f_h(\omega)|^2 \frac{d\omega}{2\pi} - \epsilon \int_\Omega |f_h(\omega)|^2 \left(\|p(\omega)\|^2 + 1\right) \frac{d\omega}{2\pi} < \int_\Omega \|z(\omega)\|^2\,|f_h(\omega)|^2 \frac{d\omega}{2\pi} \leq \operatorname*{ess\,sup}_\Omega \|z(\omega)\|^2.$$

Using (17), the assumptions on f_h and the fact that $\|q\|_2$ can be uniformly bounded above by

$$\beta \doteq \left\|(I - M_{11}\Delta)^{-1}\right\|_{\ell_2 \text{ind}} \|M_{12}\|_\infty$$

over the class $\mathcal{B}\mathbf{\Delta}_\nu^{\text{SLTV}}$ ([7], Appendix B), the left hand side of the previous inequality can be further bounded below as follows:

$$\begin{aligned}
&\left(\int_\Omega \|f_h(\omega)\tilde{q}(\omega)\|^2 - \|f_h(\omega)\tilde{p}(\omega)\|^2 \frac{d\omega}{2\pi}\right) + (1+\gamma) - \epsilon \int_\Omega \|f_h(\omega)p(\omega)\|^2 \frac{d\omega}{2\pi} - \epsilon \\
&\geq \left(\int_\Omega \|f_h(\omega)\tilde{q}(\omega)\|^2 - \|f_h(\omega)\tilde{p}(\omega)\|^2 \frac{d\omega}{2\pi}\right) + (1+\gamma) - \epsilon\|f_h\|_\infty^2 \|p\|_2^2 - \epsilon \\
&\geq \left(\int_\Omega \|f_h(\omega)\tilde{q}(\omega)\|^2 - \|f_h(\omega)\tilde{p}(\omega)\|^2 \frac{d\omega}{2\pi}\right) + (1+\gamma) - \epsilon\left(\|f_h\|_\infty^2 \beta^2 + 1\right).
\end{aligned}$$

Selecting

$$\epsilon < \frac{\gamma}{2\left(\|f_h\|_\infty^2 \beta^2 + 1\right)}$$

and, following Lemma 1, the variation rate ν^* so that

$$\int_\Omega \|f_h(\omega)\tilde{q}(\omega)\|^2 - \|f_h(\omega)\tilde{p}(\omega)\|^2 \frac{d\omega}{2\pi} > -\frac{\gamma}{2}$$

then

$$\|(M \star \Delta)\mathbf{1}\|_\infty > 1 \qquad \forall \Delta \subset \mathcal{B}\mathbf{\Delta}_\nu^{\text{SLTV}}$$

with $\nu \leq \nu^*$. ■

Remark 2: Straightforward application of the S-procedure shows that (15) is also necessary for the model to be invalidated by the experimental data in the case of uncertainty structures with no more than 2 full blocks. On the other hand, for an arbitrary number of uncertainty blocks, proceeding as in the proof of Theorem 1, it can be shown that if, at a given frequency ω_o, $\hat{y}(\omega_o) < 1$, there exists some $h^* > 0$, $\nu^* > 0$ such that, for all $h < h^*$, $\nu < \nu^*$, there exists $\Delta^{(h)} \in \mathcal{B}\mathbf{\Delta}_\nu^{\text{SLTV}}$ such that

$$\frac{1}{h} \int_{\omega_o}^{\omega_o + h} \left\|(M \star \Delta^{(h)})\mathbf{1}\right\|^2 d\omega < 1.$$

However, since $z^{(h)}(\omega) \doteq (M \star \Delta^{(h)})\mathbf{1}$, is a function of h, one cannot conclude that the aforementioned condition implies that $\|z^{(h)}(\omega)\|^2 < 1$, $\omega \in [\omega_o, \omega + h]$. Thus, the issue of whether (8) and (15) are also necessary in the case of general uncertainty structures is still open. It is worth pointing out that a counterexample will have to involve uncertainty structures where the S-procedure is lossy, for instance with three or more full uncertainty blocks.

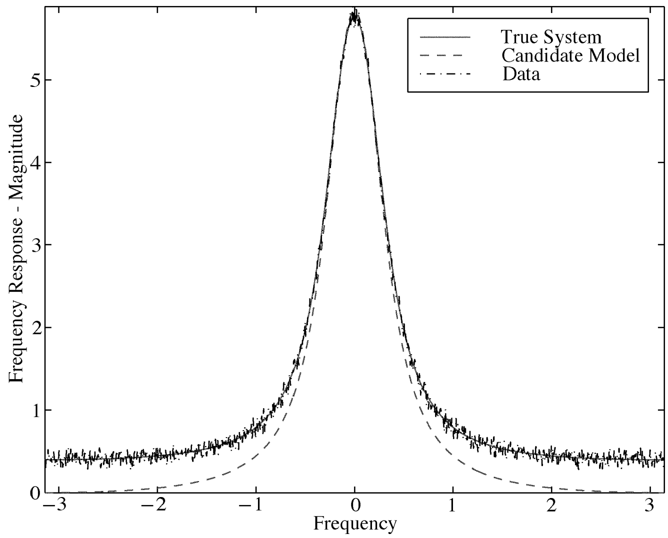

Fig. 2. Model, actual plant, and samples.

V. Simple Example

In order to illustrate the proposed method, consider the following *true* LTI system $P \star \hat{\Delta}$, with:

$$P_{11}(z) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad P_{12}(z) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$P_{21}(z) = [1 \quad 0 \quad -1] \quad P_{22}(z) = \frac{3.5(z+1)^2}{18.6z^2 - 48.8z + 32.6}$$

and

$$\begin{aligned} \hat{\Delta}(z) &= \mathrm{diag}\left(\hat{\Delta}_1(z), \hat{\Delta}_2(z), \hat{\Delta}_3(z)\right) \\ \hat{\Delta}_1(z) &= \frac{0.85(5.1 - 4.9z)}{(6.375 - 3.6250z)} \\ \hat{\Delta}_2(z) &= \frac{0.65(5.001 - 4.9990z)}{(6.15 - 3.85z)} \\ \hat{\Delta}_3(z) &= \frac{0.95(5.15 - 4.85z)}{(6.95 - 3.05z)}. \end{aligned} \tag{19}$$

Assume we are given $P_{22}(z)$ as a candidate model for $P \star \hat{\Delta}$, together with a description of the uncertainty type[5] and how it enters the model in terms of the blocks (P_{11}, P_{12}, P_{21}) respectively. Our "experimental" data,[6] $s(e^{j\omega})$, consists of a set of $N = 1000$ samples of the frequency response of $P \star \hat{\Delta}$, corrupted by complex additive noise in $\mathcal{N} \doteq \overline{\mathcal{BL}_2}(\epsilon)$, with[7] $\epsilon = 0.0894$. The plant, the model and the samples are shown in Fig. 2.

The goal is to check whether the given model subject to structured SLTV uncertainty is able to reproduce the experimental evidence within the assumed noise bound, i.e., whether there exists at least one $\Delta \in \mathcal{B}\mathbf{\Delta}_\nu^{\mathrm{SLTV}}$ so that the *equivalent* closed-loop model as defined in (5) satisfies $\|(M \star \Delta)\mathbf{1}\|_2 \le 1$. If the answer is affirmative, it is also of interest to quantify the minimum size of the uncertainty γ, so that the model remains not invalidated by the data.

TABLE I
Results of the Model (In)Validation Test

$\gamma \doteq \|\Delta\|_{\ell_2 \mathrm{ind}}$	$\mathcal{N} \doteq \overline{\mathcal{BL}_2}(\epsilon)$, $I(\hat{y}) \approx$	$\mathcal{N} \doteq \overline{\mathcal{BL}_\infty}(\epsilon)$, $\sup_{[0,2\pi)} \hat{y}(\omega) \ge$
0.2000	2.5887	11.0814
0.2167	1.9169	9.4002
0.2333	1.3577	7.8160
0.2500	0.9093	6.3498
0.2667	0.5683	5.0048
0.2833	0.3274	3.7927
0.3000	0.1703	2.7330
0.3167	0.0769	1.8259
0.3333	0.0281	1.0901
0.3500	0.0077	0.5334

To this end, we evaluated (10) at a grid of 1000 frequency points over the interval $[0, 2\pi)$. In this particular example, the frequency dependent LMI in (8) becomes:

$$\begin{bmatrix} -\left(\frac{\gamma}{\epsilon}\right)^2 & 0 & \left(\frac{\gamma}{\epsilon}\right)^2 & \frac{\gamma}{\epsilon}H(e^{j\omega}) \\ 0 & \gamma^2 d_1 & 0 & 0 \\ \left(\frac{\gamma}{\epsilon}\right)^2 & 0 & -\left(\frac{\gamma}{\epsilon}\right)^2 & -\frac{\gamma}{\epsilon}H(e^{j\omega}) \\ \frac{\gamma}{\epsilon}H^*(e^{j\omega}) & 0 & -\frac{\gamma}{\epsilon}H^*(e^{j\omega}) & -\left|H(e^{j\omega})\right|^2 \end{bmatrix} + \begin{bmatrix} -d_1 & 0 & 0 & 0 \\ 0 & -d_2 & 0 & 0 \\ 0 & 0 & -d_3 & 0 \\ 0 & 0 & 0 & d_2 + d_3 + y(\omega) \end{bmatrix} < 0$$

where d_1, d_2, and d_3 are strictly positive frequency dependent scalars, $X(\omega) \doteq \mathrm{diag}(d_1(\omega), d_2(\omega), d_3(\omega))$ and $H(e^{j\omega}) \doteq (s(e^{j\omega}) - P_{22}(e^{j\omega}))/\epsilon$.

The second column of Table I displays the results of the proposed (in)validation test for increasing values of the uncertainty size γ in the interval [0.2, 0.35]. According to Lemma 1, the model remains invalidated by the available evidence, i.e., $\|s - (P \star \Delta)v\|_2 > \epsilon$, for $\gamma \le 0.2333$; starting at $\gamma = 0.25$ the integral $I(\hat{y}) \doteq \int_0^{2\pi} \hat{y}(\omega)(dw/2\pi)$ does not exceed 1 and therefore there exists at least one admissible uncertainty in $\mathcal{B}\mathbf{\Delta}_\nu^{\mathrm{SLTV}}(\gamma)$ so that the interconnection $P \star \Delta$ can explain the experimental data.[8]

On the other hand, if we consider noise in $\mathcal{N} \doteq \overline{\mathcal{BL}_\infty}(\epsilon)$, now with $\epsilon = 0.1529$, we get the results shown by the third column of Table I. In this case, we conclude that the model is invalidated[9] by the data for $\gamma \le 0.3333$.

VI. Conclusion and Further Research

This paper presents a frequency domain test for (in)validation of LTI models subject to SLTV structured diagonal uncertainties. By characterizing the noise in terms of its $\mathcal{L}_2$ norm and

[5]Note that $\|\hat{\Delta}\|_\infty = 0.95$.

[6]In this example, we have generated the output noise samples as complex numbers with uniformly distributed random phase (between $[0, 2\pi)$) and (bounded) magnitude. We assume however that these frequency domain samples belong to some system in $\mathcal{RH}_\infty$ and, therefore, satisfy the conjugate symmetry property.

[7]This noise upper bound represents a 5% of the true frequency response energy.

[8]Note that in practice we can only compute $\hat{y}(\omega)$ over a finite grid of frequencies; the finer the grid the better the estimate of $I(\hat{y})$.

[9]In this case, we actually have a lower bound on $\sup_{[0,2\pi)} \hat{y}(\omega)$. Though we are performing the test over a finite grid of frequencies, the model is indeed invalidated by the data.

allowing an arbitrarily small variation rate of the uncertainty operator, and at the expense of relaxing the causality requirement, we obtained a set of frequency dependent LMI based conditions, that are necessary and sufficient for the experimental data to invalidate a given model-uncertainty description. Efforts are currently under way to remove the noncausality limitation of the proposed method, by addressing the (in)validation problem in the time domain.

In the case where the noise is characterized in terms of its $\mathcal{L}_\infty$ norm we have shown that similar conditions are sufficient for the model to be invalid, thus extending the conditions in [6] to the SLTV case. However, at this point it is not known whether these conditions are also necessary for uncertainty structures where the S-procedure is no longer lossless.

Appendix
Proof of Theorem 1

Before proceeding with the proof, we need the following preliminary result (see also [13]).

Lemma 2: Let

$$M(e^{j\omega}) = \begin{cases} M_0, & \omega \in [\omega_0, \omega_0 + h] \\ 0, & \text{otherwise.} \end{cases}$$

If the following LMI:

$$M_0^* \begin{bmatrix} X & 0 \\ 0 & -I \end{bmatrix} M_0 - \begin{bmatrix} X & 0 \\ 0 & -1 \end{bmatrix} < 0 \tag{20}$$

does not have a positive–definite solution X, then there exist signals $r = [p^* \ v^*]^*$, $v(e^{j\omega}) = 1$ and $s = [q^* \ z^*]^*$ supported in $[\omega_0, \omega_0 + h]$ such that

$$s = Mr \quad \|q_k\|_2^2 \geq \|p_k\|_2^2, \qquad k = 1, \ldots, n \quad \|z\|_2^2 \leq \|v\|_2^2. \tag{21}$$

Proof: Let P_k and Q_k be matrices of the form $[0 \ \cdots \ 0 \ I \ 0 \ \cdots 0]$, such that:

$$P_k r = \begin{cases} p_k & k = 1, \ldots, n \\ v & k = n+1 \end{cases} \quad Q_k s = \begin{cases} q_k & k = 1, \ldots, n \\ z & k = n+1. \end{cases}$$

According to [5, Lemma III.1], if (20) is not feasible then the following dual LMI has always a solution $W = W^* \geq 0$, $W \neq 0$:

$$\text{trace}\left(W M_0^* \begin{bmatrix} X & 0 \\ 0 & -I \end{bmatrix} M_0 - W \begin{bmatrix} X & 0 \\ 0 & -1 \end{bmatrix} \right) \geq 0 \qquad \forall X \tag{22}$$

$$\begin{aligned} \Leftrightarrow \ & \text{trace}\left(M_0 W M_0^* \begin{bmatrix} X & 0 \\ 0 & -I \end{bmatrix} - W \begin{bmatrix} X & 0 \\ 0 & -1 \end{bmatrix} \right) \\ & \geq 0 \qquad \forall X \\ \Rightarrow \ & \text{trace}\left(Q_k M_0 W M_0^* Q_k^* - P_k W P_k^* \right) \\ & \geq 0, \qquad k = 1, \ldots, n \end{aligned} \tag{23}$$

$$P_{n+1} W P_{n+1}^* - Q_{n+1} M_0 W M_0^* Q_{n+1}^* \geq 0. \tag{24}$$

Let $m \doteq \text{rank}(W)$ and factor W as RR^*, where $R = [R_1 \ R_2 \ \ldots \ R_m] \in \mathbf{C}^{(n+1)\times m}$. Replacing the expression of W, (23) and (24) become

$$\begin{aligned} & \text{trace}\left(\sum_{i=1}^m Q_k M_0 R_i R_i^* M_0^* Q_k^* - \sum_{i=1}^m P_k R_i R_i^* P_k^* \right) \\ & \quad \geq 0, \quad k = 1, \ldots, n \\ & \sum_{i=1}^m P_{n+1} R_i R_i^* P_{n+1}^* - \sum_{i=1}^m Q_{n+1} M_0 R_i R_i^* M_0^* Q_{n+1}^* \\ & \quad \geq 0. \end{aligned} \tag{25}$$

Since $P_{n+1} W P_{n+1}^* \neq 0$ (otherwise robust stability would be violated),[10] we can always scale W so that $P_{n+1} W P_{n+1}^* = \sum_{i=1}^m |R_{n+1,i}|^2 = 1$. Moreover, we can always choose the elements $R_{n+1,i} \neq 0$, $i = 1, \ldots, m$ (e.g., by right multiplying R by a unitary matrix U).

Define the signal r over non overlapping frequency intervals of length $h|R_{n+1,i}|^2$

$$r(e^{j\omega}) = \begin{cases} \frac{R_i}{R_{n+1,i}}, & \omega \in [\omega_{i-1}, \omega_i], \qquad i = 1, \ldots, m \\ 0, & \text{otherwise} \end{cases}$$

with $\omega_i = \omega_{i-1} + h|R_{n+1,i}|^2$. Now, by construction $v(e^{j\omega}) = 1$ in $[\omega_0, \omega_0 + h]$ and:

$$\begin{aligned} & \int_0^{2\pi} r(e^{j\omega}) r(e^{j\omega})^* \frac{d\omega}{2\pi} \\ & \quad = \sum_{i=1}^m \int_{\omega_{i-1}}^{\omega_i} \frac{R_i R_i^*}{|R_{n+1,i}|^2} \frac{d\omega}{2\pi} = \frac{h}{2\pi} \sum_{i=1}^m R_i R_i^* \\ & \int_0^{2\pi} M(e^{j\omega}) r(e^{j\omega}) r(e^{j\omega})^* M(e^{j\omega})^* \frac{d\omega}{2\pi} \\ & \quad = \frac{h}{2\pi} \sum_{i=1}^m M_0 R_i R_i^* M_0^* \end{aligned}$$

which together with (25) yields (21). ■

Proof: [Theorem 1, Sufficiency]: Assume (8) and (9) hold, i.e., for any $\epsilon > 0$ and $\forall \omega \in [0, 2\pi)$, there exist $X(\omega) = X(\omega)^* > 0$ and a positive transfer function $y(\omega)$ so that

$$\begin{aligned} & M(e^{j\omega})^* \begin{bmatrix} X(\omega) & 0 \\ 0 & -I \end{bmatrix} M(e^{j\omega}) - \begin{bmatrix} X(\omega) & 0 \\ 0 & -y(\omega) \end{bmatrix} - \epsilon I < 0 \\ & \int_0^{2\pi} y(\omega) \frac{d\omega}{2\pi} > 1. \end{aligned} \tag{26}$$

[10]Consider the input signal r over nonoverlapping frequency intervals of length h/m

$$r(e^{j\omega}) = \begin{cases} R_i, & \omega \in [\omega_{i-1}, \omega_i], \qquad i = 1, \ldots, m \\ 0, & \text{otherwise} \end{cases}$$

with $\omega_i = \omega_{i-1} + (h/m)$. If $P_{n+1} W P_{n+1}$, then $\|v\|_2 = 0$ while (p, q) are not zero signals, violating robust stability against the class $\mathcal{B}\mathbf{\Delta}_\nu^{\text{SLTV}}$.

Following a reasoning similar to [9, Lemma 2.3] or [7, Ch. 6, Lemma 6.7], given $X(\omega)$ and the fact that $M(z)$ is rational, it is possible to construct a function $D(z) \in \mathcal{RH}_\infty$ that preserves the structure of X, such that $D^{-1}(z) \in \mathcal{RH}_\infty$ and $\forall\omega, X(\omega) = D(e^{j\omega})^* D(e^{j\omega})$. Since $D(z) \in \mathcal{RH}_\infty$, it admits the expansion $\sum_{i=0}^{\infty} D_i z^i$ where the sequence $\{D_i\}$ converges exponentially to zero. Denote by D the corresponding (LTI, causal) operator in $\mathcal{L}(\ell_2)$, $D \doteq \sum_{i=0}^{\infty} D_i \lambda^i$.

Pick $x(e^{j\omega}) = [p^*(e^{j\omega})\ \mathbf{1}]^*$. Multiplying (26) from the left and from the right by $x(e^{j\omega})^*$ and $x(e^{j\omega})$, respectively, rearranging terms and integrating over $[0, 2\pi]$ yields

$$\left[\|Dq\|_2^2 - \|Dp\|_2^2 - \epsilon\left(\|p\|_2^2 + 1\right)\right] + \int_0^{2\pi} y(\omega)\frac{d\omega}{2\pi} < \|z\|_2^2. \tag{27}$$

Consider the term between brackets on the left hand side of the above equation. Following [7, Ch. 6, p. 91], we have that

$$\begin{aligned} \|D\Delta D^{-1}\|_{\ell_2 \mathrm{ind}} &\le 1 + \nu\kappa(D) \\ \kappa(D) &\doteq \|D^{-1}\|_{\ell_2 \mathrm{ind}} \sum_{i=0}^{\infty} i\bar{\sigma}(D_i) < \infty. \end{aligned} \tag{28}$$

Letting $u = Dq$:

$$\begin{aligned} &\left\|(D\Delta D^{-1})u\right\|_2^2 = \|Dp\|_2^2 \le [1 + \nu\kappa(D)]^2 \|Dq\|_2^2 \Leftrightarrow \\ &0 \le \|Dq\|_2^2 - \|Dp\|_2^2 + \left[\nu^2\kappa(D)^2 + 2\nu\kappa(D)\right]\|Dq\|_2^2 \\ &\quad \le \|Dq\|_2^2 - \|Dp\|_2^2 + \left[\nu^2\kappa(D)^2 + 2\nu\kappa(D)\right]\|D\|_{\ell_2 \mathrm{ind}}^2 \|q\|_2^2. \end{aligned}$$

Denote $\alpha(\nu) \doteq [\nu^2\kappa(D)^2 + 2\nu\kappa(D)]\|D\|_{\ell_2 \mathrm{ind}}^2$. Clearly, $\alpha(\nu) \to 0$ as $\nu \to 0$. On the other hand, $\|q\|_2$ is uniformly bounded above by $\beta \doteq \|(I - M_{11}\Delta)^{-1}\|_{\ell_2 \mathrm{ind}} \|M_{12}\|_\infty$ over the class $\mathcal{B}\boldsymbol{\Delta}_\nu^{\mathrm{SLTV}}$ [7, App. B], and by assumption $\int_0^{2\pi} y(\omega)(dw/2\pi) = 1 + \gamma, \gamma > 0$. Back to the term between brackets in (27)

$$\begin{aligned} &\left(\|Dq\|_2^2 - \|Dp\|_2^2 + \alpha(\nu)\|q\|_2^2\right) - \epsilon\left(\|p\|_2^2 + 1\right) - \alpha(\nu)\|q\|_2^2 \\ &\quad \ge -\epsilon\left(\|q\|_2^2 + 1\right) - \alpha(\nu)\|q\|_2^2 \\ &\quad \ge -\epsilon(\beta^2 + 1) - \alpha(\nu)\beta^2 > -\gamma. \end{aligned} \tag{29}$$

Choosing $\epsilon < \gamma/(2(1 + \beta^2))$ and ν^* sufficiently small so that $\alpha(\nu^*) < \gamma/(2\beta^2)$ renders the left-hand side of (27) always greater than 1 and yields the desired result

$$1 < \|z\|_2^2 = \|(M \star \Delta)\mathbf{1}\|_2^2 \tag{30}$$

for any $\Delta \in \mathcal{B}\boldsymbol{\Delta}_\nu^{\mathrm{SLTV}}$ with $\nu \le \nu^*$. ■

Proof: [Theorem 1, Necessity]: Following [13], define at each frequency ω:

$$\begin{aligned} \hat{y}(\omega) &\doteq \sup\{y : \text{condition (8) hold}\} \\ y_k &\doteq \max_{\omega \in [kh, (k+1)h]} \hat{y}(\omega) \end{aligned}$$

for any partition over $[0, 2\pi]$. Note that since $y \le M_{22}^* M_{22}$, $\hat{y}$ is well defined. Assume that condition (9) fails, i.e., $\int_0^{2\pi} \hat{y}(\omega)(dw/2\pi) \le 1$. Consider first the case $\int_0^{2\pi} \hat{y}(\omega)(dw/2\pi) < 1$. Then, given an arbitrarily small $\epsilon > 0$ there exists a $h_1(\epsilon) > 0$ so that

$$\int_0^{2\pi} \hat{y}(\omega)\frac{d\omega}{2\pi} \le \sum_k y_k \frac{h}{2\pi} \le 1 - \gamma(\epsilon), \quad \forall h \le h_1. \tag{31}$$

Pick $\gamma(\epsilon) \doteq \epsilon(2 - (\epsilon/4)) > 0$.

Using the facts that the interconnection $M \star \Delta$ is uniformly robustly stable for the class $\mathcal{B}\boldsymbol{\Delta}_\nu^{\mathrm{SLTV}}$ (see [7, App. B, Cor. B.5]) and system M continuous on $[0, 2\pi)$, there exists a $h_2 > 0$ so that

$$\left\|(M \star \Delta) - (\hat{M} \star \Delta)\right\|_{\ell_2 \mathrm{ind}} \le \frac{\epsilon}{2} \qquad \forall h \le h_2, \forall\Delta \in \mathcal{B}\boldsymbol{\Delta}_\nu^{\mathrm{SLTV}} \tag{32}$$

where $\hat{M}(e^{j\omega}) \doteq M(e^{jkh})$ for $\omega \in [kh, (k+1)h]$.

Next, note that if $(X(\omega), y(\omega))$ solve LMI (8), then so do $X_\alpha(\omega) \doteq \alpha X(\omega)$ and $y_\alpha(\omega) \doteq \alpha y(\omega)$ for any $\alpha \in (0, 1)$.[11] Thus, it follows that $\hat{y}(\omega) \ge 0$. Define now the narrow band system:

$$M_k \doteq M(e^{jkh})\begin{bmatrix} I & 0 \\ 0 & \frac{1}{\sqrt{y_k + \epsilon}} \end{bmatrix}, \quad \text{with} \quad h \le \min(h_1, h_2)$$

by assumption

$$\begin{aligned} &\begin{bmatrix} I & 0 \\ 0 & \frac{1}{\sqrt{y_k+\epsilon}} \end{bmatrix}\left(M(e^{jkh})^*\begin{bmatrix} X(kh) & 0 \\ 0 & -I \end{bmatrix} M(e^{jkh}) \right. \\ &\qquad \left. - \begin{bmatrix} X(kh) & 0 \\ 0 & -(y_k + \epsilon) \end{bmatrix}\right)\begin{bmatrix} I & 0 \\ 0 & \frac{1}{\sqrt{y_k+\epsilon}} \end{bmatrix} \\ &\quad = M_k^*\begin{bmatrix} X(kh) & 0 \\ 0 & -I \end{bmatrix} M_k - \begin{bmatrix} X(kh) & 0 \\ 0 & -1 \end{bmatrix} < 0 \end{aligned}$$

is not feasible. Applying Lemma 3, there exist (piecewise constant) signals supported in $[kh, (k+1)h]$, $r^k = [(p^k)^*(v^k)^*]^*$, $v^k = 1$ and $s^k = [(q^k)^*(z^k)^*]^*$, so that

$$\begin{aligned} s^k = M(e^{jkh})r^k \qquad &\left\|q_i^k\right\|_2^2 \ge \left\|p_i^k\right\|_2^2 \quad i = 1, \ldots, n \\ &\|z^k\|_2^2 \le (y_k + \epsilon)\|v^k\|_2^2. \end{aligned} \tag{33}$$

Consider the following piecewise constant signals with support in $[0, 2\pi)$:

$$\left.\begin{aligned} \hat{p}(e^{j\omega}) &\doteq p^k \\ \hat{q}(e^{j\omega}) &\doteq q^k \\ \hat{v}(e^{j\omega}) &\doteq v^k \\ \hat{z}(e^{j\omega}) &\doteq z^k \end{aligned}\right. \quad \omega \in [kh, (k+1)h]$$

[11]This follows from noting that $\forall(p, q, v, z)$ and $\alpha \in (0, 1)$:

$$\begin{aligned} 0 \ge\ &\left|X(\omega)^{\frac{1}{2}} q(e^{j\omega})\right|^2 - \left|X(\omega)^{\frac{1}{2}} p(e^{j\omega})\right|^2 \\ &+ y(\omega)\left|v(e^{j\omega})\right|^2 - \left|z(e^{j\omega})\right|^2 \\ >\ &\left|X(\omega)^{\frac{1}{2}} q(e^{j\omega})\right|^2 - \left|X(\omega)^{\frac{1}{2}} p(e^{j\omega})\right|^2 \\ &+ y(\omega)\left|v(e^{j\omega})\right|^2 - \frac{1}{\alpha}\left|z(e^{j\omega})\right|^2. \end{aligned}$$

the system $\hat{M}(e^{j\omega}) \doteq M(e^{jkh})$ for $\omega \in [kh, (k+1)h]$ and the perturbation $\hat{\Delta}$

$$\hat{\Delta} = \mathrm{diag}(\hat{\Delta}_1, \ldots, \hat{\Delta}_n), \quad \hat{\Delta}_i u \doteq \sum_k \frac{p_i^k \left\langle q_i^k, u_i \right\rangle}{\left\| q_i^k \right\|_2^2}. \quad (34)$$

By construction, $p^k = \hat{\Delta} q^k$ and $\hat{z} = (\hat{M} \star \hat{\Delta})\hat{v}$. It can also be shown that $\hat{\Delta} \in \mathcal{B}\mathbf{\Delta}_\nu^{\mathrm{SLTV}}$, with $\nu \doteq 2\sin(h/2)$. According to Lemma 2 and (33), for an impulsive input $v(e^{j\omega}) = 1$

$$\left\| (\hat{M} \star \hat{\Delta}) v \right\|_2^2 \le \sum_k (y_k + \epsilon) \frac{h}{2\pi}. \quad (35)$$

Using (32), for this particular $\hat{\Delta}$ and this particular input v

$$\begin{aligned}\left\| (M \star \hat{\Delta}) v \right\|_2 &\le \left\| (M \star \hat{\Delta}) v - (\hat{M} \star \hat{\Delta}) v \right\|_2 + \left\| (\hat{M} \star \hat{\Delta}) v \right\|_2 \\ &\le \frac{\epsilon}{2} \|v\|_2 + \left(\sum_k (y_k + \epsilon) \frac{h}{2\pi} \right)^{\frac{1}{2}} \\ &\le \frac{\epsilon}{2} + (1 - \gamma(\epsilon) + \epsilon)^{\frac{1}{2}} = 1. \end{aligned} \quad (36)$$

Then[12] there exists at least one $\hat{\Delta}$ of variation $\nu \doteq 2\sin(h/2)$ with $h \le \min(h_1, h_2)$, so that

$$\left\| (M \star \hat{\Delta}) v \right\|_2^2 \le 1.$$

On the other hand, if $\int_0^{2\pi} \hat{y}(\omega)(dw/2\pi) \le 1$ then given any $\epsilon > 0$ there exists $h_1(\epsilon) > 0$ so that

$$\int_0^{2\pi} \hat{y}(\omega) \frac{d\omega}{2\pi} \le \sum_k y_k \frac{h}{2\pi} \le 1 + \gamma(\epsilon) \qquad \forall h \le h_1.$$

Pick $\gamma(\epsilon) \doteq \epsilon^2/4$. By following the same reasoning as before, we can construct a $\hat{\Delta}$ of variation $\nu \doteq 2\sin(h/2)$ with $h \le \min(h_1, h_2)$, so that:

$$\left\| (M \star \hat{\Delta}) v \right\|_2^2 \le (1 + \epsilon).$$

Since ϵ is arbitrarily small, we conclude that the model remains not invalidated against the class $\mathcal{B}\mathbf{\Delta}_\nu^{\mathrm{SLTV}}$ of *arbitrarily small* variation ν, i.e.,

$$\lim_{\nu \to 0} \left\| (M \star \hat{\Delta}) v \right\|_2^2 \le 1.$$

■

[12]By selecting $\gamma(\epsilon) \doteq \epsilon(3 - \epsilon) > 0$, $\|(M \star \hat{\Delta})v\|_2^2$ can be made strictly less than 1.

Proof of Necessity in Theorem 2

Assume that condition (13) fails. Then, from Theorem 1 it follows that, for each j there exist some signals $p^{(j)}, q^{(j)} \in \mathcal{L}_2$ such that

$$\begin{aligned} \begin{bmatrix} q^{(j)} \\ z^{(j)} \end{bmatrix} &= M^{(j)} \begin{bmatrix} p^{(j)} \\ \mathbf{1} \end{bmatrix} \\ \int_{s_k}^{s_k+h} \left\| p_i^{(j)} \right\|^2 d\omega &\le \int_{s_k}^{s_k+h} \left\| q_i^{(j)} \right\|^2 d\omega \\ \frac{1}{2\pi} \int_0^{2\pi} \left\| z^{(j)} \right\|^2 d\omega &\le 1 \end{aligned} \quad (37)$$

for some partition $\{s_k\}$ of $[0, 2\pi]$, and where the input p and output q have been partitioned according to the uncertainty structure. Let $p_N \doteq P_N p$ and $q_N \doteq P_N q$. Since $p^{(j)}, q^{(j)}$ belong to $\mathcal{L}_2$ it follows that there exist N_1, N_2 large enough such that, for all $1 \le j \le n_u$

$$\begin{aligned} \left\| q_{N_1}^{(j)} \right\|_2 &\ge \left\| q^{(j)} \right\|_2 - \epsilon \\ \left\| p_{N_2}^{(j)} \right\|_2 &\ge \left\| p^{(j)} \right\|_2 - \epsilon \\ \int_{s_k}^{s_k+h} \left\| p_{N_2}^{(j)} \right\|^2 d\omega &\le \int_{s_k}^{s_k+h} \left\| q_{N_1}^{(j)} \right\|^2 d\omega. \end{aligned} \quad (38)$$

Consider now the perturbation $\tilde{\Delta} \doteq \sum_{i=1}^{n_u} \lambda^{2iN} \Delta_i \lambda^{-2iN}$, with $N = \max\{N_1, N_2\}$ and

$$(\Delta_i u) = p_{N_2}^{(i)} \frac{\left\langle q_{N_1}^{(i)}, u \right\rangle}{\left\| q_{N_1}^{(i)} \right\|^2}. \quad (39)$$

Since by construction the signals $\{\lambda^{2Ni} q_{N_1}^{(i)}, \lambda^{2Nj} q_{N_1}^{(j)}\}$ are orthogonal, it follows that $\tilde{\Delta} \lambda^{2iN} q_{N_1}^{(i)} = \lambda^{2iN} p_{N_2}^{(i)}$. Moreover, proceeding as in [13] it can be easily shown that $\tilde{\Delta} \in \mathcal{B}\mathbf{\Delta}_\nu^{\mathrm{SLTV}}$. Finally, since $\|(I - M_{11}\Delta)^{-1}\|_{\ell_2 \mathrm{ind}}$ is uniformly bounded over $\mathcal{B}\mathbf{\Delta}_\nu^{\mathrm{SLTV}}$ from (37) and (38) it follows that

$$\left\| \left(M^{(i)} \star \tilde{\Delta} \right) \lambda^{2Ni} \mathbf{1} \right\|_2^2 = \left\| z^{(i)} \right\|_2^2 + \mathcal{O}(\epsilon) \le 1 + \mathcal{O}(\epsilon) \quad (40)$$

for $i = 1, \ldots, n_u$. Since ϵ is arbitrary, this last inequality implies that the model has not been invalidated by the available experimental data.

Proof of Lemma 1

Proof: Since f_h is continuous on $[0, 2\pi]$, it is bounded, i.e., $\|f_h\|_\infty < \infty$. From Weierstrass' approximation theorem [16, Ch. 5] there exists a polynomial[13] f_n that converges uniformly to f_h on $[0, 2\pi]$ as $n \to \infty$, i.e., given any $\epsilon_1 > 0$ there exists some N such that if $n \ge N$ then $|f_h(e^{j\omega}) - f_n(e^{j\omega})| <$

[13]The polynomial f_n can be computed for example as the arithmetic mean of the first $n+1$ partial sums of the Fourier series of f_h (Fejèr theorem [16, Ch. 5]).

ϵ_1, $\forall\omega$ in $[0, 2\pi]$. Pick $\epsilon_1 \doteq \epsilon/(4\|f_h\|_\infty)$. Consider now the image of $f_h q$

$$\tilde{p} \doteq \Delta f_h q = \Delta f_n q + \Delta(f_h - f_n)q = \Delta \sum_{k=-N}^{N} c_k \lambda^k q + \Delta(f_h - f_n)q.$$

Let $\mathcal{C}^n = \Delta\lambda^n - \lambda^n\Delta$. Since $\Delta \in \mathcal{B}\mathbf{\Delta}_\nu^{\mathrm{SLTV}}$ we have that $\|\mathcal{C}^n\|_{\ell_2\mathrm{ind}} \le n\nu$. Also

$$\|\Delta(f_h - f_n)q\|_2 \le \|f_h - f_n\|_\infty \|q\|_2 \le \epsilon_1$$

since by assumption $q \in \mathcal{B}\mathcal{L}_2$.

Direct computation of $\tilde{p}$ yields

$$\begin{aligned}\tilde{p} &= \sum_{k=-N}^{N} c_k \Delta \lambda^k q + \Delta(f_h - f_n)q \\ &= \sum_{k=-N}^{N} c_k \lambda^k \Delta q + \sum_{k=-N}^{N} c_k \mathcal{C}^k q + \Delta(f_h - f_n)q \\ &= f_h p + g(p, q) \end{aligned} \tag{41}$$

where

$$g(p, q) = -(f_h - f_n)\Delta q + \sum_{k=-N}^{N} c_k \mathcal{C}^k q + \Delta(f_h - f_n)q.$$

Thus

$$\|\tilde{p}\|_2^2 \ge \|f_h p\|_2^2 + \|g(p, q)\|_2^2 - 2\|f_h p\|_2 \, \|g(p, q)\|_2 .$$

Since $\Delta \in \mathcal{B}\mathbf{\Delta}_\nu^{\mathrm{SLTV}}$ and $q \in \mathcal{B}\mathcal{L}_2$ we have that

$$\begin{aligned}\int_s^{s+h} &\left\|f_h(e^{j\omega})q(e^{j\omega})\right\|^2 - \left\|f_h(e^{j\omega})p(e^{j\omega})\right\|^2 \frac{d\omega}{2\pi} \\ &= \|f_h q\|_2^2 - \|f_h p\|_2^2 \\ &\ge \|\Delta(f_h q)\|_2^2 - \|f_h p\|_2^2 \\ &= \|\tilde{p}\|_2^2 - \|f_h p\|_2^2 \\ &\ge \|g(p, q)\|_2^2 - 2\|f_h p\|_2 \, \|g(p, q)\|_2 \\ &\ge -2\|f_h\|_\infty \|q\|_2 \, \|g(p, q)\|_2 . \end{aligned} \tag{42}$$

To complete the proof, note that

$$\begin{aligned}\|g(p, q)\|_2 &\le \sum_{k=-N}^{N} |c_k| \|\mathcal{C}^k\|_{\ell_2\mathrm{ind}} \|q\|_2 \\ &\quad + 2\|f_h - f_n\|_\infty \|q\|_2 \end{aligned} \tag{43}$$

$$\le \left(2N\nu \sup_{k\in\mathbf{Z}} |c_k| + 2\epsilon_1\right) \|q\|_2 \to \epsilon \tag{44}$$

as $\nu \to 0$. ■

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