On the Design of Robust Controllers for Arbitrary Uncertainty Structures

Constantino M. Lagoa, Xiang Li, and Mario Sznaier

***Abstract*—The focal point of this note is the design of robust controllers for linear time-invariant uncertain systems. Given bounds on performance (defined by a convex performance evaluator) the algorithm converges to a controller that robustly satisfies the specifications. The procedure introduced has its basis on stochastic gradient algorithms and it is proven that the probability of performance violation tends to zero with probability one. Moreover, this algorithm can be applied to any uncertain plant, independently of the uncertainty structure. As an example of application of this new approach, we demonstrate its usefulness in the design of robust H_2 controllers.**

***Index Terms*—Robust control, stochastic approximation.**

I. Introduction

The design of robust controllers has long been considered one of the more important problems in the control systems area. Well known approaches to address this problem include, among others, H_∞ theory and the structured singular value, e.g., see [17]. However, results to date are only applicable to specific uncertainty structures or/and can be conservative. To overcome these difficulties, recently a new approach has been developed to address the problem of robustness analysis and robust controller design. This new approach relies on well known results in probability theory and it has shown that classical robustness theory can be very conservative, i.e., one can greatly reduce the order of the controller and/or enlarge the admissible set of uncertainties and still have a very low risk of performance violation, e.g., see [1], [3], [7], [16], and [22]. Moreover, these methods can be applied to arbitrary uncertainty structures, as long as random uncertainty samples can be generated. Several stochastic approaches have also been developed for robust controller design which exploit the advantages of randomization mentioned above. The work in [8], [18], and [21] uses random sampling of the control parameters to look for the one with best performance. In [6], [10], [12], and [15] a stochastic gradient approach is applied to robust controller design. The problem of robust controller design was also addressed in [13] and [19], where genetic algorithms were used to determine the controller parameters. Also, in [14], simulated annealing algorithms were used for controller design. The main motivation for the work presented in this note is provided by the work in [6] and [15], where the structure of linear matrix inequalities was exploited to develop fast stochastic gradient algorithms. Building on this work, we develop algorithms based on a nonstandard stochastic approximation for the design of robust output feedback controllers.

In the sequel, the problem of robust output feedback controller design for linear systems with arbitrary dependence on the uncertain parameters is addressed. Consider an uncertain plant $G(z, \Delta)$, where $\Delta \in \mathbf{\Delta}$ represents uncertainty and $\mathbf{\Delta}$ is the compact uncertainty support set. The uncertainty can be either static or dynamic and no assumptions are made on the way $G(z, \Delta)$ depends on Δ. The only assumption is that random samples $\Delta^k \in \mathbf{\Delta}$ can be generated. Now, consider the closed-loop system in Fig. 1, whose closed-loop transfer function is denoted by $T_{CL}(z, \Delta, C)$. Given a convex objective function $g(\cdot)$ whose subgradient can be computed and a performance level γ, the objective is to design a controller $C^*(z)$ such that

$$g\left[T_{CL}(z, \Delta, C^*)\right] \leq \gamma$$

for all $\Delta \in \mathbf{\Delta}$. Note that the assumption on the availability of a procedure for generating random uncertainty samples is a rather mild one. Algorithms have been developed for generating both random samples of static uncertain parameters (e.g., see [3], [5], and [9]) as well as dynamic uncertain parameters; see [4], [11]. Given the probability measure underlying the random samples generation, the algorithm provided produces a sequence of controllers C_k having the property that the risk of violating the performance specification

$$P_k \doteq \mathrm{Prob}\{g\left[T_{CL}(z, \Delta, C_k)\right] > \gamma\}$$

tends to zero as $k \to \infty$. Moreover, it is proven that $\sum_{k=0}^{\infty} P_k < \infty$. Hence, P_k tends to zero asymptotically faster than $1/k$. The general nature of the algorithm provided, enables one to address many problems in robust controller design. In particular, these procedure can be used to solve the open problem of robust H_2 controller design. An example illustrating this particular instance of our algorithm is also provided. This algorithm is a nonstandard stochastic approximation algorithm in the sense that, in each step, it does not directly adapt the controller coefficients. Instead, for each given admissible value of the uncertain parameters, it uses the Youla parametrization of the closed-loop transfer function to indirectly adapt the controller coefficients. This new approach requires the use of a new "distance measure" of controllers to establish the convergence of the algorithm, i.e., convergence is proven by using the *robust controller gap* (see Section IV-A).

The note is organized as follows. In Section II, the notation used in this note is introduced and some ancillary results are provided. The problem formulation and the algorithm are presented in Section III. The main result concerning the convergence of the algorithm is provided and proven in Section IV. The algorithm is applied to robust H_2 design in Section V and concluding remarks are presented in Section VI.

II. Notation and Preliminaries

We now state the notation used throughout this note as well as some standard results needed for the presentation of our results.

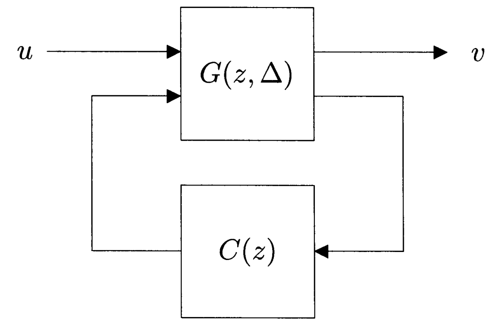

Fig. 1. Closed-loop system.

Manuscript received March 27, 2003; revised June 25, 2003. Recommended by Associate Editor T. Iwasaki. This work was supported by the National Science Foundation under Grant ECS-9984260 and Grant ECS-0115946, and by the Air Force Office of Scientific Research under Grant AFSOR-F49620-00-1-0020.

The authors are with the Electrical Engineering Department, The Pennsylvania State University, University Park, PA 16802 USA (e-mail: lagoa@engr.psu.edu; xiangli@psu.edu; msznaier@frodo.ee.psu.edu).

Digital Object Identifier 10.1109/TAC.2003.819304

A. Notation

Let $H_2^{n\times m}$ denote the Hilbert space of functions $H: \mathbf{C} \rightarrow \mathbf{C}^{n\times m}$ analytic in the set $\{z \in \mathbf{C}: |z| \geq 1\}$, equipped with the inner product

$$\langle H, T\rangle = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\{\mathrm{Trace}[H(e^{j\theta})^* T(e^{j\theta})]\} d\theta$$

where Re denotes the real part, $\mathrm{Trace}(A)$ is the trace of the matrix A and A^* denotes the conjugate transpose of A. Hence, the H_2 space has norm $\|T\|_2 = \sqrt{\langle T, T\rangle}$. Also, let RH_2 denote the subspace of all rational functions in H_2 analytic in $\{z \in \mathbf{C}: |z| \geq 1\}$. Moreover, define the space $\mathcal{G}$ as the space of rational functions $G: \mathbf{C} \rightarrow \mathbf{C}^{n\times m}$ that can be represented as

$$G(z) = G_s(z) + G_u(z)$$

where $G_s(z)$ analytic in the set $\{z \in \mathbf{C}: |z| \geq \alpha\}$ and $G_u(z)$ is strictly proper, analytic in the set $\{z \in \mathbf{C}: |z| < \alpha\}$ and $0 < \beta < \alpha < 1$. Now, given two functions $G, H \in \mathcal{G}$ define the distance function d as

$$d(G,H) \doteq \left(\|G_s(z) - H_s(z)\|_2^2 + \|G_u\left(\frac{\beta}{z}\right) - H_u\left(\frac{\beta}{z}\right)\|_2^2\right)^{1/2}.$$

The results later in the note that make use of this distance function are similar for any value of α. However, α is usually taken to be very close to one. Finally, define the projection $\pi_s: \mathcal{G} \rightarrow H_2$

$$\pi_s(G) \doteq G_s.$$

Now, consider a convex function $g: H_2 \rightarrow \mathbf{R}$. Given any $G_0 \in H_2$, there exists a $\partial_G g(G_0) \in H_2$ such that

$$g(G) - g(G_0) \geq \langle \partial g_G(G_0), G - G_0\rangle \tag{1}$$

for all $G \in H_2$. The quantity $\partial_G g(G_0)$ is said to be a subgradient of g at the point G_0.

B. Closed-Loop Transfer Function Parametrization

Central to the results presented in this note is the parametrization of all closed-loop transfer functions. Consider the closed-loop plant in Fig. 1 with uncertain parameters $\Delta \in \mathbf{\Delta}$. The uncertainty Δ can include static uncertainty, uncertain transfer function matrices or a combination of both. The Youla parametrization (e.g., see [17]) indicates that, given $\Delta \in \mathbf{\Delta}$ and a stabilizing controller $C \in \mathcal{G}$, the closed-loop transfer function can be represented as

$$T_{CL}(z,\Delta,C) = T_\Delta^1(z) + T_\Delta^2(z) Q_{\Delta,C}(z) T_\Delta^3(z) \tag{2}$$

where $T_\Delta^1, T_\Delta^2, T_\Delta^3 \in RH_2$ are determined by the plant $G(z,\Delta)$ (and, hence, they also depend on the uncertainty Δ) and $Q_{\Delta,C} \in RH_2$ depends on both the open-loop plant $G(z,\Delta)$ and the controller $C(z)$. Also, given any $Q_{\Delta,C}(s) \in RH_2$, there exists a controller $C \in \mathcal{G}$ such that the equality above is satisfied. This parametrization also holds for all closed-loop transfer functions, stable and unstable. Using a frequency scaling reasoning, one can prove the following result: Given $\Delta \in \mathbf{\Delta}$ and a controller $C \in \mathcal{G}$, the closed-loop transfer function can be represented as (2) where $T_\Delta^1, T_\Delta^2, T_\Delta^3 \in RH_2$ are the same as above and $Q_{\Delta,C}(s) \in \mathcal{G}$. Furthermore, given any $Q_{\Delta,C}(s) \in \mathcal{G}$ there exists a controller $C \in \mathcal{G}$ such that the equality above is satisfied. Note that the mapping from Δ to $T_\Delta^1, T_\Delta^2, T_\Delta^3$ is not unique. In what follows, we assume that a unique mapping has been selected. Results to follow do not depend on how this mapping is chosen.

III. Controller Design Algorithm

Before providing the controller design algorithm, we first provide a precise definition of the problem to be solved and the assumptions that are made.

A. Problem Statement

Consider the closed-loop system in Fig. 1 and a convex objective function $g: H_2 \rightarrow \mathbf{R}$. As mentioned in Section I, given a performance value γ, we aim at designing a controller $C^*(z)$ such that the closed-loop system $T_{CL}(z,\Delta,C^*)$ is stable for all admissible values of the uncertainty and satisfies

$$g\left[T_{CL}(z,\Delta,C^*)\right] \leq \gamma$$

for all $\Delta \in \mathbf{\Delta}$. Throughout this note, we will assume that the problem above is feasible. More precisely, the following assumption is made.

Assumption 1: There exists a controller C^* and an $\varepsilon > 0$ such that, for all $\Delta \in \mathbf{\Delta}$

$$d(Q_{\Delta,C^*}, Q) < \varepsilon \Rightarrow g\left[T_\Delta^1(z) + T_\Delta^2(z) Q(z) T_\Delta^3(z)\right] \leq \gamma.$$

B. Controller Design Algorithm

We now state the proposed robust controller design algorithm. This algorithm has a free parameter η that has to be specified. This parameter can be arbitrarily chosen from the interval (0,2). Although this algorithm uses the same step as the algorithms in [6] and [15], it is different in the sense that the update at each step does not directly involve the design variables (i.e., the controller) but rather it updates the Youla parameter corresponding to the uncertainty sample.

Step 0 Let $k=0$. Pick a controller $C_0(z)$.
Step 1 Draw sample Δ^k. Given $G(z,\Delta^k)$, compute $T_{\Delta^k}^1(z)$, $T_{\Delta^k}^2(z)$, $T_{\Delta^k}^3(z)$ as described in **[17]**.
Step 2 Let $Q_k(z)$ be such that the closed-loop transfer function using controller $C_k(s)$ is

$$T_{CL}(z,\Delta^k,\ C_k) = T_{\Delta^k}^1(z) + T_{\Delta^k}^2(z) Q_k(z) T_{\Delta^k}^3(z).$$

Step 3 Do the stabilizing projection[1]

$$Q_{k,s}(z) = \pi_s(Q_k(z)).$$

Step 4 Perform update

$$Q_{k\rightarrow k+1}(z) = Q_{k,s}(z) - \alpha_k(Q_{k,s},\Delta^k)(z)\partial_Q g(T_{CL}(z,\Delta^k,Q))|_{Q_{k,s}} \tag{3}$$

where **(4)**, as shown at the bottom of the next page, holds.
Step 5 Determine the controller $C_{k+1}(z)$ so that

$$Q_{\Delta^k,C_{k+1}} = Q_{k\rightarrow k+1}.$$

Step 6 Let $k=k+1$. Go to Step 1.

[1]Note that, since C_k is not guaranteed to be a robustly stabilizing controller, Q_k might not be stable.

C. Remark

In the previous algorithm, we assume the knowledge of the quantity ε. If the value of ε is not available, one can instead use a decreasing sequence $\varepsilon_k > 0$ whose limit is zero and

$$\sum_{k=1}^{\infty} \varepsilon_k^2 = \infty.$$

The results presented in this note can be easily altered to allow for this modification. However, if the value of ε is available, one should use it since the introduction of the sequence ε_k reduces the speed of convergence.

D. Stopping Criterion

In a practical implementation of the aforementioned algorithm, a possible stopping criterion is the following: Periodically perform a Monte Carlo simulation to estimate the risk of performance violation and stop if the risk is below a given threshold.

IV. Main Result

We now present the main result of this note, i.e., the algorithm described in Section III converges to a controller that robustly satisfies the performance specifications. The exact statement is given as follows.

Theorem 1: Let $g\colon H_2 \to \mathbf{R}$ be a convex function with subgradient $\partial g \in RH_2$ and let $\gamma > 0$ be given. Define the risk of performance violation as

$$P_k \doteq \mathrm{Prob}\{g(T_{CL}(z, \Delta, C_k)) > \gamma\}.$$

Then, if Assumption 1 holds, the algorithm described in Section III-B generates a sequence of controllers C_k for which the risk of performance violation satisfies the equality

$$\sum_{k=0}^{\infty} p_k < \infty.$$

Hence, risk tends to zero as $k \to \infty$.

Before the proof of Theorem 1 is presented, we introduce the concept of robust controller gap. robust controller gap.

A. Robust Controller Gap

In order to prove the previous result, one needs a measure of how far is a controller from the optimal. Hence, the concept of *robust controller gap* is introduced. This "distance" measure uses the difference between closed-loop transfer functions as an indication of how far are the controllers. Let f be the probability density function used to generate the samples in the controller design algorithm. Then, given two controllers C_1 and C_2, the robust gap is

$$r_{\mathrm{gap}}(C_1, C_2) = \int d^2(Q_{\Delta,C_1}, Q_{\Delta,C_2}) f(\Delta) d\Delta.$$

Hence, given three controllers C_1, C_2 and C^*, we have

$$\begin{aligned} d^2(Q_{\Delta,C_1}, Q_{\Delta,C^*}) - d^2(Q_{\Delta,C_2}, Q_{\Delta,C^*}) = & \, r_{\mathrm{gap}}(C_1, C_*) \\ & - r_{\mathrm{gap}}(C_2, C_*) \\ & + V \end{aligned}$$

with $\mathbf{E}[V|C_1, C_2, C^*] = 0$ where $\mathbf{E}[X|Y]$ denotes the conditional expectation of X given Y.

We are now ready to present the proof.

B. Proof of Theorem 1

The first part of the proof is similar to the one in [6] and [15], and is, therefore, omitted. Let C^* be a controller which achieves the robust performance specification and let $T_{\Delta^k,Q} \doteq T_\Delta^1 + T_\Delta^2 Q T_\Delta^3$. Using Assumption 1 and using the same reasoning as in [6], one can prove that if $g(T_{\Delta^k,Q_{k,s}}) > \gamma$, one gets

$$d(Q_{k\to k+1}, Q_{\Delta^k,C^*})^2 \le \|Q_{k,s} - Q_{\Delta^k,C^*}\|_2^2 - \varepsilon^2\eta(2-\eta).$$

Now, define the indicator function

$$\mathbf{I}_{\{g(T_{\Delta^k,Q_{k,s}})>\gamma\}} \doteq \begin{cases} 1, & \text{if } g(T_{\Delta^k,Q_{k,s}}) > \gamma \\ 0, & \text{otherwise} \end{cases}$$

and obtain the following inequality:

$$\begin{aligned} d(Q_{k\to k+1}, Q_{\Delta^k,C^*})^2 \le & \, \|Q_{k,s} - Q_{\Delta^k,C^*}\|_2^2 \\ & - \varepsilon^2\eta(2-\eta)\mathbf{I}_{\{g(T_{\Delta^k,Q_{k,s}})>\gamma\}}. \end{aligned}$$

Now, let $Q_{k,u} = Q_k - Q_{k,s}$. Since

$$\|Q_{k,s} - Q_{\Delta^k,C^*}\|_2^2 + \left\| Q_{k,u}\left(\frac{1}{z}\right)\right\|_2^2 = d(Q_k, Q_{\Delta^k,C^*})^2$$

we have

$$\begin{aligned} d(Q_{k\to k+1}, Q_{\Delta^k,C^*})^2 \le & \, d(Q_k, Q_{\Delta^k,C^*})^2 \\ & - \varepsilon^2\eta(2-\eta)\mathbf{I}_{\{g(T_{\Delta^k,Q_{k,s}})>\gamma\}} - \left\| Q_{k,u}\left(\frac{1}{z}\right)\right\|_2^2. \end{aligned}$$

Given the definition of robust gap, provided in Section III, the previous equation can be rewritten in the following form:

$$\begin{aligned} r_{\mathrm{gap}}(C_{k+1}, C^*) \le & \, r_{\mathrm{gap}}(C_k, C^*) \\ & - \varepsilon^2\eta(2-\eta)\mathbf{I}_{\{g(T_{\Delta^k,Q_{k,s}})>\gamma\}} - \left\| Q_{k,u}\left(\frac{1}{z}\right)\right\|_2^2 + V_k \end{aligned}$$

where $\mathbf{E}[V_k|C_{k+1}, C_k, C^*] = 0$. now let $\mathcal{F}_k = \sigma(r_{\mathrm{gap}}(C_1, C^*), \ldots, r_{\mathrm{gap}}(C_k, C^*))$ be the σ-algebra generated by $r_{\mathrm{gap}}(C_1, C^*)$, $r_{\mathrm{gap}}(C_2, C^*)$, $\ldots$, $r_{\mathrm{gap}}(C_k, C^*)$, e.g., see [20] for a precise definition of a σ-algebra. Now, take expectation conditioned on $\mathcal{F}_k$.Then

$$\begin{aligned} \mathbf{E}[r_{\mathrm{gap}}(C_{k+1}, C^*)|\mathcal{F}_k] \le & \, r_{\mathrm{gap}}(C_k, C^*) - \varepsilon^2\eta(2-\eta) \\ & \times \mathrm{Prob}\{g(T_{\Delta,Q_{k,s}}) > \gamma|\mathcal{F}_k\} \\ & - \mathbf{E}\left[\left\| Q_{\Delta,C_k,u}\left(\frac{1}{z}\right)\right\|_2^2 |\mathcal{F}_k\right] \end{aligned} \tag{5}$$

where $0 < \eta < 2$. Now, note that $r_{\mathrm{gap}}(C_k, C^*) \ge 0$. Furthermore, one can easily prove that $\mathbf{E}[r_{\mathrm{gap}}(C_0, C^*)] < \infty$. Hence, the process $\{r_{\mathrm{gap}}(C_k, C^*),\ k \ge 1\}$ is a *supermartingale* and it is bounded in $\mathcal{L}^1$, e.g., see [20] for a precise definition of a supermartingale. Therefore, $r_{\mathrm{gap}}(C_k, C^*)$ converges to a finite value with probability one, e.g., see [20]. Hence, its expectation also converges to a finite value. Now, compute the expected value of both sides of (5). Let $P_k = \mathrm{Prob}\{g(T_{\Delta,Q_{k,s}}) > \gamma\}$, and, using the fact that $\mathbf{E}[\mathbf{E}[X|Y]] = \mathbf{E}[X]$, get

$$\begin{aligned} \mathbf{E}[r_{\mathrm{gap}}(C_{k+1}, C^*)] \le & \, \mathbf{E}[r_{\mathrm{gap}}(C_k, C^*)] \\ & - \varepsilon^2\eta(2-\eta)P_k - \mathbf{E}\left[\left\| Q_{\Delta,C_k,u}\left(\frac{1}{z}\right)\right\|_2^2\right]. \end{aligned}$$

$$\alpha_k(Q_k, \Delta) = \begin{cases} \eta\dfrac{g(T_{CL}(z,\Delta,Q_k)) - \gamma + \varepsilon\, \|\partial_Q g(T_{CL}(z,\Delta,Q))|_{Q_k}\|_2}{\|\partial_Q g(T_{CL}(z,\Delta,Q))|_{Q_k}\|_2^2}, & \text{if } g(T_{CL}(z,\Delta,Q_k)) > \gamma; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Hence

$$\mathbf{E}[r_{\text{gap}}(C_{k+1}, C^*)] \leq r_{\text{gap}}(C_0, C^*) - \varepsilon^2 \eta(2-\eta) \sum_{i=0}^{k} P_i - \sum_{i=0}^{k} \mathbf{E}\left[\left\| Q_{\Delta, C_i, u}\left(\frac{1}{z}\right) \right\|_2^2 \right].$$

Given the fact that $\mathbf{E}[r_{\text{gap}}(C_{k+1}, C^*)]$ converges to a finite value, we have to have

$$\sum_{i=0}^{\infty} P_i < \infty \text{ and } \sum_{i=0}^{\infty} \mathbf{E}\left[\left\| Q_{\Delta, C_i, u}\left(\frac{1}{z}\right) \right\|_2^2 \right] < \infty.$$

Q.E.D.

V. Robust H_2 Design

We now turn our attention to the case of robust weighted H_2 controller design. For simplicity of exposition, we are going to consider the single-input–single-output case. A straightforward extension can be done to the case of multiple inputs and/or outputs. Given $W \in RH_2$, consider the weighted H_2 norm defined as

$$g(G) = \left(\frac{1}{2\pi} \int_0^{2\pi} |W(e^{j\theta}) G(e^{j\theta})|^2 d\theta \right)^{1/2}.$$

Now, since we are considering the case of a single input/ single output system, given a controller C and an uncertainty value $\Delta \in \mathbf{\Delta}$, the closed-loop transfer function can be represented in the form

$$T_{CL}(z, \Delta, C) = T_\Delta^1(z) + T_\Delta^2(z) Q_{\Delta, C}(z).$$

Now, the results in [2] indicate that, in this case, the subgradient with respect to Q of the objective function is given by

$$\partial_Q g(T_{CL}(z, \Delta^k, C))(Q) = \frac{1}{2\pi \|T_{CL}(z, \Delta, Q)\|_2} \times T_{CL}(z, \Delta, C)\, T_\Delta^2(z) W(z).$$

A. Numerical Example

As a numerical example of the application of the ideas put forth in this note, consider the problem of designing a robust discrete time controller for a dc armature-controlled servomotor. More precisely, consider the closed-loop sampled data system in Fig. 2 where $G(s)$ represents the dc motor to be controlled and $C(z)$ the discrete-time controller to be designed. The transfer function of the dc motor is of the form

$$G(s, \Delta) = \frac{\omega^2}{s(s + 2\delta\omega)}$$

where the value of $\Delta = (\omega, \delta)$ is not exactly known. The only information available is that $\omega \in [5, 7]$ and $\delta \in [0.2, 0.4]$. For a given $\gamma > 0$, the objective is to design a controller $C(z)$ such that

$$\|W(z)(1 + C(z)G(z, \Delta))^{-1}\|_2 \leq \gamma$$

for all admissible values of Δ, where $W(z)$ is a given weighting function and $G(z, \Delta)$ is the discretized version of the plant $G(s, \Delta)$. Throughout this example, we assume that the sampling time is $T = 0.5$ s and that the weighting function is

$$W(z) = \frac{0.033\,33z + 0.045\,36}{z - 0.6065}.$$

To design a controller using the algorithms presented in this note, a performance level of $\gamma = 0.15$ was chosen and uniform distributions for the uncertain parameters were used. After 30 000 iterations, the following controller was obtained:

$$C_1(z) = \frac{0.3043z^3 - 0.4643z^2 + 0.2114z - 0.0102}{z^4 + 0.6533z^3 - 0.9549z^2 + 0.002\,239z + 0.413}.$$

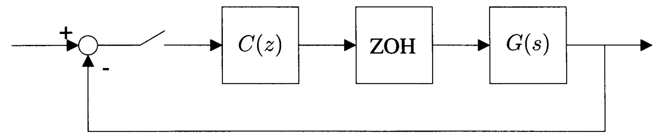

Fig. 2. Closed-loop system.

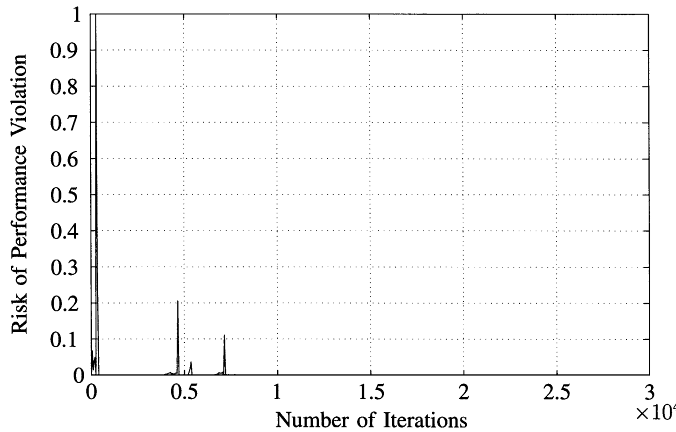

Fig. 3. Risk of performance violation.

To assess the performance of this controller, a Monte Carlo simulation with 100 000 samples was performed and the estimated risk of performance violation was found to be *zero*. We also estimated the risk of performance violation as a function of the iteration number. These estimates were obtained through Monte Carlo simulations with 3,000 samples each and the results obtained are shown in Fig. 3. As it can be seen, the risk decreases rapidly to very low levels. At the 7 000th iteration, the risk is approximately zero.

VI. Concluding Remarks

In this note, we address the problem of robust controller design for linear time invariant systems with arbitrary uncertainty structure. Given bounds on a convex performance function, the proposed algorithm converges to an output feedback controller that robustly satisfies the specifications. Moreover, it is proven that this stochastic gradient like procedure produces a sequence of controllers with a risk of performance violation that decreases to zero asymptotically faster than $1/k$, where k is the number of iterations. As an example, the problem of robust H_2 performance is considered and a numerical example is provided.

The results presented are just a first step and, hence, there are many open problems. Effort is currently being put in the elimination of the assumption of the knowledge that a robust controller exists for the given performance level. Preliminary results indicate that the algorithm presented in this note can be modified to minimize the expected performance. Of interest is also the problem of order of the controller. Since there are no restrictions on the order of the controllers designed using the algorithm presented in this note, it would be of interest to modify it so that it would take the maximum order of the controller as one of the specifications. Also, the procedure presented does not assure that, at each iteration, one has a controller that robustly stabilizes the system. In many cases, this is a "hard" constraint in the sense that the final design should lead to a robustly stable system. Hence, we believe that effort should be put in the study of this problem.

References

[1] B. R. Barmish and C. M. Lagoa, "The uniform distribution: A rigorous justification for its use in robustness analysis," *Math. Control, Signals, Syst.*, vol. 10, pp. 203–222, 1997.

[2] S. Boyd and C. Barratt, *Linear Controller Design—Limits of Performance*. Upper Saddle River, NJ: Prentice-Hall, 1991.
[3] G. C. Calafiore, F. Dabbene, and R. Tempo, "Randomized algorithms for probabilistic robustness with real and complex structured uncertainty," *IEEE Trans.Automat. Contr.*, vol. 45, pp. 2218–2235, Dec. 2000.
[4] G. C. Calafiore and F. Dabbene, "Randomization in RH_∞: An approach to control design with hard/soft performance specications," in *Proc. 41st IEEE Conf. Decision Control*, Las Vegas, NV, 2002, pp. 2260–2265.
[5] G. Calafiore, F. Dabbene, and R. Tempo, "Uniform sample generation in l_p balls for probabilistic robustness analysis," in *Proc. 37th IEEE Conf. Decision Control*, Tampa, FL, 1998, pp. 3335–3340.
[6] G. Calafiore and B. T. Polyak, "Stochastic algorithms for exact and approximate feasibility of robust LMI's," *IEEE Trans. Automat. Contr.*, vol. 46, pp. 1755–1759, Nov. 2001.
[7] X. Chen and K. Zhou, "A probabilistic approach to robust control," in *Proc. 36th IEEE Conf. Decision Control*, San Diego, CA, 1997, pp. 4894–4895.
[8] ——, "Constrained optimal synthesis and robustness analysis by randomized algorithms," in *Proc. 1998 Amer. Control Conf.*, vol. 3, Philadelphia, PA, 1998, pp. 1429–1433.
[9] L. Devroye, "Random variate generation for multivariate unimodal densities," *ACM Trans. Modeling Comput. Simul.*, vol. 7, pp. 447–477, 1997.
[10] Y. Fujisaki, F. Dabbene, and R. Tempo, "Probabilistic robust design of LPV control systems," in *Proc. 40th IEEE Conf. Decision Control*, Orlando, FL, 2001, pp. 2019–2024.
[11] C. M. Lagoa, M. Sznaier, and B. R. Barmish, "An algorithm for generating transfer functions uniformly distributed over H_∞ balls," in *Proc. 40th IEEE Conf. Decision Control*, Orlando, FL, 2001, pp. 5038–5043.
[12] C. I. Marrison and R. F. Stengel, "Stochastic robustness synthesis applied to a benchmark control problem," *Int. J. Robust Nonlinear Control*, vol. 5, no. 1, pp. 13–31, 1995.
[13] ——, "Robust control system design using random search and genetic algorithms," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 835–839, June 1997.
[14] T. Motoda, R. F. Stengel, and Y. Miyazawa, "Robust control system design using simulated annealing," *J. Guid., Control, Dyna.*, vol. 25, no. 2, pp. 267–274, Mar. 2002.
[15] B. T. Polyak and R. Tempo, "Probabilistic robust design with linear quadratic regulators," in *Proc. 39th IEEE Conf. Decision Control*, Sydney, NSW, Australia, 2000, pp. 1037–1042.
[16] L. R. Ray and R. F. Stengel, "A Monte Carlo approach to the analysis of control system robustness," *Automatica*, vol. 29, no. 1, pp. 229–236, 1993.
[17] R. S. Sánchez-Peña and M. Sznaier, *Robust Systems: Theory and Applications*. New York: Wiley, 1998.
[18] R. F. Stengel and L. E. Ryan, "Stochastic robustness of linear time-invariant control systems," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 82–87, Jan. 1991.
[19] Q. Wang and R. F. Stengel, "Robust control of nonlinear systems with parametric uncertainty," *Automatica*, vol. 38, no. 9, pp. 1591–1599, September 2002.
[20] D. Williams, *Probability With Martingales*. New York: Cambridge Univ. Press, 1991.
[21] A. Yoon, P. Khargonekar, and K. Hebbale, "Design of computer experiments for open-loop control and robustness analysis of clutch-to-clutch shifts in automatic transmissions," in *Proc. 1997 Amer. Control Conf.*, vol. 5, Albuquerque, NM, 1997, pp. 3359–3364.
[22] A. Yoon and P. Khargonekar, "Computational experiments in robust stability analysis," in *Proc. 36th IEEE Conf. Decision Control*, San Diego, CA, 1997, pp. 3260–3265.

Invariant Subspaces for LPV Systems and Their Applications

Gary Balas, József Bokor, and Zoltán Szabó

***Abstract*—The aim of this note is to extend the notion of invariant subspaces known in the geometric control theory of the linear time invariant systems to the linear parameter-varying (LPV) systems by introducing the concept of parameter-varying invariant subspaces. For LPV systems affine in their parameters, algorithms are given to compute many parameter varying subspaces relevant in the solution of state feedback and observer design problems.**

***Index Terms*—Distributions, input affine systems, invariant subspaces, linear parameter-varying (LPV) systems.**

I. Introduction

Important engineering processes involve time-varying linear and nonlinear models. A general theory for the robust control of nonlinear systems is not computationally tractable and useful progress requires an intermediate level of complexity. Linear parameter-varying (LPV) modeling techniques have gained a lot of interest as they provide a systematic means of computing gain-scheduled controllers, especially those related to vehicle and aerospace control, [2], [4], [11], [18], [23].

Many of the control system design techniques using LPV models can be cast or recast as convex feasibility problem with infinite constraints that involve linear matrix inequalities (LMIs). This problem can be addressed by using affine LPV modeling that reduces the infinite constraints imposed on the LMI formulation to a finite number, [1], [30].

The pure LPV model is not quite matched for practical problems, e.g., to the flight control problem, where the scheduling variables are in fact system states (e.g. airspeed and angle of attack), rather than bounded external variables. An approach to this problem is to generate so-called quasi-LPV models, which are applicable when the scheduling variables are measured states, the dynamics are linear in the inputs and other states, and there exist inputs to regulate the scheduling variables to arbitrary equilibrium values.

The mathematically dual concepts of (A,B)(or controlled)-invariance and (C,A) (or conditioned)-invariance play an important role in the geometric theory of linear time-invariant (LTI) systems, [6], [32]. These concepts were used to study some fundamental problems of LTI control theory, such as disturbance decoupling (DDP), unknown input observer design, fault detection (FPRG), [19], [20], [32]. The nonlinear version of this geometrical approach is much more complex and deals with certain locally controlled or conditioned invariant distributions and codistributions, [14], [15], [24], [25].

The aim of this note is to extend these notions for the parameter-varying systems by introducing the notion of *parameter-varying* (A, B)*-invariant*, *parameter-varying* (C, A)*-invariant*, controllability and unobservability subspaces, and to give some algorithms to compute these subspaces if certain conditions are fulfilled.

Manuscript received November 19, 2002; revised July 22, 2003. Recommended by Associate Editor P. A. Iglesias. This work was supported in part by NASA Langley under NASA Grant NCC-1-337 (Dr. C. M. Belcastro Technical Monitor) and in part by the Hungarian National Science Foundation (OTKA) under Grant T 030182.

G. Balas is with the Department of Aerospace Engineering and Mechanics, University of Minnesota, Minneapolis, MN 55455 USA.

J. Bokor and Z. Szabó are with the Computer and Automation Research Institute, Hungarian Academy of Sciences, H-1111 Budapest, Hungary (e-mail: bokor@sztaki.hu; szaboz@decst.scl.sztaki.hu).

Digital Object Identifier 10.1109/TAC.2003.819647