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Application of Probabilistically Constrained Linear Programs to Risk-Adjusted Controller Design†

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Abstract

The focal point of this paper is the Probabilistically Constrained Linear Program (PCLP) and how it can be applied to control system design under risk constraints. The PCLP is the counterpart of the classical linear program, where it is assumed that there is random uncertainty in the constraints and, therefore, the deterministic constraints are replaced by probabilistic ones. It is shown that for a wide class of distributions, called log-concave symmetric distributions, the PCLP is a convex program. A deterministic equivalent of the PCLP is presented which provides insight on numerical implementation. Finally, this concept is applied to control system design. It is shown how the PCLP can be applied to the design of controllers for discrete-time systems to obtain a closed loop system with a well-defined risk of violating the so-called property of *super stability*. Furthermore, we address the problem of risk-adjusted pole placement.

1 Introduction

Recently there has been a growing interest in the development of control system design procedures which are able to handle risk constraints. The main motivation for considering this problem is that "classical" robust controllers tend to be complex; i.e., often "classical" controller design algorithms produce high order controllers. The hope is that, if one is willing to tolerate a small well-defined risk of violation of performance specifications, one would obtain a significantly less complex controller. Also, there are several problems which naturally lead to a formulation involving risk constraints. An example is emergency system operation. Here one would like to be able to allow for a small well-defined risk of violation of stability in exchange for a better chance of handling the emergency situation. For example, in the control of an aircraft, one may want to have more power available to avoid a collision, although this might lead to a small risk of engine failure. For situations like this, a different kind of controller should be designed. It should be a controller that maximizes performance at the expense of a small well-defined risk of instability.

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The problem of designing risk-adjusted controllers has been considered in several papers; e.g., see [1], [2], [3] and [4]. However, there is a fundamental difference between the work mentioned above and the results presented in this paper. In contrast to the work presented in the papers mentioned above, where the search for the controller parameters is done using randomized algorithms, this paper is integrated in a new line of research which aims at developing fast deterministic algorithms for risk-adjusted controller design. The work presented in [5], [6] and [7] indicates that there are several risk-adjusted controller design problems which are convex and, hence, numerically solvable. In this paper, we extend the class of risk-adjusted design problems which are known to be convex. More precisely, we extend the results in [7] and show how they can be used in a systems design context. The main paradigm underlying the results presented is the concept of Probabilistically Constrained Linear Program.

1.1 Probabilistically Constrained Linear Program: The main result of this paper concerns the convexity of the so-called Probabilistically Constrained Linear Program (PCLP). We show that, for a large class of probability distributions, the probabilistic version of the classical linear program is convex. Furthermore, we show how it can be applied in a controller design context. The class of distributions that is considered in this paper is the class of *log-concave symmetric* distributions which include many of the "typical" distributions used to date in the area of probabilistic robustness such as uniform and normal distributions.

Indeed, consider the "classical" linear program described by

$$\min c^T x$$

subject to

$$x^T a^i \leq b_i; \quad i = 1, 2, \ldots, k$$

where $c, x, a^i \in \mathbf{R}^\ell$ and $b_i \in \mathbf{R}$, $i = 1, 2, \ldots, k$. In the PCLP framework, the constraint vectors a^i and b above are treated as random and the deterministic constraints are replaced by probabilistic constraints. There are a number of versions of the PCLP problem and the one that is used in this paper is the same that is used in [7].

1.2 PCLP: Given acceptable risk levels $0 \leq \varepsilon_i \leq 1$, $i = 1,2,\ldots,k$, find

$$\min c^T x$$

subject to

$$\text{Prob}\{x^T a^i \leq b_i\} \geq 1 - \varepsilon_i; \;\; i = 1,2,\ldots,k$$

where $c, x \in \mathbf{R}^\ell$ and a^i, b are random vectors of appropriate dimensions.

1.3 Convexity of the Feasible Set: A fundamental question about the PCLP is the following: Is the PCLP a convex program? In other words, is the feasible set

$$\mathcal{X}_\varepsilon \doteq \{x \in \mathbf{R}^\ell : \text{Prob}\{x^T a^i \leq b_i\} \geq 1 - \varepsilon_i, i = 1,\ldots,k\}$$

convex? It turns out that, without additional conditions on the distribution of the pair (a^i, b_i), one can easily generate examples where the answer is "no."

In this paper, we prove that the PCLP is a convex program when the distribution of the random parameters is log-concave and symmetric; see Section 2 for a precise definition of this class of distributions. Convexity results are available for other kinds of distributions: In [13], it is proven that $\mathcal{X}_\varepsilon$ is convex when $0 \leq \varepsilon_i \leq 1/2$ and b_i and the components of the a^i are independent and normally distributed. This result was later extended for the case when a^i and b_i have stable distributions; e.g., see [14]. Finally, the work in [7] shows that, for $0 \leq \varepsilon_i \leq 1/2$, the PCLP is convex if the uncertain parameters are uniformly distributed over a convex symmetric set. In this paper we extend the results in [7]. We prove that for a large class of distributions (which includes uniform distributions over convex symmetric sets), the PCLP is a convex program. Also, we show how to apply it in a controller design context.

1.4 The Sequel: Section 2 is dedicated to the definition of the class of admissible distributions for the uncertain parameters: log-concave symmetric distributions. The main result of this paper is presented in Section 3 which states that the PCLP is a convex program. In Section 4, we provide some insights on a numerical implementation of the PCLP. Section 5 is dedicated to the application of the results in this paper in the context of control system design. Finally, in Section 6, some concluding remarks are presented and several directions for further research are outlined.

2 Preliminaries: Log-concavity

In order to communicate the main result, we need to elaborate on what probability density functions are admissible for the uncertain parameters. To this end, we require a definition of log-concave functions; see [15].

2.1 Log-concave Functions and Probability Densities: A function $f : \mathbf{R}^\ell \to [0,\infty)$ is said to be *log-concave* if the following condition holds: Given any $x^0, x^1 \in \mathbf{R}^\ell$ and $\lambda \in [0,1]$,

$$f((1-\lambda)x^0 + \lambda x^1) \geq [f(x^0)]^{1-\lambda}[f(x^1)]^\lambda.$$

In the sequel, let $\mathcal{F}$ denote the class of log-concave symmetric probability density functions. Without loss of generality, one can assume that the center of symmetry is the origin; i.e., if $f \in \mathcal{F}$ then for any $x \in \mathbf{R}^\ell$, we have $f(x) = f(-x)$. Throughout this paper, we assume that the probability density function f of the vector of uncertain parameters is log-concave and symmetric; i.e., $f \in \mathcal{F}$. It is important to note that the class $\mathcal{F}$ is quite rich. Most "common" probability density functions (such as uniform or normal) are readily shown to be log-concave and symmetric. Hence, the main result to follow applies to typical density functions used in the probabilistic robustness literature to date.

3 Convexity of the PCLP

In this section, we present the main result of this paper. Theorem 3.1 to follow indicates that, if the distribution of the uncertain parameters is log-concave and symmetric and for risk levels satisfying $0 \leq \varepsilon_i \leq 1/2$, the PCLP is a convex program. Although the result below only involves the convexity of a PCLP with one constraint, the extension to the case with an arbitrary number of constraints is immediate. This extension is a consequence of the fact that an intersection of convex sets is still convex. Throughout this paper, we write $a^i = a^i_0 + \Delta a^i; i = 1,2,\ldots,k$ and $b = b_0 + \Delta b$ and assume that the pair $(\Delta a^i, \Delta b_i)$ has a log-concave symmetric distribution function. For simplicity, it is assumed that the vector b is deterministic; i.e., $b = b_0$. However, it is noted that the formulation and the results presented can be easily generalized for the case when b is random.

3.1 Theorem: *Let $a_0 \in \mathbf{R}^\ell$, $b \in \mathbf{R}$ and the risk level $0 \leq \varepsilon \leq 1/2$ be given. Also, let the random vector Δa have a log-concave symmetric distribution. Then, the set*

$$\mathcal{X}_\varepsilon \doteq \{x \in \mathbf{R}^\ell : \text{Prob}\{x^T(a_0 + \Delta a) \leq b\} \geq 1 - \varepsilon\}$$

is convex.

Proof: See Appendix.

4 Deterministic Equivalent of the PCLP

The result in the previous section indicates that the PCLP is a convex program. However, it does not provide any indication on how to solve the resulting optimization problem. In this section, we present the concept of *floating body* which provides some insights on how one can solve the PCLP.

4.1 Floating Body: Central to the results presented in this paper is the concept of *floating body* of a probability measure. Given $0 < \varepsilon < 1/2$, the floating body K_ε of a probability distribution is a convex symmetric set for which each supporting hyper-plane "cuts-off" a set of probability ε. More precisely, given $0 < \varepsilon < 1/2$ and $u \in \mathbf{R}^\ell$, $\|u\|_2 = 1$, let $H(u, \varepsilon)$ be the supporting hyper-plane of K_ε normal to u. Also, let $\mathcal{H}^+(u, \varepsilon)$ be the half-space defined by $H(u, \varepsilon)$ which does not contain the origin. Then, K_ε is a floating body of the given probability measure if

$$\mathrm{Prob}(\mathcal{H}^+(u, \varepsilon)) = \varepsilon.$$

for all $\|u\|_2 = 1$. Not every probability measure has a floating body. However, the results in [8] indicate that every probability distribution in the class $f \in \mathcal{F}$ does have a floating body K_ε for any $0 < \varepsilon < 1/2$.

4.2 Additional Notation: Let $\|\cdot\|$ be a norm in $\mathbf{R}^\ell$. We define the dual norm as

$$\|x\|_* \doteq \max\{x^T y : \|y\| \leq 1\}.$$

Now, recalling that the probability distribution of Δa is log-concave and symmetric, define the norm associated with its floating body K_ε as

$$\|\Delta a\|_\varepsilon \doteq \inf\{\rho \in \mathbf{R}^+ : \Delta a \in \rho K_\varepsilon\}$$

and let $\|\cdot\|_{\varepsilon,*}$ denote its dual norm as defined above.

4.3 Deterministic Equivalent of the PCLP: Since

$$\{\Delta a \in \mathbf{R}^\ell : x^T(a_0 + \Delta a) \leq b\}$$

is an half-space, the definition of the floating body presented in Section 4.1 indicates that requiring

$$\mathrm{Prob}\{x^T(a_0 + \Delta a) \leq b\} \geq 1 - \varepsilon$$

for $0 < \varepsilon < 1/2$ is equivalent to requiring

$$x^T(a_0 + \Delta a) \leq b$$

for all $\Delta a \in K_\varepsilon$, where K_ε is the floating body of the probability distribution of Δa as defined in Section 4.1. Now, given the definition of dual norm above, this is equivalent to

$$\|x\|_{\varepsilon,*} \leq b - x^T a_0.$$

Therefore, the probabilistic constraints of the PCLP can be replaced by deterministic ones of the form above. Hence, if the quantity $\|x\|_{\varepsilon,*}$ can be easily determined, this leads to an immediate numerical implementation for solving the PCLP.

4.4 Elliptical Log-concave Distributions: It turns out that there are cases where $\|x\|_{\varepsilon,*}$ is easily determined. An example is the case where the probability distribution of the uncertain parameters is an *elliptical log-concave* distribution. An elliptical log-concave distribution is a distribution whose probability density function is of the form

$$f(y) = g(y^T W y)$$

where $g : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ is a log-concave non-increasing function and W is a positive definite matrix. Examples of such distributions are multivariable normal distributions and uniform distributions over hyper-spheres. For such probability distributions it is easy to prove that the convex floating body is an ellipsoid with the same aspect ratio as the ellipsoid

$$\mathcal{E} \doteq \{\Delta a \in \mathbf{R}^\ell : \Delta a^T W \Delta a \leq 1\}.$$

The actual "radius" of the ellipsoid K_ε can be determined analytically for some probability distributions. If one cannot determine this radius analytically, an easy one line search optimization problem can be setup to numerically obtain this value. Therefore, for such probability distributions, the PCLP reduces to a convex quadratic optimization problem. More precisely, consider an elliptical log-concave probability density function of the form above. Then, for any $0 < \varepsilon < 1/2$, the floating body K_ε is of the form

$$K_\varepsilon = \{\Delta a \in \mathbf{R}^\ell : \Delta a^T W \Delta a \leq r^2(\varepsilon)\}$$

for some $r(\varepsilon) > 0$. It can be easily shown that requiring

$$x^T(a_0 + \Delta a) \leq b$$

for all $\Delta a \in K_\varepsilon$ is equivalent to requiring

$$\|r(\varepsilon) W^{-1/2} x\|_2 \leq b - x^T a_0$$

which is a convex quadratic constraint on x.

5 Application to Control Systems Design

We now show how the PCLP can be used in the context of controller design. First, we apply the PCLP to the design of super stable systems. A second example shows how the theory in this paper can be applied to robust pole assignment.

5.1 Super Stability: In contrast to the concept of stability, where only asymptotic behavior is considered, super stability allows for computing the worst-case value of the ℓ^∞ norm of the output due to ℓ^∞ bounded disturbances and initial conditions. It also provides an upper bound on the ℓ^∞ induced norm of the system (which is exact for FIR systems). We now briefly review some of the properties of super stable systems; see [9] and [10] for proofs and additional properties. Consider a discrete-time linear time invariant system

$$y(q) = G(q)w(q), \; G(q) = b(q)/(1 + a(q))$$

where w are exogenous disturbances, y is the output, q is the delay operator: $qx[k] = x[k-1]$ and where the polynomial $a(q)$ does not have a constant term, i.e.

$$a(q) = a_1 q + a_2 q^2 + \cdots + a_n q^n;$$

Defining $||a||_1 = \sum_{i=1}^{n} |a_i|$, a system is said to be super stable if $||a||_1 < 1$. Moreover, in [9], it is shown that in this case the ℓ^∞ induced norm of the system is bounded by

$$||G(q)||_{\ell^\infty \to \ell^\infty} \leq \frac{||b||_1}{1 - ||a||_1}$$

This property was exploited in [9] to synthesize low order ℓ^1 controllers. Synthesizing a controller such that the ℓ^1 norm of the closed–loop system is less or equal than a given μ reduces to finding the parameters of the controller transfer function such that

$$\mu||d_{cl}||_1 + ||n_{cl}||_1 \leq \mu.$$

where d_{cl} and n_{cl} are the coefficients of the denominator and numerator of the closed loop transfer function. This problem can be easily recast in an LP format. Moreover, as noted in [9], this approach can also address the problem of fragility exhibited by some optimal control design methods [11]. Assume that the plant is subject to parametric uncertainty of the form

$$G(q) = \frac{b(q)}{1 + a(q)} = \frac{\sum_{i=0}^{m} (b_{0,i} + \Delta b_i) q^i}{1 + \sum_{j=1}^{n} (a_{0,j} + \Delta a_j) q^j}$$

where $b_{0,i}$ and $a_{0,i}$ are the nominal values of the coefficients and Δb_i and Δa_i represent the uncertainty. Also, consider a controller of the form[1]

$$G_c(q) = \frac{b_c(q)}{1 + a_c(q)}.$$

In this case robust performance is achieved if

$$\mu||d_{cl}||_1 + ||n_{cl}||_1 \leq \mu$$

holds for all admissible values of the uncertainty, a problem that can be easily recast as finding a feasible point of a set of linear inequalities on the coefficients of the controller. However, there is a major difference between the nominal and robust performance case: while it can be shown that the former always admits a solution if the controller order is chosen to be at least equal to the order of the plant, the later may not have a solution even for high order controllers. On the other hand, as we illustrate next with a simple example, it might be possible to find low order risk–adjusted controllers, even for very small values of ε, the probability of violating the constraints. These controllers can be found by solving the risk-adjusted counterpart of the LP problem described in [9], which is easily seen to be a PCLP.

5.2 Numerical Example: We now consider the example in [9]. The discrete time system presented has nominal transfer function

$$P(q) = \frac{n(q)}{d(q)} = \frac{q - 2.5q^2 + 1.501q^3}{1 - 2.7q + 23.5q^2 - 4.6q^3}$$

[1]For notational simplicity, here we assume that the controller is not subject to uncertainty, but the proposed procedure can be easily modified to take controller uncertainty into account.

and we assume that all coefficients are subject to uncertainty. Moreover, we assume that the uncertainty vector is uniformly distributed on a hyper-sphere with radius 0.05. We assume that the controller has the form

$$G_c(q) = \frac{b_c(q)}{1 + a_c(q)} = \frac{b_{c,0} + b_{c,1}q + \ldots + b_{c,m_c} q^{m_c}}{1 + a_{c,1}q + \ldots + a_{c,n_c} q^{n_c}}.$$

We first tried to design a controller that will results in a robustly super stable closed loop system. We tried controllers up until order $m_c = n_c = 6$ and were not able to find one. Then we allowed for a risk of $\varepsilon = 1.25 \times 10^{-4}$. We were then able to find the following risk-adjusted controller

$$C(q) = \frac{4.5819 - 17.7802q - 1.0245q^2 + 0.8795q^3}{1 - 1.8819q + 0.6538q^2 + 0.287q^3}$$

which has order 3. Having this results, a Monte Carlo simulation was performed to compute the risk of violating super stability (recall that ε is the risk of violating each inequality). The number of samples used was 10^7 and the estimated probability of violating super stability obtained is 0.78%, showing that one can obtain a low order controller even for small risk levels.

5.3 Robust Pole Assignment: We now describe how one can apply the results in this paper to the problem of robust pole assignment. We start with a continuous uncertain open loop plant described by the following transfer function

$$G(s) = \frac{(b_{0,0} + \Delta b_0) + (b_{0,1} + \Delta b_1)s + \cdots + (b_{0,m} + \Delta b_m)s^m}{(a_{0,0} + \Delta a_0) + (a_{0,1} + \Delta a_1)s + \cdots + (a_{0,n} + \Delta a_n)s^n}$$

where $b_{0,i}$ and $a_{0,i}$ are the nominal values of the coefficients of the numerator and denominator respectively and Δb_i and Δa_i represent the uncertainty. Now, since uncertainty is present, one cannot determine a controller that will assign the closed loop poles to a specific location. As in [12], one instead aims at designing a controller such that the the closed loop poles lead to the satisfaction of the desired specifications. In other words, each of the coefficients of the closed loop characteristic polynomial should belong to a given interval. More precisely, given a controller of the form

$$G_c(s) = \frac{b_{c,0} + b_{c,1}s + \cdots + b_{c,m_c} s^{m_c}}{a_{c,0} + a_{c,1}s + \cdots + a_{c,n_c} s^{n_c}}$$

one aims at finding the coefficients of the controller such that the closed loop characteristic polynomial belongs to the family of polynomials

$$s^{n_{cl}} + [\delta^-_{n_{cl}-1}, \delta^+_{n_{cl}-1}] s^{n_{cl}-1} + \cdots + [\delta^-_1, \delta^+_1] s + [\delta^-_0, \delta^+_0]$$

for all admissible uncertainty values, where $n_{cl} = n_c + n$ is the degree of the closed loop characteristic polynomial. Therefore, the search for the coefficients of the controller reduces to finding a feasible solution to a set of linear inequalities to be satisfied for all admissible values of $\Delta a_0, \ldots, \Delta a_n$ and $\Delta b_0, \ldots, \Delta b_m$. For most common types of uncertainties, the problem above is easily proven to be convex. However,

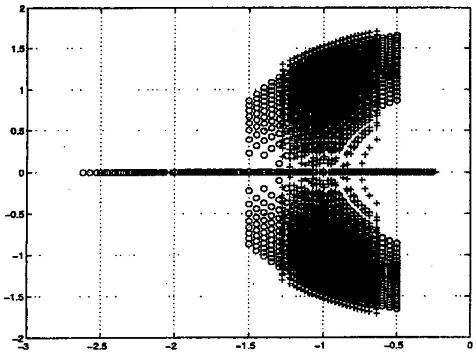

Figure 1: Desired pole location "o" and actual one "+".

the designing of a robust controller can result in controllers which are complex. Therefore, we take a risk-adjusted point of view; i.e., instead of requiring that each inequality is satisfied for all admissible values of the uncertain parameters, we require that the risk of violating each of the inequalities is less than or equal to a prescribed risk level ε. In other words, we solve a PCLP version of the problem above.

5.4 Numerical Example: The example presented here is a modification of one of the examples in [12]. Consider an uncertain plant with transfer function

$$G(s) = \frac{(0.75+\Delta b_1)s+1.25+\Delta b_0}{s^2+(0.75+4\Delta a_1)s+\Delta a_0}$$

where the uncertain parameter vector is uniformly distributed over the hyper-sphere of radius 0.25. We now aim at designing a controller such that the closed loop polynomial belongs to the family

$$\Delta_T(s) = s^2+[1,3]s+[1,3].$$

Therefore, the controller transfer function is constant $G_c(s) = b_0$. We tried to find a robust controller for the system above. In this case, this was not possible. Then, a risk of $\varepsilon = 0.02$ was allowed in the PCLP version of the problem above. In this case a risk-adjusted constant controller exists and has the form $G_s(s) = 1.555$. The pole cluster distributions of the desired system and the actual closed loop system are shown in Figure 1. A Monte Carlo simulation was performed to estimate the actual risk of violating the specifications. The estimated value of the risk is approximately 3.6%, showing that, even for low risk values, one can obtain risk-adjusted controllers in cases where a robust controller does not exist. Furthermore, in this case, we obtain robust stability as an added benefit; see Figure 1.

6 Conclusions and Further Research

In this paper, we extend the results in [7] and show that the probabilistically constrained linear program is a convex optimization problem for any log-concave symmetric distribution. Also, a deterministic equivalent was provided which can be easily implemented in the case of elliptical distributions, such as normal or uniform over an hyper-sphere. Finally, this result was applied in the systems design context, showing that, even for very low levels of risk, one can obtain controllers that are substantially less complex than their robust counterparts.

The results in this paper suggest several directions for further research. First, the authors believe that effort should be put in the development of numerical tools for solving the PCLP when the distribution is other than elliptical. Also, it seems that the "ratio" between the complexity of a robust controller and the complexity of the risk-adjusted controller increases with the dimension of the uncertainty vector. Therefore, it would be of interest to quantify how does complexity depend on the uncertainty dimension. Finally, we note that results to date only deal with risk constraints. It would be of great interest to develop design procedures that would take into account both risk and robust constraints.

7 Appendix: Proof of Theorem 3.1

The proof is identical to the one presented in [7] and it is presented here for the sake of completeness. For a given $0 \le \varepsilon \le 1/2$, note that proving the convexity of the set

$$X_\varepsilon \doteq \{x \in \mathbf{R}^\ell : \mathrm{Prob}\{x^T(a_0+\Delta a) \le b\} \ge 1-\varepsilon\}$$

is equivalent to proving the quasi-concavity of the function

$$\varphi(x) \doteq \mathrm{Prob}\{x^T(a_0+\Delta a) \le b\}$$

on the set

$$\mathcal{D} \doteq \{x \in \mathbf{R}^\ell : \mathrm{Prob}\{x^T(a_0+\Delta a) \le b\} \ge 1/2\}.$$

Hence given $x^0, x^1 \in \mathcal{D}$, we must prove that

$$\varphi((1-\lambda)x^0+\lambda x^1) \ge \min\{\varphi(x^0), \varphi(x^1)\}$$

for all $0 \le \lambda \le 1$. Notice that the definition above only makes sense if the set $\mathcal{D}$ is convex. Proceeding by contradiction and assume that the set $\mathcal{D}$ is not convex. Given the fact that $\varphi(x)$ is continuous, non convexity of $\mathcal{D}$ implies the existence of $x^0, x^1 \in \mathbf{R}^\ell$ and $0 < \lambda < 1$ such that $\varphi(x^0) = \varphi(x^1) = 1/2$ and $\varphi((1-\lambda)x^0+\lambda x^1) < 1/2$. Now, defining

$$Q_{good}(x) = \{\Delta a \in \mathbf{R}^\ell : x^T(a_0+\Delta a) \le b\},$$

the symmetry of the distribution of Δa and the assumptions on x^0, x^1 and λ imply that

$$0 \in Q_{good}(x^0) \cap Q_{good}(x^1);\; 0 \notin Q_{good}((1-\lambda)x^0+\lambda x^1).$$

However, it can be easily shown that

$$Q_{good}(x^0) \cap Q_{good}(x^1) \subseteq Q_{good}((1-\lambda)x^0+\lambda x^1).$$

This contradicts $0 \notin Q_{good}((1-\lambda)x^0 + \lambda x^1)$. Therefore, the set $\mathcal{D}$ is convex. We now proceed with the proof of quasi-concavity of $\varphi(x)$. Proceeding by contradiction, assume there exist $x^0, x^1 \in \mathcal{D}$ and $0 < \lambda < 1$ such that

$$\varphi((1-\lambda)x^0 + \lambda x^1) < \min\{\varphi(x^0), \varphi(x^1)\}.$$

Without loss of generality, we assume that $\varphi(x^0) \leq \varphi(x^1)$ and recall that $\varphi(x)$ is a continuous function of x. Therefore, there exists a $\lambda < \lambda^* \leq 1$ such that

$$\varphi((1-\lambda^*)x^0 + \lambda^* x^1) = \varphi(x^0).$$

Note that λ^* is strictly greater than λ since we assumed that $\varphi((1-\lambda)x^0 + \lambda x^1) < \varphi(x^0)$. Letting $y^0 = x^0$, $y^1 = (1-\lambda^*)x^0 + \lambda^* x^1$ and $\zeta = \lambda/\lambda^*$, we obtain

$$(1-\lambda)x^0 + \lambda x^1 = (1-\zeta)y^0 + \zeta y^1.$$

Hence, we have

$$\varphi((1-\zeta)y^0 + \zeta y^1) < \varphi(y^0) = \varphi(y^1).$$

Now, define $y^\zeta \doteq (1-\zeta)y^0 + \zeta y^1$. Then

$$\text{Prob}(Q_{good}(y^\zeta)) < \text{Prob}(Q_{good}(y^0)) = \text{Prob}(Q_{good}(y^1)).$$

Let $\gamma = 1 - \text{Prob}(Q_{good}(y^0))$. Since $y^0 \in \mathcal{D}$, then $0 \leq \gamma \leq 1/2$. To establish quasi-concavity of $\varphi(x)$ for $x \in \mathcal{D}$ we consider several cases. In the case of $\gamma = 0$ or $\gamma = 1/2$, a contradiction is reached since the robust linear program (risk $\gamma = 0$) is a convex program and the set $\mathcal{D}$ is convex. For the intermediate case when $0 < \gamma < 1/2$, since Δa has a log-concave symmetric distribution, Proposition 2 in [8] indicate that for this range of values of γ, the floating body K_γ exists and is a convex symmetric set. Therefore,

$$K_\gamma \subseteq Q_{good}(y^0) \cap Q_{good}(y^1).$$

Now, given that $Q_{good}(y^0) \cap Q_{good}(y^1) \subseteq Q_{good}(y^\zeta)$, we have $K_\gamma \subseteq Q_{good}(y^\zeta)$. Recall that

$$\text{Prob}(Q_{good}(y^\zeta)) < 1 - \gamma \Rightarrow \text{Prob}(Q^c_{good}(y^\zeta)) > \gamma.$$

However, given the definition of K_γ, we have

$$K_\gamma \cap Q^c_{good}(y^\zeta) \neq \emptyset$$

and we reach a contradiction. Since we reached a contradiction in all of the cases above, we conclude that the function $\varphi(x)$ is quasi-concave for all $x \in \mathcal{D}$.

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