

## PROBABILISTICALLY CONSTRAINED LINEAR PROGRAMS AND RISK-ADJUSTED CONTROLLER DESIGN\*

CONSTANTINO M. LAGOA<sup>†</sup>, XIANG LI<sup>†</sup>, AND MARIO SZNAIER<sup>†</sup>

**Abstract.** The focal point of this paper is the probabilistically constrained linear program (PCLP) and how it can be applied to control system design under risk constraints. The PCLP is the counterpart of the classical linear program, where it is assumed that there is random uncertainty in the constraints and, therefore, the deterministic constraints are replaced by probabilistic ones. It is shown that for a wide class of probability density functions, called log-concave symmetric densities, the PCLP is a convex program. An equivalent formulation of the PCLP is also presented which provides insight into numerical implementation. This concept is applied to control system design. It is shown how the results in this paper can be applied to the design of controllers for discrete-time systems to obtain a closed loop system with a well-defined risk of violating the so-called property of *superstability*. Furthermore, we address the problem of risk-adjusted pole placement.

**Key words.** probabilistic constraints, linear program, convexity, risk-adjusted controller

**AMS subject classifications.** 90C15, 90C25, 90C05, 93C99

**DOI.** 10.1137/S1052623403430099

**1. Introduction.** Recently there has been a growing interest in the development of control systems design procedures which are able to handle risk constraints. The main motivation for considering this problem is that “classical” robust controllers tend to be complex; i.e., often “classical” controller design algorithms produce high order controllers. The hope is that, if one is willing to tolerate a small, well-defined risk of violation of performance specifications, one would obtain a significantly less complex controller. Also, there are several problems which naturally lead to a formulation involving risk constraints. An example is emergency system operation. Here one would like to allow for a small, well-defined risk of violation of stability in exchange for a better chance of handling the emergency situation. For example, in the control of an aircraft, one may want to have more power available to avoid a collision, although this might lead to a small risk of engine failure. For situations like this, a different kind of controller should be designed. It should be a controller that maximizes performance at the expense of a very small, well-defined risk of instability. In other words, we relax the conditions on stability in exchange for an improvement in performance.

The problem of designing risk-adjusted controllers has been considered in several papers, e.g., see [1], [2], [3], and [4]. However, there is a fundamental difference between the work mentioned above and the results presented in this paper. In contrast to the work presented in the papers mentioned above, where the search for the controller parameters is done using randomized algorithms, this paper integrates a new line of research which aims at developing fast deterministic algorithms for risk-adjusted controller design. The work presented in [5] and [6] indicates that there are several risk-adjusted controller design problems which are convex and, hence, numer-

---

\*Received by the editors June 19, 2003; accepted for publication (in revised form) July 19, 2004; published electronically April 22, 2005. Funding for this research was provided by the National Science Foundation under grants ECS-9984260 and ECS-0115946 and the Air Force Office of Scientific Research under grant AFSOR-F49620-00-1-0020.

<http://www.siam.org/journals/siopt/15-3/43009.html>

<sup>†</sup>Electrical Engineering Department, The Pennsylvania State University, University Park, PA 16802 (lagoa@engr.psu.edu, xxl124@psu.edu, msznaier@frodo.ee.psu.edu).

ically solvable. In this paper, we extend the class of risk-adjusted design problems which are known to be convex. The main paradigm underlying the results presented is the concept of the probabilistically constrained linear program.

**1.1. Probabilistically constrained linear program.** The main result of this paper concerns the convexity of the so-called probabilistically constrained linear program (PCLP). We show that, for a large class of probability distributions, the probabilistic version of the classical linear program is convex. Furthermore, we show how it can be applied in a controller design context.

Indeed, consider the “classical” linear program described by

$$\min c^T x$$

subject to

$$x^T a^i \leq b_i; \quad i = 1, 2, \dots, k,$$

where  $c, x, a^i \in \mathbf{R}^\ell$ , and  $b_i \in \mathbf{R}$ ,  $i = 1, 2, \dots, k$ .

In the PCLP framework, the constraint vectors  $a^i$  and  $b$  above are treated as random and the deterministic constraints are replaced by probabilistic constraints. There are a number of versions of the PCLP problem and the one that is used throughout this paper is described below.

DEFINITION 1.1. *Given acceptable risk levels  $0 \leq \varepsilon_i \leq 1$ ,  $i = 1, 2, \dots, k$ , the PCLP is defined as the following optimization problem:*

$$\min c^T x$$

subject to

$$\text{Prob}\{x^T a^i \leq b_i\} \geq 1 - \varepsilon_i; \quad i = 1, 2, \dots, k,$$

where  $c, x \in \mathbf{R}^\ell$ , and  $a^i, b$  are random vectors of appropriate dimensions.

**1.2. Convexity of the feasible set.** A fundamental question about the PCLP is the following: Is the PCLP a convex program? In other words, is the feasible set

$$\mathcal{X}_\varepsilon \doteq \{x \in \mathbf{R}^\ell : \text{Prob}\{x^T a^i \leq b_i\} \geq 1 - \varepsilon_i \text{ for } i = 1, 2, \dots, k\}$$

convex? It turns out that, without additional conditions on the distribution of the pair  $(a^i, b_i)$ , one can easily generate examples in which the answer to this question is “no.”

In this paper, we prove that the PCLP is a convex program when the probability density function of the random parameters is log-concave and symmetric; see section 2 for a precise definition of this class of distributions.

**1.3. Why consider individual probability constraints?** In the setup above, one considers only individual probability constraints; i.e., one looks at the probability of satisfaction of each of the constraints independently. In this paper, only this case is considered since joint probability constraints result, in general, in a nonconvex feasible set. As an example, consider the feasible set

$$\mathcal{X}_{0.45} \doteq \{x \in \mathbf{R}^2 : \text{Prob}\{x^T [\Delta a_1 \ \Delta a_2] \leq 0.15 \text{ and } x^T [0 \ \Delta a_2] \leq 0.15\} \geq 1 - 0.45\},$$

where  $\Delta a = [\Delta a_1 \ \Delta a_2]^T$  is uniformly distributed over the set

$$\{\Delta a \in \mathbf{R}^2 : |\Delta a_1| \leq 1, |\Delta a_2| \leq 1\}.$$

One can see that both  $x^1 = [1 \ 0]^T$  and  $x^2 = [0 \ 1]^T$  belong to the set  $\mathcal{X}_{0.45}$ . Now consider

$$x^3 = \frac{1}{2}x^1 + \frac{1}{2}x^2 = [0.5 \ 0.5]^T.$$

It can be proven that

$$\text{Prob}\{(x^3)^T[\Delta a_1 \ \Delta a_2]^T \leq 0.15 \text{ and } (x^3)^T[0 \ \Delta a_2] \leq 0.15\} = 0.525,$$

and hence  $x^3$  does not belong to  $\mathcal{X}_{0.45}$ . In other words, the set  $\mathcal{X}_{0.45}$  is not convex.

**1.4. Previous results.** Convexity results are available for specific kinds of distributions. For example, in [15] it is proven that  $\mathcal{X}_\varepsilon$  is convex when  $0 \leq \varepsilon_i \leq 1/2$  and  $b_i$  and the components of the  $a^i$  are independent and normally distributed. This result was later extended to the case when  $a^i$  and  $b_i$  have stable distributions, e.g., see [16]. The convexity of the PCLP has also been established for the case when  $a^i$  are deterministic and  $b_i$  are random with a log-concave distribution; see [17]. Using recent results from geometry, in this paper we extend these earlier results and prove that, for a large class of joint distributions for  $a^i$  and  $b_i$  (which includes many of the commonly used distributions), the PCLP is a convex program.

Several results are also available for PCLPs with joint probability constraints. It has been shown that there are two instances in which the jointly constrained PCLP is convex: (i) when the quantities  $a^i$  are deterministic and the vector  $b$  has a log-concave distribution (not necessarily symmetric) or a so-called  $\alpha$ -concave distribution; and (ii) when  $a^i$  and the vector  $b$  have a joint Gaussian distribution with some restrictions on the covariance matrix. Also, there are several results involving quasi-concave constraint functions instead of the linear functions considered in this paper. For an overview of such results see [17].

**1.5. The sequel.** Section 2 is dedicated to the definition of the class of admissible distributions for the uncertain parameters: log-concave symmetric distributions. The main result of this paper is presented in section 3, which states that the PCLP is a convex program. In section 4, we provide some insight into a numerical implementation of the PCLP. Section 5 is dedicated to the application of our results to control system design. Finally, in section 6, some concluding remarks are presented and several directions for further research are outlined.

**2. Preliminaries: Log-concavity.** In order to communicate the main result, we need to elaborate on which probability density functions are admissible for the vector of uncertain parameters. To this end, we require a definition of log-concave functions, e.g., see [17]. In this section, we also introduce a concept that is central to the results presented in this paper: floating body of a probability measure.

**2.1. Log-concave functions and probability densities.** A function  $f : \mathbf{R}^\ell \rightarrow [0, \infty)$  is said to be *log-concave* if the following condition holds: Given any  $x^0, x^1 \in \mathbf{R}^\ell$ , and  $\lambda \in [0, 1]$ , it follows that

$$f((1 - \lambda)x^0 + \lambda x^1) \geq [f(x^0)]^{1-\lambda}[f(x^1)]^\lambda.$$

In what follows, we let  $\mathcal{L}$  denote the class of log-concave symmetric probability density functions. Without loss of generality, one can assume that the center of symmetry is the origin; i.e., if  $f \in \mathcal{L}$ , then for any  $x \in \mathbf{R}^\ell$ , we have

$$f(x) = f(-x).$$

Throughout this paper we assume that the probability density function  $f$  of the vector of uncertain parameters is log-concave and symmetric, i.e.,  $f \in \mathcal{L}$ .

**2.2. Remarks.** It is important to note that the class  $\mathcal{L}$  of log-concave symmetric density functions is quite rich. For example, most “common” probability density functions used in the control field (such as uniform distribution over compact convex symmetric sets or Gaussian) are readily shown to be log-concave and symmetric. In other words, the main result to follow applies to typical density functions which have appeared in the probabilistic robustness literature to date.

**2.3. Floating body.** Central to the results presented in this paper is the concept of a *floating body* of a probability measure. This concept was first introduced by Dupin [7] in the context of uniform distributions. Given a risk level  $0 < \varepsilon < 1/2$ , the floating body  $K_\varepsilon$  of a probability distribution is a convex symmetric set for which each supporting hyperplane “cuts off” a set of probability  $\varepsilon$ .

More precisely, let  $0 < \varepsilon < 1/2$  be given and consider a set  $K_\varepsilon$ . For any direction  $u \in \mathbf{R}^\ell$ ,  $\|u\|_2 = 1$ , let  $H(u)$  be the supporting hyperplane of  $K_\varepsilon$  normal to  $u$ . Also, let  $\mathcal{H}^+(u)$  be the half-space defined by  $H(u)$  which does not contain the origin. Then,  $K_\varepsilon$  is the *floating body associated with  $\varepsilon$*  of the given probability measure if, given any direction  $\|u\|_2 = 1$ ,

$$\text{Prob}(\mathcal{H}^+(u)) = \varepsilon.$$

Not every probability measure has a floating body. However, the results in [9] indicate that every probability distribution with a probability density function belonging to the class  $\mathcal{L}$  does have a compact convex symmetric floating body  $K_\varepsilon$  for any  $0 < \varepsilon < 1/2$ .

**2.4. Why consider only symmetric probability density functions?** It turns out that symmetry of the probability density function plays an important role in the results to follow. In general, nonsymmetric probability density functions result in a distribution that does not have a floating body. In other words, they result in a PCLP which is not a convex program. As an example, consider uncertainty  $\Delta a = [\Delta a_1 \ \Delta a_2]^T$  uniformly distributed over the set

$$\{\Delta a \in \mathbf{R}^2 : \Delta a_1^2 + \Delta a_2^2 \leq 1, \Delta a_1 \geq 0, \Delta a_2 \geq 0\},$$

which has a nonsymmetric log-concave probability density function. Now, consider a risk level  $\varepsilon = 0.45$ . One can easily see that for  $x^1 = [1 \ 0]^T$  and  $x^2 = [0 \ 1]^T$ , one has

$$\text{Prob}\{(x^1)^T \Delta a \leq 0.45\} = \text{Prob}\{(x^2)^T \Delta a \leq 0.45\} = 0.552 \geq 1 - \varepsilon.$$

Moreover, if one takes

$$x^3 = \frac{1}{2}x^1 + \frac{1}{2}x^2 = [0.5 \ 0.5]^T,$$

we have

$$\text{Prob}\{(x^3)^T \Delta a \leq 0.45\} = 0.5157.$$

Now, let

$$\mathcal{A}_i \doteq \{\Delta a \in \mathbf{R}^2 : (x^i)^T \Delta a \leq 0.45\},$$

and, proceeding by contradiction, assume that the floating body  $K_{0.45}$  exists for this probability distribution. From the results above, we have  $K_{0.45} \subseteq \mathcal{A}_1$  and  $K_{0.45} \subseteq \mathcal{A}_2$ . Hence,

$$K_{0.45} \subseteq \mathcal{A}_1 \cap \mathcal{A}_2.$$

However,  $K_{0.45} \not\subseteq \mathcal{A}_3$ . This leads to a contradiction since  $\mathcal{A}_1 \cap \mathcal{A}_2 \subseteq \mathcal{A}_3$ . Hence, the floating body  $K_{0.45}$  does not exist for the distribution above.

**2.5. Additional notation.** Let  $\|\cdot\|$  be a norm in  $\mathbf{R}^\ell$ . We define the dual norm as

$$\|x\|_* \doteq \max\{x^T y : \|y\| \leq 1\}.$$

Now, recalling that the probability density functions considered in this paper are log-concave and symmetric, given a probability density function  $f \in \mathcal{L}$ , let  $\|\cdot\|_\varepsilon$  be the Minkowski functional associated with its floating body  $K_\varepsilon \neq \emptyset$ , i.e.,

$$\|x\|_\varepsilon \doteq \inf \left\{ \rho \in \mathbf{R}^+ : \frac{1}{\rho} x \in K_\varepsilon \right\}.$$

Note that the quantity above is a norm since the set  $K_\varepsilon$  is compact, convex, and symmetric. Furthermore, let  $\|\cdot\|_{\varepsilon,*}$  denote its dual norm as defined above.

**3. Convexity of the PCLP.** In this section, we present the main result of the paper. Theorem 3.1 to follow indicates that, if the distribution of the uncertain parameters is log-concave and symmetric and if the risk levels satisfy  $0 \leq \varepsilon_i \leq 1/2$ , the PCLP is a convex program. Although the result below involves only the convexity of a PCLP with one constraint, the extension to the case with an arbitrary number of constraints is immediate. This extension is a consequence of the fact that an intersection of convex sets is still convex. More precisely, for multiple constraints defining  $\mathcal{X}_\varepsilon$  as an intersection of

$$\mathcal{X}_{\varepsilon,i} \doteq \{x \in \mathbf{R}^\ell : \text{Prob}\{x^T a^i \leq b_i\} \geq 1 - \varepsilon_i\},$$

the convexity result is immediate because each  $\mathcal{X}_{\varepsilon,i}$  is readily shown to be convex.

Throughout this paper, we write

$$a^i = a_0^i + \Delta a^i; \quad i = 1, 2, \dots, k,$$

and

$$b = b_0 + \Delta b,$$

where  $a_0$  and  $b_0$  are the nominal values of the random parameters  $a_i$  and  $b_i$ , respectively. We assume that the pair  $(\Delta a^i, \Delta b_i)$  has a log-concave symmetric probability density function. For simplicity of exposition, it is assumed that the vector  $b$  is deterministic, i.e.,  $b = b_0$ . The more general result can be easily obtained by noting that

$$\begin{aligned} \mathcal{X}_{\varepsilon,i} &= \{x \in \mathbf{R}^\ell : \text{Prob}\{x^T a^i - \Delta b_i \leq b_{0,i}\} \geq 1 - \varepsilon_i\} \\ &= \{\tilde{x} \in \mathbf{R}^{\ell+1} : \text{Prob}\{\tilde{x}^T \tilde{a}^i \leq b_{0,i}\} \geq 1 - \varepsilon_i\} \cap \{\tilde{x} \in \mathbf{R}^{\ell+1} : \tilde{x}_{\ell+1} = -1\}, \end{aligned}$$

where  $\tilde{a}^i = [a^i{}^T \Delta b_i]^T$ .

**THEOREM 3.1.** *Let  $a_0 \in \mathbf{R}^\ell$ ,  $b \in \mathbf{R}$ , and the risk level  $0 \leq \varepsilon \leq 1/2$  be given. Also, let the random vector  $\Delta a$  have a log-concave symmetric distribution. Then, the set*

$$\mathcal{X}_\varepsilon \doteq \{x \in \mathbf{R}^\ell : \text{Prob}\{x^T(a_0 + \Delta a) \leq b\} \geq 1 - \varepsilon\}$$

is convex.

*Proof.* For the given  $0 \leq \varepsilon \leq 1/2$ , we first note that proving the convexity of the set

$$\mathcal{X}_\varepsilon \doteq \{x \in \mathbf{R}^\ell : \text{Prob}\{x^T(a_0 + \Delta a) \leq b\} \geq 1 - \varepsilon\}$$

is equivalent to proving the quasi concavity of the function

$$\varphi(x) \doteq \text{Prob}\{x^T(a_0 + \Delta a) \leq b\}$$

on the set

$$\mathcal{D} = \{x \in \mathbf{R}^\ell : \varphi(x) \geq 1/2\}.$$

In other words, given  $x^0, x^1 \in \mathcal{D}$  one must show that

$$\varphi((1 - \lambda)x^0 + \lambda x^1) \geq \min\{\varphi(x^0), \varphi(x^1)\}$$

for all  $0 \leq \lambda \leq 1$ . Now, define

$$Q_{good}(x) \doteq \{\Delta a \in \mathbf{R}^\ell : x^T(a_0 + \Delta a) \leq b\}$$

leading to

$$\varphi(x) = \text{Prob}\{Q_{good}(x)\}$$

and extend the definition of the convex floating body to include risk levels  $\varepsilon = 1/2$  and  $\varepsilon = 0$  as

$$K_{1/2} = \{0\}; \quad K_0 = \{\Delta a \in \mathbf{R}^\ell : f(\Delta a) \neq 0\},$$

where  $f(\cdot)$  denotes the probability density function. Note that both are convex symmetric sets and that, for any  $0 \leq \varepsilon \leq 1/2$ ,

$$\varphi(x) \geq 1 - \varepsilon \Leftrightarrow Q_{good}(x) \supseteq K_\varepsilon.$$

Now, take  $x^0, x^1 \in \mathcal{D}$  and let  $0 \leq \varepsilon \leq 1/2$  be defined as

$$\varepsilon \doteq 1 - \min\{\varphi(x^0), \varphi(x^1)\}.$$

Now, since

$$\min\{\varphi(x^0), \varphi(x^1)\} = 1 - \varepsilon,$$

the definition of convex floating body implies the inclusions

$$Q_{good}(x^0) \supseteq K_\varepsilon; \quad Q_{good}(x^1) \supseteq K_\varepsilon.$$

Hence, for any  $\lambda \in [0, 1]$ ,

$$Q_{good}((1 - \lambda)x^0 + \lambda x^1) \supseteq Q_{good}(x^0) \cap Q_{good}(x^1) \supseteq K_\varepsilon.$$

Therefore,

$$\varphi((1 - \lambda)x^0 + \lambda x^1) \geq 1 - \varepsilon = \min\{\varphi(x^0), \varphi(x^1)\}$$

or, in other words,  $\varphi(x)$  is a quasi-concave function for all  $x \in \mathcal{D}$ . This completes the proof.  $\square$

**4. On the numerical implementation of the PCLP.** The result in the previous section indicates that the PCLP is a convex program. However, it does not provide any indication of how to numerically solve the resulting optimization problem. In this section, we use the concept of *floating body* introduced in section 2.3 to provide some insight into how one can numerically solve the PCLP.

**4.1. A closed-form formulation of the PCLP.** Since the set

$$\{\Delta a \in \mathbf{R}^\ell : x^T(a_0 + \Delta a) \leq b\}$$

is a half-space, the definition of the floating body presented in section 2.3 indicates that requiring

$$\text{Prob}\{x^T(a_0 + \Delta a) \leq b\} \geq 1 - \varepsilon$$

for  $0 < \varepsilon < 1/2$  is equivalent to requiring

$$x^T(a_0 + \Delta a) \leq b$$

for all  $\Delta a \in K_\varepsilon$ , where  $K_\varepsilon$  is the floating body of the probability distribution of  $\Delta a$  as defined in section 2.3. Now, given the definition of dual norm, this is equivalent to

$$\|x\|_{\varepsilon,*} \leq b - x^T a_0.$$

Therefore, the probabilistic constraints of the PCLP can be replaced by convex ones of the form above. Hence, if the quantity  $\|x\|_{\varepsilon,*}$  can be easily determined, this leads to an immediate numerical implementation for solving the PCLP.

**4.2. Elliptical log-concave distributions.** It turns out that there are cases where  $\|x\|_{\varepsilon,*}$  is easily determined. An example is the case where the probability distribution of the uncertain parameters is an *elliptical log-concave* distribution. An elliptical log-concave distribution is a distribution whose probability density function is of the form

$$f(y) = g(y^T W y),$$

where  $g : \mathbf{R}_0^+ \rightarrow \mathbf{R}_0^+$  is a log-concave nonincreasing function and  $W$  is a positive definite matrix. Examples of such distributions are multivariable normal distributions and uniform distributions over balls. Note that the function above is indeed log-concave. This can be proven by using the fact that the composition of a log-concave nonincreasing function with a convex function is itself log-concave. For such probability distributions one can determine the “shape” of the floating body. This result is presented below.

LEMMA 4.1. *Consider an elliptical log-concave distribution. Then, for any  $0 < \varepsilon < 1/2$ , there exists a radius  $r(\varepsilon)$  leading to*

$$K_\varepsilon = \{\Delta a \in \mathbf{R}^\ell : \Delta a^T W \Delta a \leq r^2(\varepsilon)\}.$$

*Sketch of proof.* As mentioned in the definition of an elliptical log-concave distribution, its probability density function is of the form

$$f(\Delta a) = g((\Delta a)^T W \Delta a),$$

where  $g(\cdot)$  is log-concave. Now let

$$\Delta\tilde{a} = \frac{1}{\det(W^{1/2})} W^{1/2} \Delta a,$$

where the symmetric matrix  $W^{1/2}$  is such that  $W = W^{1/2}W^{1/2}$ . Since the Jacobian of the linear transformation above can be easily shown to be 1, it follows that the probability density function of  $\Delta\tilde{a}$  is given by

$$f_{\Delta\tilde{a}}(\Delta\tilde{a}) = g \left[ \det^2(W^{1/2})(\Delta\tilde{a})^T \Delta\tilde{a} \right].$$

The contours of  $f_{\Delta\tilde{a}}$  are hyperspheres and, hence, this probability distribution is “direction independent”; i.e., given any constant  $b$  and any two vectors  $\|u\| = 1$  and  $\|v\| = 1$ , we have

$$\text{Prob}\{u^T \Delta\tilde{a} \leq b\} = \text{Prob}\{v^T \Delta\tilde{a} \leq b\}.$$

Hence, given any  $0 < \varepsilon < 1/2$ , the floating body of this distribution is of the form

$$\tilde{K}_\varepsilon = \{\Delta\tilde{a} \in \mathbf{R}^\ell : \Delta\tilde{a}^T \Delta\tilde{a} \leq \tilde{r}^2(\varepsilon)\}$$

for some  $\tilde{r}(\varepsilon)$ . Since there is a linear relationship between  $\Delta a$  and  $\Delta\tilde{a}$ , one can see that the floating body of the probability distribution of  $\Delta a$  is given by

$$K_\varepsilon = \{\Delta a \in \mathbf{R}^\ell : \Delta a^T W \Delta a \leq \tilde{r}^2(\varepsilon) \det^2(W^{1/2})\}.$$

This completes the proof.  $\square$

The actual “radius” of the ellipsoid  $K_\varepsilon$  can be determined analytically for some probability distributions. If one cannot determine this radius analytically, a one line search optimization problem can be set up to numerically approximate this value by performing several Monte Carlo simulations. Therefore, for such probability distributions, one can easily see that the PCLP reduces to a convex quadratic optimization problem, i.e., minimization of a linear function subject to quadratic constraints. More precisely, consider an elliptical log-concave probability density function of the form described above. Then, for any  $0 < \varepsilon < 1/2$ , the floating body  $K_\varepsilon$  is of the form

$$K_\varepsilon = \{\Delta a \in \mathbf{R}^\ell : \Delta a^T W \Delta a \leq r^2(\varepsilon)\},$$

where  $r(\varepsilon) > 0$  is such that the half-space  $H = \{\Delta a \in \mathbf{R}^\ell : \Delta a_1 \leq \zeta\}$  with

$$\text{Prob}(H) = 1 - \varepsilon$$

results in a hyperplane  $\{\Delta a \in \mathbf{R}^\ell : \Delta a_1 = \zeta\}$  tangent to the ellipsoid

$$\{\Delta a \in \mathbf{R}^\ell : \Delta a^T W \Delta a \leq r^2(\varepsilon)\}.$$

Now let

$$\Delta\gamma = \frac{1}{r(\varepsilon)} W^{1/2} \Delta a,$$

where  $W^{1/2}$  is a positive definite matrix satisfying  $W^{1/2}W^{1/2} = W$ . Note that  $\Delta a \in K_\varepsilon$  if and only if  $\|\Delta\gamma\|_2 \leq 1$ . Then, requiring

$$x^T (a_0 + \Delta a) \leq b$$



for all  $\Delta a \in K_\varepsilon$  is equivalent to requiring

$$x^T(a_0 + r(\varepsilon)W^{-1/2}\Delta\gamma) \leq b$$

for all  $\|\Delta\gamma\|_2 \leq 1$ . Hence, since the dual of the 2-norm is the 2-norm itself, the expression above is equivalent to

$$\|r(\varepsilon)W^{-1/2}x\|_2 \leq b - x^T a_0,$$

which is a convex quadratic constraint on  $x$ .

**5. Application to control systems design.** In this section, we exemplify how the concept of a PCLP can be applied to controller design. We start by applying the PCLP to the design of superstable systems. A second example shows how the theory in this paper can be applied to robust pole assignment.

**5.1. Superstability.** In contrast to the concept of  $\ell^\infty$  stability, where only input/output behavior is considered, superstability allows for computing the worst-case value of the  $\ell^\infty$  norm of the output due to  $\ell^\infty$  bounded disturbances and initial conditions. In addition, it provides an upper bound on the  $\ell^\infty$  induced norm of the system (this bound is exact in the case of finite impulse response (FIR) systems). The reader is referred to [10] and [11] for proofs and additional properties.

Consider a discrete-time linear time invariant system

$$y(z) = G(z)w(z), \quad G(z) = \frac{b(z)}{1 + a(z)},$$

where the  $w$  are exogenous disturbances,  $y$  is the output,  $z$  is the delay operator  $zx[k] = x[k - 1]$ , and where the polynomial  $a(z)$  does not have a constant term, i.e.,

$$a(z) = a_1z + a_2z^2 + \cdots + a_nz^n.$$

Define

$$\|a\|_1 = \sum_{i=1}^n |a_i|.$$

A system is said to be superstable if

$$\|a\|_1 < 1.$$

Moreover, it can be shown (see [10]) that in this case the  $\ell^\infty$  induced norm of the system is bounded by

$$\|G(z)\|_{\ell^\infty \rightarrow \ell^\infty} \leq \frac{\|b\|_1}{1 - \|a\|_1}.$$

This property was exploited in [10] to synthesize low order  $\ell^1$  controllers as follows. Consider a feedback controller of the form

$$G_c(z) = \frac{b_c(z)}{1 + a_c(z)}.$$

Synthesizing a controller of the form described above such that the  $\ell^1$  norm of the closed loop system is less than or equal to a given  $\mu$  reduces to finding controller coefficients satisfying

$$\mu \|d_{cl}\|_1 + \|n_{cl}\|_1 \leq \mu,$$

where

$$d_{cl}(z) = a(z) + a_c(z) + a(z)a_c(z) + b(z)b_c(z); \quad n_{cl}(z) = b(z)b_c(z).$$

Hence, in this case, controller design is equivalent to finding a feasible point of a set of linear inequalities.

Now, assume that the plant is subject to parametric uncertainty of the form

$$G(z) = \frac{b(z)}{1+a(z)} = \frac{b_{0,0} + \Delta b_0 + (b_{0,1} + \Delta b_1)z + \dots + (b_{0,m} + \Delta b_m)z^m}{1 + (a_{0,1} + \Delta a_1)z + \dots + (a_{0,n} + \Delta a_n)z^n},$$

where  $b_{0,i}$  and  $a_{0,i}$  are the nominal values of the coefficients of the numerator and denominator, respectively, and  $\Delta b_i$  and  $\Delta a_i$  represent the uncertainty.<sup>1</sup> In this case robust performance is achieved if the inequality

$$\mu \|d_{cl}\|_1 + \|n_{cl}\|_1 \leq \mu$$

holds for all admissible values of the uncertainty, which is a linear program. More precisely, given  $\mu > 0$ , the problem above can be recast as

$$A(\Delta a_0, \dots, \Delta a_n, \Delta b_0, \dots, \Delta b_m) \begin{bmatrix} a_{c,0} \\ \vdots \\ a_{c,n_c} \\ b_{c,0} \\ \vdots \\ b_{c,m_c} \end{bmatrix} \leq \mu \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}^T,$$

where  $a_{c,0}, \dots, a_{c,n_c}$  and  $b_{c,0}, \dots, b_{c,m_c}$  denote the coefficients of the denominator and the numerator of the controller, respectively. However, there is a major difference between the nominal and robust performance case: while it can be shown that the former always admits a solution if the controller order is chosen to be at least equal to the order of the plant, the latter may not have a solution even for high order controllers. On the other hand, as we illustrate next with a simple example, it might be possible to find low order risk-adjusted controllers, even for very small risk values of violating the constraints; i.e., given a distribution for the uncertain parameters and a small, well-defined risk  $\varepsilon$ , one may find a low order controller such that the probability of violating each of the resulting linear constraints is less than  $\varepsilon$ . In other words, in order to reduce controller complexity, one solves a PCLP for a given risk level  $\varepsilon$ .

<sup>1</sup>For notational simplicity, here we assume that the controller is not subject to uncertainty; however, the proposed procedure can be easily modified to take controller uncertainty into account.

**5.2. Numerical example.** We now consider the example in [10]. The discrete-time system presented has nominal transfer function

$$P(z) = \frac{n(z)}{d(z)} = \frac{z - 2.5z^2 + 1.501z^3}{1 - 2.7z + 23.5z^2 - 4.6z^3}$$

and we assume that all coefficients are subject to uncertainty. Moreover, we assume that the uncertainty vector is uniformly distributed on a ball with radius 0.05. In searching for the controller, we assume that it has the form

$$G_c(z) = \frac{b_c(z)}{1 + a_c(z)} = \frac{b_{c,0} + b_{c,1}z + \cdots + b_{c,m_c}z^{m_c}}{1 + a_{c,1}z + \cdots + a_{c,n_c}z^{n_c}}.$$

It can be shown that there is no controller of order less than or equal to  $n_c = 6$  that renders the closed loop system robustly superstable. Next, we allowed for a risk of  $\varepsilon = 1.25 \times 10^{-4}$ . Since, in this case, the distribution of the uncertainty is an elliptical log-concave distribution, we can use the results in section 4 to determine a risk-adjusted controller. As mentioned in section 4.2, the floating body for this distribution is a ball and, hence, one can set up a quadratic program to solve the problem at hand. Matlab was used to solve the resulting quadratic program, and the third order risk-adjusted controller

$$C(z) = \frac{4.5819 - 17.7802z - 1.0245z^2 + 0.8795z^3}{1 - 1.8819z + 0.6538z^2 + 0.287z^3}$$

was obtained. Next, a Monte Carlo simulation was performed to assess the risk of violating superstability (recall that  $\varepsilon$  is the risk of violating each inequality, not the actual risk of violating at least one of the inequalities). The number of samples used was  $10^7$  and the estimated probability of violating superstability obtained is 0.78%, showing that one can obtain a low order controller even for small risk levels.

**5.3. Robust pole assignment.** We now describe how one can apply the results in this paper to the problem of robust pole assignment. We start with a continuous uncertain open loop plant described by the transfer function

$$G(s) = \frac{n(s)}{d(s)} = \frac{(b_{0,0} + \Delta b_0) + (b_{0,1} + \Delta b_1)s + \cdots + (b_{0,m} + \Delta b_m)s^m}{(a_{0,0} + \Delta a_0) + (a_{0,1} + \Delta a_1)s + \cdots + (a_{0,n} + \Delta a_n)s^n},$$

where  $b_{0,i}$  and  $a_{0,i}$  are the nominal values of the coefficients of the numerator and denominator, respectively, and  $\Delta b_i$  and  $\Delta a_i$  represent the uncertainty. Now, since uncertainty is present, one cannot determine a controller that will assign the closed loop poles to a specific location. As in [13], one instead aims at designing a controller such that the closed loop characteristic polynomial belongs to a set defined by the performance specifications. More precisely, each of the coefficients of the closed loop characteristic polynomial should belong to a given interval; i.e., given a controller of the form

$$G_c(s) = \frac{n_c(s)}{d_c(s)} = \frac{b_{c,0} + b_{c,1}s + \cdots + b_{c,m_c}s^{m_c}}{a_{c,0} + a_{c,1}s + \cdots + a_{c,n_c}s^{n_c}},$$

one aims at determining the coefficients of the closed loop controller such that the closed loop characteristic polynomial

$$n_c(s)n(s) + d_c(s)d(s)$$

belongs to the family of polynomials

$$s^{n_{cl}} + [\delta_{n_{cl}-1}^-, \delta_{n_{cl}-1}^+]s^{n_{cl}-1} + \dots + [\delta_1^-, \delta_1^+]s + [\delta_0^-, \delta_0^+]$$

for all admissible uncertainty values, where  $n_{cl} = n_c + n$  is the degree of the closed loop characteristic polynomial. As can be easily seen, the search for the coefficients of the controller reduces to finding a feasible solution to a set of linear inequalities of the form

$$\begin{bmatrix} \delta_0^- \\ \vdots \\ \delta_{n_{cl}-1}^- \end{bmatrix} \leq A(\Delta a_0, \dots, \Delta a_n, \Delta b_0, \dots, \Delta b_m) \begin{bmatrix} a_{c,0} \\ \vdots \\ a_{c,n_c} \\ b_{c,0} \\ \vdots \\ b_{c,m_c} \end{bmatrix} \leq \begin{bmatrix} \delta_0^+ \\ \vdots \\ \delta_{n_{cl}-1}^+ \end{bmatrix},$$

where  $A(\Delta a_0, \dots, \Delta a_n, \Delta b_0, \dots, \Delta b_m)$  is an affine function of its arguments. To satisfy the performance specifications, the inequalities above should be satisfied for all admissible values of  $\Delta a_0, \dots, \Delta a_n$  and  $\Delta b_0, \dots, \Delta b_m$ . If the uncertainty support set is convex and symmetric, the problem above turns out to be convex. This is easily seen if one rewrites the inequalities above as follows: Let  $q = [\Delta a_0, \dots, \Delta a_n, \Delta b_0, \dots, \Delta b_m]^T$  and  $x = [a_{c,0}, \dots, a_{c,n_c}, b_{c,0}, \dots, b_{c,m_c}]^T$ . Then, for the uncertainty vector  $q$ , whose support is a convex symmetric set  $Q$ , the inequalities above can be written in the form

$$g_i(x)^T q \leq h_i(x) + k_i; \quad i = 1, 2, \dots, n_c + m_c + 2,$$

for all  $\|q\| \leq 1$ , where  $\|\cdot\|$  is the norm whose unit ball is the set  $Q$  and  $g_i(x)$  and  $f_i(x)$  are affine functions of  $x$ . Hence, we can rewrite the inequalities above as

$$\|g_i(x)\|_* \leq h_i(x) + k_i; \quad i = 1, 2, \dots, n_c + m_c + 2,$$

which is easily seen to be a convex constraint in  $x$ .

However, even though in many cases the problem above is convex, it can lead to complex controllers. To avoid this, we take a risk-adjusted point of view of the problem above; i.e., rather than requiring that each inequality be satisfied for all admissible values of the uncertain parameters, we require that the risk of violating each of the inequalities be less than or equal to a prescribed risk level  $\varepsilon$ . In other words, we solve a PCLP version of the problem above.

**5.4. Numerical example.** The example presented here is a small modification of one of the examples in [13]. Consider an uncertain plant with transfer function

$$G(s) = \frac{(0.75 + \Delta b_1)s + 1.25 + \Delta b_0}{s^2 + (0.75 + \Delta a_1)s + \Delta a_0},$$

where the uncertain parameter vector is uniformly distributed over an ellipsoid. More precisely

$$\begin{bmatrix} \Delta a_0 \\ \Delta a_1 \\ \Delta b_0 \\ \Delta b_1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Delta \alpha,$$

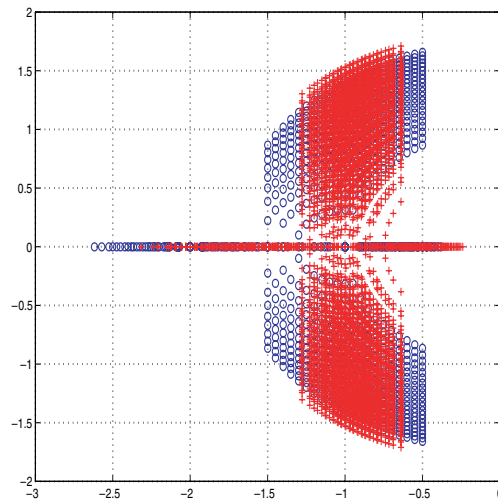


FIG. 5.1. Desired pole location “o” and actual pole location “+”.

where  $\Delta\alpha \in \mathbf{R}^4$  is uniformly distributed over the ball of radius 0.25. We now aim at designing a controller such that the closed loop polynomial belongs to the family

$$\Delta_T(s) = s^2 + [1, 3]s + [1, 3].$$

Therefore, the controller transfer function is the constant

$$G_c(s) = b_0.$$

It can be proven that no constant controller can meet the specification for all admissible values of the uncertain parameters. Next, we allowed for a risk of  $\varepsilon = 0.02$ . As in the previous numerical example, the distribution of the uncertainty is elliptical and log-concave and, hence, one can use the results in section 4 to determine a risk-adjusted controller. Again, the floating body for this distribution is a ball, and one can set up a quadratic program to solve the problem at hand. Matlab was used to solve the resulting quadratic program and the risk-adjusted constant controller

$$G_s(s) = 1.555$$

was obtained. The pole cluster distributions of the desired system and the actual closed loop system are shown in Figure 5.1. A Monte Carlo simulation was performed to estimate the actual risk of violating the specifications. The estimated value of the risk obtained was approximately 3.6%, illustrating another instance where, even for low risk values, one can obtain risk-adjusted controllers in cases where a robust controller does not exist. Furthermore, in this case, we were able to obtain robust stability as an added benefit; see Figure 5.1.

**6. Conclusions and further research.** In this paper, we show that the PCLP is a convex optimization problem for any log-concave symmetric distribution. Also, a closed-form formulation of the PCLP was provided which can be easily implemented in the case of elliptical distributions, such as normal or uniform over a ball. This result was applied to systems design, showing that classical control theory can lead to complex controllers. Namely, we have presented examples where, even for very low

levels of risk, one can obtain controllers that are substantially less complex than their robust counterparts.

The results in this paper suggest several directions for further research. First, we believe that effort should be put into the development of numerical tools for solving the PCLP when the distribution is other than elliptical. Also, it seems that the “ratio” between the complexity of a robust controller and the complexity of the risk-adjusted controller increases with the dimension of the uncertainty vector. Therefore, it would be of interest to quantify how this difference in complexity depends on the uncertainty dimension. Finally, we note that results to date deal only with risk constraints. It would be of great interest to develop design procedures that would take into account both risk and robust constraints. For example, one might want to design a control system that would result in a robustly stable system having a risk of violating the performance specification that is less than or equal to a given risk level.

## REFERENCES

- [1] L. R. RAY AND R. F. STENGEL, *A Monte Carlo approach to the analysis of control system robustness*, *Automatica*, 29 (1993), pp. 229–236.
- [2] R. F. STENGEL AND L. R. RAY, *Stochastic robustness of linear time-invariant control systems*, *IEEE Trans. Automat. Control*, 36 (1991), pp. 82–87.
- [3] A. YOON, P. KHARGONEKAR, AND K. HEBBALE, *Design of computer experiments for open-loop control and robustness analysis of clutch-to-clutch shifts in automatic transmissions*, in *Proceedings of the American Control Conference (Albuquerque, NM)*, Vol. 5, IEEE, Piscataway, NJ, 1997, pp. 3359–3364.
- [4] X. CHEN AND K. ZHOU, *Constrained optimal synthesis and robustness analysis by randomized algorithms*, in *Proceedings of the American Control Conference (Philadelphia, PA)*, Vol. 3, IEEE, Piscataway, NJ, 1998, pp. 1429–1433.
- [5] B. R. BARMISH AND C. M. LAGOA, *On convexity of the probabilistic design problem for quadratic stabilizability*, in *Proceedings of the American Control Conference (San Diego, CA)*, Vol. 1, IEEE, Piscataway, NJ, 1999, pp. 430–434.
- [6] C. M. LAGOA, *A convex parameterization of risk-adjusted stabilizing controllers*, in *Proceedings of the 38th Annual IEEE Conference on Decision and Control*, Phoenix, AZ, Vol. 1, 1999, pp. 534–539.
- [7] C. DUPIN, *Application de Géométrie et de Mécanique à la Marine*, aux Ponts et Chaussées, Paris, 1822.
- [8] S. MEYER AND S. REISNER, *A geometric property of the boundary of symmetric convex bodies and convexity of flotation surfaces*, *Geom. Dedicata*, 37 (1991), pp. 327–337.
- [9] M. MEYER AND S. REISNER, *Characterizations of affinely-rotation-invariant log-concave measures by section-centroid location*, in *Geometric Aspects of Functional Analysis (1989–90)*, *Lecture Notes in Math.* 1469, Springer, Berlin, 1991, pp. 145–152.
- [10] F. BLANCHINI AND M. SZNAIER, *A convex optimization approach to fixed-order controller design for disturbance rejection in SISO systems*, in *Proceedings of the 36th Annual IEEE Conference on Decision and Control*, San Diego, CA, 1997, pp. 1540–1545.
- [11] B. T. POLYAK AND M. E. HALPERN, *The use of a new optimization criterion for discrete-time feedback controller design*, in *Proceedings of the 38th Annual IEEE Conference on Decision and Control*, Phoenix, AZ, 1999, pp. 1540–1545.
- [12] L. H. KEEL AND S. P. BHATTACHARYYA, *Robust, fragile or optimal*, *IEEE Trans. Automat. Control*, 42 (1997), pp. 1098–1105.
- [13] L. H. KEEL AND S. P. BHATTACHARYYA, *A linear programming approach to controller design*, in *Proceedings of the 36th Annual IEEE Conference of Decision and Control*, San Diego, CA, 1997, pp. 2139–2148.
- [14] B. R. BARMISH AND C. M. LAGOA, *The uniform distribution: A rigorous justification for its use in robustness analysis*, *Math. Control Signals Systems*, 10 (1997), pp. 203–222.
- [15] C. VAN DE PANNE AND W. POPP, *Minimum-cost cattle feed under probabilistic protein constraints*, *Management Sci.*, 9 (1963), pp. 405–430.
- [16] S. VAJDA, *Probabilistic Programming*, Academic Press, New York, 1972.
- [17] A. PRÉKOPA, *Stochastic Programming*, Kluwer Academic Publishers, Boston, 1995.