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An Algorithm for Generating Transfer Functions Uniformly Distributed Over $\mathcal{H}_\infty$ Balls

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Abstract

Probabilistic methods have recently been the subject of considerable attention in the context of robust performance assessment. However, in spite of their potential, these methods have been limited to the case of parametric uncertainty; the problem of sampling causal bounded operators is largely open. In this paper, we take steps towards removing this limitation by providing a computationally efficient algorithm aimed at uniform sampling over balls contained in suitably chosen proper subspaces of $\mathcal{H}_\infty$. As shown in the paper, samples generated from these balls can be used, for instance by Monte Carlo methods, to assess robust performance for uncertainty models involving the $\mathcal{H}_\infty$ norm.

1 Introduction

A large number of control problems of practical importance can be reduced to the robust performance analysis framework illustrated in Figure 1. The family of systems under consideration consists of the interconnection of a known stable LTI plant with some bounded uncertainty $\Delta \subset \mathcal{D}$, and the goal is to compute the worst-case, with respect to $\mathcal{D}$, of the norm of the output to some class of exogenous disturbances.

Depending on the choice of models for the input signals and on the criteria used to assess performance, this prototype problem leads to different mathematical formulations such as $\mathcal{H}_\infty$, ℓ^1, $\mathcal{H}_2$ and ℓ^∞ control. A common feature to all these problems is that, with the notable exception of the $\mathcal{H}_\infty$ case, no tight performance bounds are available for systems with uncertainty Δ being a causal bounded LTI operator[1]. Moreover, even in the $\mathcal{H}_\infty$ case, the problem of computing a tight perfor-

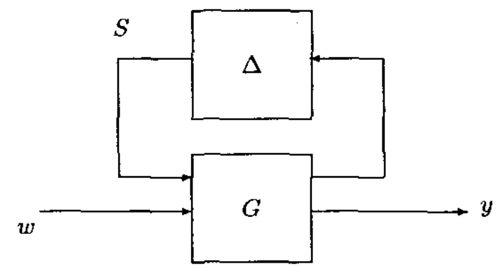

Figure 1: The Robust Performance Analysis Problem

mance bound is known to be NP–hard in the case of structured uncertainty, with more than two uncertainty blocks [7].

Given the difficulty of computing these bounds, over the past few years, considerable attention has been devoted to the use of probabilistic methods. This approach furnishes, rather than worst case bounds, risk–adjusted bounds; i.e., bounds for which the probability of performance violation is no larger than a prescribed risk level ϵ. An appealing feature of this approach is that, contrary to the worst–case approach case, here, the computational burden grows moderately with the size of the problem. Moreover, in many cases, worst–case bounds can be too conservative, in the sense that performance can be substantially improved by allowing for a small level of performance violation. The application of Monte Carlo methods to the analysis of control systems was recently in the work by Stengel, Ray and Marrison in [18, 21, 23] and was followed, among others, by [3, 4, 8, 9, 15, 24, 26, 29]. The design of controllers under risk specifications is also considered in some of the work above as well as in [5, 10, 17, 25, 27].

At the present time the domain of applicability of Monte Carlo techniques is largely restricted to the finite–dimensional parametric uncertainty case. The main reason for this limitation resides in the fact that up

[1]Recently some tight bounds have been proposed for the $\mathcal{H}_2$ case, but these bounds do not take causality into account; see [19].

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to now, the problem of sampling causal bounded operators (rather than vectors or matrices) has not appeared in the systems literature. In this paper, we provide an algorithm aimed at removing this limitation when the set $\mathcal{D}$ consists of balls in $\mathcal{H}_\infty$. The main idea of the paper is based on the fact that in this case, performance can be computed by considering sequences of balls contained in suitable subspaces of $\mathcal{H}_\infty$[2]. By relying on matrix dilation and Carathéodory-Fejér interpolation, we replace the problem of generating operators uniformly distributed over the unit balls of interest to that of generating finite–dimensional vectors uniformly distributed over a convex set. In principle, reduction to sampling over convex sets is still an hard problem and many of the solution methods available require designing a random walk whose stationary distribution is the required one [11, 12]. In this paper, we provide an alternative approach that reduces the problem of uniform sample generation over a convex set to the generation of a sequence of uniform samples over intervals. As shown in the sequel, for the classes of problems addressed in this paper, this leads to a computationally efficient sampling algorithm.

2 Preliminaries

2.1 Notation: By $\mathcal{L}_\infty$, we denote the Lebesgue space of complex-valued matrix functions essentially bounded on the unit circle, equipped with the norm $\|G(z)\|_\infty \doteq ess\sup_{|z|=1} \overline{\sigma}(G(z))$, where $\overline{\sigma}$ represents the largest singular value. By $\mathcal{H}_\infty$, we denote the subspace of functions in $\mathcal{L}_\infty$ with bounded analytic continuation inside the unit disk, equipped with the norm $\|G(z)\|_\infty \doteq ess\sup_{|z|<1} \overline{\sigma}(G(z))$. Also of interest is the space $\mathcal{H}_{\infty,\rho}$ of transfer matrices in $\mathcal{H}_\infty$ which have analytic continuation inside the disk of radius $\rho > 1$, equipped with the norm $\|G(z)\|_{\infty,\rho} \doteq \sup_{|z|<\rho} \overline{\sigma}(G(z))$. Finally, we use $\mathcal{B}$ and $\mathcal{R}$ to denote unit balls and subspaces composed of real rational transfer matrices, respectively.

Given a matrix M, the notation M^T and M^* is used for the transpose and Hermitian conjugate respectively. As usual $M > 0$ ($M \geq 0$) indicates that M is positive definite (positive semi-definite), and $M < 0$ that M is negative definite. Given two transfer function matrices M and Δ of compatible dimensions, we denote by $M \star \Delta$ the upper LFT $\mathcal{F}_u(M, \Delta)$; i.e., it denotes the closed loop transfer function matrix of the system depicted in Figure 1; i.e.,

$$M \star \Delta \doteq M_{22} + M_{21}\Delta\left(I - M_{11}\Delta\right)^{-1} M_{12}$$

where

$$M = \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right]$$

is the adequate partitioning of the transfer function matrix M. Finally, we let $\lfloor x \rfloor$ denote the largest integer smaller or equal to x.

2.2 Statement of the Problem: The use of Monte Carlo methods for risk assessment and volume estimation has been widely studied in the probability literature; e.g., see [22, 13] and references therein. However, these methods rely on the ability to generate samples of a random variable with the appropriate distribution. Hence, if one wants to apply Monte Carlo methods for risk assessment in the presence of dynamic uncertainty, a problem of the form below is to be solved;

Problem 1 *Given some set $\mathcal{D} \subset \mathcal{H}_\infty$, generate uniformly distributed transfer functions $F(z) \in \mathcal{D}$.*

A fundamental difficulty with this formulation is that since $\mathcal{BH}_\infty$ is infinite dimensional, it is unclear what is meant by "uniformly distributed." To circumvent this difficulty, in this paper we consider a slightly different problem: generating uniformly distributed finite impulse response (FIR) filters $F(z)$ *that can be completed* to belong to $\mathcal{BH}_\infty$. Specifically, the problem under consideration is:

Problem 2 *Given n, generate uniformly distributed samples over some appropriate finite-dimensional representation of the set:*

$$\begin{aligned}\mathcal{F}_n \doteq \{&H(z) = h_o + h_1 z + \ldots h_{n-1} z^{n-1} : \\ &H(z) + z^n G(z) \in \mathcal{BH}_\infty, \text{ for some } G(z) \in \mathcal{H}_\infty\}.\end{aligned}$$

As shown in the sequel, solving Problem 2 indirectly addresses Problem 1, in the sense that the solution to the latter can be used as a surrogate solution to the former.

3 Main Results

3.1 Reduction to Sampling Over Convex Sets From the Carathéodory-Fejér Theorem (for example, see [2]), it follows that given $\mathbf{h} \doteq [h_0, h_1, h_2, \ldots, h_{n-1}]$, the corresponding $H(z)$ belongs to $\mathcal{F}_n$ if and only if $\overline{\sigma}[H(\mathbf{h})] \leq 1$, where

$$H(\mathbf{h}) \doteq \begin{bmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ 0 & h_0 & \cdots & h_{n-2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h_0 \end{bmatrix}.$$

[2]These balls are also interesting in their own right, since they allow for incorporating smoothness constraints in the uncertainty description.

Thus, a natural representation for $\mathcal{F}_n$ in Problem 2 is the set

$$\mathcal{C}_{\mathcal{F}_n} \doteq \{\mathbf{h} : \overline{\sigma}[H(\mathbf{h})] \leq 1\}.$$

This leads to the problem below.

Problem 3 *Given $n > 0$, generate uniform samples over the convex set $\mathcal{C}_{\mathcal{F}_n} \doteq \{\mathbf{h} : \overline{\sigma}[H(\mathbf{h})] \leq 1\}$.*

In the section below, an algorithm for generating uniform samples over an arbitrary finite–dimensional compact convex set is given and we solve Problem 3 as a special case.

3.2 Uniform Samples Over Convex Sets: Let $\mathcal{C} \subset \mathbf{R}^n$ denote an arbitrary convex set and consider the following algorithm:

Algorithm 1 *1. Let $k = 0$. Generate N samples uniformly distributed over the interval $[m_1, M_1]$, where*

$$\begin{aligned} m_0 \doteq \quad & \min\{h_0 : (h_0, h_1, \ldots, h_{n-1}) \in \mathcal{C} \\ & \textit{for some } (h_1, \ldots, h_{n-1}) \in \mathbf{R}^{n-1}\} \\ M_0 \doteq \quad & \max\{h_0 : (h_0, h_1, \ldots, h_{n-1}) \in \mathcal{C} \\ & \textit{for some } (h_1, \ldots, h_{n-1}) \in \mathbf{R}^{n-1}\} \end{aligned}$$

2. Let $k = k + 1$. For every generated sample $(h_0^l, h_1^l, \ldots, h_{k-1}^l)$, generate $\lfloor N(M_k^l - m_k^l) \rfloor$ samples uniformly over the interval $[m_k^l, M_k^l]$ where

$$\begin{aligned} m_k^l \doteq \quad & \min\{h_k : (h_0^l, \ldots, h_{k-1}^l, h_k, \ldots, h_{n-1}) \in \mathcal{C} \\ & \textit{for some } (h_{k+1}, \ldots, h_{n-1}) \in \mathbf{R}^{n-1-k}\} \\ M_k^l \doteq \quad & \max\{h_k : (h_0^l, \ldots, h_{k-1}^l, h_k, \ldots, h_{n-1}) \in \mathcal{C} \\ & \textit{for some } (h_{k+1}, \ldots, h_{n-1}) \in \mathbf{R}^{n-1-k}\} \end{aligned}$$

3. If $k < n$ go to step 2. Else stop.

Theorem 1 *as $N \to \infty$, the probability distribution of the samples generated by Algorithm 1 above converges with probability one to a uniform distribution.*

Proof: Due to constraints on the length of the paper, the proof is not provided. To obtain it, please contact the first author. His email address is given in the first page.

3.3 Remarks: The main reason which prevents the algorithm above from producing truly uniformly distributed samples is the fact that, at step s,

$$\frac{\lfloor N l_s(X_1^k, X_2^m, \ldots, X_{s-1}^n) \rfloor}{N} \neq l_s(X_1^k, X_2^m, \ldots, X_{s-1}^n)$$

where

$$\begin{aligned} & l_s(y_1, \ldots, y_{s-1}) \doteq \\ & \quad \max\{x_s : (y_1, \ldots, y_{s-1}, x_s, x_{s+1}, \ldots, x_n) \in \mathcal{C} \\ & \qquad \text{for some } (x_{s+1}, \ldots, x_n) \in \mathbf{R}^{n-s}\} \\ & \quad - \min\{x_s : (y_1, \ldots, y_{s-1}, x_s, x_{s+1}, \ldots, x_n) \in \mathcal{C} \\ & \qquad \text{for some } (x_{s+1}, \ldots, x_n) \in \mathbf{R}^{n-s}\} \end{aligned}$$

for $s = 1, 2, \ldots, n$. The difference between these values can be made very small even for relatively small values of N. Furthermore, it is noted that the main difference between "traditional" Monte Carlo sample generation and the algorithm provided in this paper is as follows: In Algorithm 1, several optimization problems have to be solved to compute the samples. However, as seen in the next section, this is facilitated by a closed form solution for these optimizations. Therefore, the computational burden required to compute reliable estimates of risk is similar to the one in "traditional" Monte Carlo simulations. For bounds on the number of samples required for reliable estimation of risk, see [15] and [24].

3.4 $\mathcal{BH}_\infty$ as a Simpler Case: For the case of a general convex set $\mathcal{C}$, Algorithm 1 requires solving on the order of 2^n convex optimization problems. However, as shown in the sequel, for sets of the form $\mathcal{C} = \{\mathbf{h} : \overline{\sigma}[H(\mathbf{h}] \leq 1\}$ these optimization problems can be solved in closed form. Since these are precisely the sets arising in the context of Problem 2, it follows that this problem can be efficiently solved by applying Algorithm 1. To obtain the desired closed form solution, we consider the problem

$$\begin{aligned} \min\{h_k : \overline{\sigma}[H(h_1^l, h_2^l, \ldots, h_{k-1}^l, h_k, h_{k+1}, \ldots, h_n)] \leq 1 \\ \text{for some } (h_{k+1}, \ldots, h_n) \in \mathbf{R}^{n-k}\}. \end{aligned}$$

where $h_1^l, h_2^l, \ldots, h_{k-1}^l$ are given. Indeed, with

$$\widetilde{H}(h_1, h_2, \ldots, h_n) \doteq \begin{bmatrix} h_n & \cdots & h_2 & h_1 \\ h_{n-1} & \cdots & h_1 & 0 \\ \vdots & & & \vdots \\ h_1 & 0 & \cdots & 0 \end{bmatrix}$$

and observing that

$$\begin{aligned} & \{h_k : \overline{\sigma}[H(h_1^l, h_2^l, \ldots, h_{k-1}^l, h_k, h_{k+1}, \ldots, h_n)] \leq 1 \\ & \qquad \text{for some } (h_{k+1}, \ldots, h_n) \in \mathbf{R}^{n-k}\} \\ & = \{h_k : \overline{\sigma}[\widetilde{H}(h_1^l, h_2^l, \ldots, h_{k-1}^l, h_k, h_{k+1}, \ldots, h_n)] \leq 1 \\ & \qquad \text{for some } (h_{k+1}, \ldots, h_n) \in \mathbf{R}^{n-k}\}, \end{aligned}$$

from Parrott's Theorem (e.g., see [28]), it follows that

$$\min_{h_n} \bar{\sigma}[\widetilde{H}] = \bar{\sigma}\begin{bmatrix} h_{n-1} & h_{n-2} & h_2^l & h_1^l \\ h_{n-2} & \cdots & h_1^l & 0 \\ \vdots & & & \vdots \\ h_1^l & 0 & \cdots & 0 \end{bmatrix}.$$

This is a consequence of the fact that the matrix $\widetilde{H}(h_1, h_2, \dots, h_n)$ is symmetric. Applying the same reasoning $k-2$ times, we obtain

$$\min_{h_n, h_{n-1}, \dots, h_{k+1}} \bar{\sigma}[\widetilde{H}] = \bar{\sigma}\begin{bmatrix} h_k & h_{k-1}^l & h_2^l & h_1^l \\ h_{k-1}^l & \cdots & h_1^l & 0 \\ \vdots & & & \vdots \\ h_1^l & 0 & \cdots & 0 \end{bmatrix}.$$

Therefore, minimizing (maximizing) h_k over the set

$$\{h_k : \bar{\sigma}[H(h_1^l, h_2^l, \dots, h_{k-1}^l, h_k, h_{k+1}, \dots, h_n)] \leq 1 \\ \text{for some } (h_{k+1}, \dots, h_n) \in \mathbf{R}^{n-k}\}$$

is equivalent to minimizing (maximizing) h_k over the set

$$\left\{ h_k : \bar{\sigma}\begin{bmatrix} h_k & h_{k-1}^l & h_2^l & h_1^l \\ h_{k-1}^l & \cdots & h_1^l & 0 \\ \vdots & & & \vdots \\ h_1 & 0 & \cdots & 0 \end{bmatrix} \leq 1 \right\}.$$

Note that characterizing all the h_k that belong to the set above is precisely the matrix dilation problem addressed in Parrott's Theorem. Hence, all feasible h_k are of the form

$$h_k \doteq -YH^lY^T + (1 - YY^T)w;\ |w| \leq 1$$

where

$$H^l \doteq H[(0, h_1, \dots, h_{k-2})]$$
$$Y \doteq \begin{bmatrix} h_{k-1} & \dots & h_2 & h_1 \end{bmatrix} (I - (H^l)^* H^l)^{-\frac{1}{2}}$$

Now, by computing the maximum and minimum h_k over the allowable values of $|w| \leq 1$, it follows that the values m_k, M_k required by Algorithm 1 are explicitly given by:

$$\begin{aligned} m_k &= -YH^lY^T - |(1 - YY^T)| \\ M_k &= -YH^lY^T + |(1 - YY^T)|. \end{aligned}$$

4 Approximately Sampling of $\mathcal{BH}_\infty$

In this section, we indicate how a solution of Problem 2 serves for Problem 1. Begin by noting that the Carathéodory-Fejér Theorem (for example, see [2]) only specifies the values of the function and its first $n-1$ derivatives at $z = 0$. However, these conditions do not impose any constraints on the smoothness of the function over the unit disk and can lead to transfer functions which do not represent a physical uncertainty. For example, $h = \begin{bmatrix} 0 & 0 & \dots & 0 & \frac{\epsilon}{(1+\epsilon)^2} \end{bmatrix}$ has all the $h_i, i \leq n-1$ arbitrarily small and satisfies the requirements of Carathéodory-Fejér Theorem. Moreover, it can be easily shown that a suitable interpolant is given by

$$H(z) = \frac{\epsilon}{1 + \epsilon - z^n}.$$

Clearly, $H(z) \in \mathcal{BH}_\infty$. Since $\|\frac{d}{dz}H(z)\| = \frac{n}{\epsilon} \to \infty$, these functions are arguably not a good abstraction of physical uncertainty, estimating worst–case performance bounds using samples from the set $\mathcal{F}_n$ can lead to conservative results. This effect can be avoided by working with the ball $\mathcal{BH}_{\infty,\rho}$, instead of $\mathcal{BH}_\infty$, since restricting all the poles of the system to the exterior of the disk $|z| \geq \rho$ induces a smoothness constraint. This leads to the following modified version of Problem 2:

Problem 4 *Given $n > 0$ and $\rho > 1, \rho \sim 1$, generate uniformly distributed samples over an appropriate finite-dimensional representation of the set*

$$\mathcal{F}_{n,\rho} \doteq \{H(z) = h_o + h_1 z + \dots h_{n-1} z^{n-1} : H(z) \\ + z^n G(z) \in \mathcal{BH}_{\infty,\rho}, \text{for some } G(z) \in \mathcal{H}_{\infty,\rho}\}.$$

As shown below, this problem readily reduces to Problem 3 and thus can be solved using Algorithm 1. To this end, note that $F(z) \in \mathcal{BH}_{\infty,\rho}$ is equivalent to $F(\frac{z}{\rho}) \in \mathcal{BH}_\infty$. Combining this observation with Carathéodory-Fejér Theorem, it follows that given $\mathbf{h} = \begin{bmatrix} h_0 & h_1 & \dots & h_{n-1} \end{bmatrix}$, then there exists $G(z) \in \mathcal{BH}_{\infty,\rho}$ such that $\sum_{i=0}^{n-1} h_i z^i + z^n G(z) \in \mathcal{BH}_{\infty,\rho}$ if and only if $\bar{\sigma}[H(\hat{\mathbf{h}})] \leq 1$ where

$$\hat{\mathbf{h}} \doteq \begin{bmatrix} h_o & \frac{h_1}{\rho} & \dots & \frac{h_{n-1}}{\rho^{n-1}} \end{bmatrix}.$$

Hence, Problem 4 reduces to Problem 3 simply with the change of variables $h_k \to \frac{h_k}{\rho^k}$.

Next, we show that the norm of the tail $\|z^n G(z)\|_\infty$ tends to zero as $n \to \infty$. Thus, sampling the set $\mathcal{F}_{n,\rho}$ approximates sampling the ball $\mathcal{BH}_{\infty,\rho}$. To establish this result, note that if $F \in \mathcal{BH}_{\infty,\rho}$, then its Markov parameters satisfy

$$f_k = \frac{1}{2\pi} \oint_{\partial D_\rho} F(z) \frac{dz}{z^{k+1}} \Rightarrow |f_k| \leq \frac{1}{\rho^k}$$

where D_ρ denotes the disk centered at the origin with radius ρ. Thus

$$\|z^n G(z)\|_\infty = \|\sum_{i=n}^{\infty} f_i z^i\|_\infty \leq \sum_{i=n}^{\infty} \frac{1}{\rho^i} = \frac{1}{\rho^{n-1}} \frac{1}{\rho - 1}.$$

From this inequality it follows that $\|F(z) - H(z)\|_\infty \leq \epsilon$ for $n \geq$ some $n_o(\epsilon)$ that can be precomputed a priori.

Finally, we conclude this section by showing that the proposed algorithm can also be used to assess performance against uncertainty in $\overline{\mathcal{RBH}_\infty}$. Consider a sequence $\rho_i \downarrow 1$ and let Δ_i be the corresponding worst-case uncertainty. Recall (see for instance Corollary B.5 in [19]) that robust stability of the LFT interconnection shown in Figure 1 implies that $(I - M_{11}\Delta)^{-1}$ is uniformly bounded over $\mathcal{BH}_\infty$. In turn, this implies that there exists some finite β such that $\|M \star \Delta\|_\infty \leq \beta$ for all $\Delta \in \mathcal{BH}_\infty$. Since $\mathcal{BH}_{\infty,\rho} \subset \mathcal{BH}_\infty$, it follows that both Δ_i and $M \star \Delta_i$ are normal families. Thus, they contain a normally convergent subsequence $\Delta_i \rightarrow \tilde{\Delta}$ and $M \star \Delta_i \rightarrow M \star \tilde{\Delta}$. It can be easily shown that $\tilde{\Delta}$ is indeed the worst case uncertainty over $\overline{\mathcal{RBH}_\infty}$. Thus, robust performance can be assessed by applying the proposed algorithm to a sequence of problems with decreasing values of ρ.

5 Conclusions

During the past two decades, considerable attention has been devoted to the problem of assessing robust performance of the interconnection shown in Figure 1. However, in spite of intense research, very few tight worst–case bounds are available. Moreover, even in cases where the problem has been solved (such as $\mathcal{H}_\infty$), computing these bounds leads to NP–hard problem, in all but the simplest cases. Probabilistic methods have the potential to address both the issue of the conservatism of worst–case bounds and the associated computational complexity problem. However, up to the present time, application of these methods has been limited to the case of finite–dimensional parametric uncertainty, largely due to the unavailability of methods for generating samples from sets of bounded causal operators.

In this paper, this limitation is addressed by proposing a computationally efficient algorithm for approximately sampling balls of the form $\mathcal{BH}_{\infty,\rho}$. Samples generated with the proposed algorithm can be used to assess, up to an arbitrary precision ϵ, robust performance against dynamic uncertainty $\Delta \in \mathcal{BH}_{\infty,\rho}$. Moreover, by considering the sequence of problems obtained as $\rho \rightarrow 1$, the method can be also applied to uncertainty in $\mathcal{BH}_\infty$, in cases where it is known that the problem can be restricted to $\overline{\mathcal{RBH}_\infty}$. An example is assessing worst-case $\mathcal{H}_\infty$ performance in the presence of structured uncertainty with an arbitrary number of blocks.

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