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A Linear Matrix Inequality Approach to Synthesizing Low Order l^1 Controllers [1]

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Abstract

The l^1 control theory is appealing, since it allows for directly incorporating time-domain specifications into the controller synthesis procedure and furnishes a complete solution to the robust performance problem. Moreover, in the SISO case, the synthesis procedure can be recast into a finite-dimensional Linear Programming problem and solved efficiently. The MIMO case can be solved iteratively by adding fictitious inputs and outputs to recast the problem into an one-block form. However, it is well known that, in contrast to the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ cases, optimal l^1 controllers can have arbitrarily high order, even when the states of the plant are available for feedback. In this paper, motivated by [11], we address the problem of designing low order suboptimal l^1 controllers using a Linear Matrix Inequality optimization approach. The main results show that, in the state-feedback case, the suboptimal controller is static, while in the output-feedback case it has the same order as that of the plant. In both cases the synthesis process involves solving an LMI feasibility problem and a scalar minimization over (0,1).

1 Introduction

A large number of control problems involve designing a controller capable of stabilizing a given linear time invariant system, while minimizing the worst case responses to some exogenous disturbances. This problem is relevant, for instance, to disturbance rejection, tracking and robustness to model uncertainty (see [14] and references therein). In the case where the signals involved are persistent bounded signals, it leads to the l^1 optimal control theory [1, 5, 6, 14, 15].

In the SISO and one-block (i.e. square) MIMO cases, by exploiting duality theory, the l^1 control problem can be recast into a finite-dimensional optimization problem and solved efficiently[5]. The resulting closed-loop system has a finite impulse response. However, neither its order nor the order of the controller is bounded by the order of the plant, even in the full state feedback case. Rather, the order of the closed-loop system or the controller can be arbitrarily high ([4], chapter 12).

In contrast, multiblock MIMO problems do not lead, in general, to finite dimensional linear programming problems. Rather, they are solved iteratively through methods furnishing sequences of upper and lower bounds [4]. At the present time, the most efficient method, delay augmentation (DA), is based upon the idea of augmenting the plant with delays to obtain an one-block problem, whose solution can be obtained using finite-dimensional linear programming. Clearly, the optimal cost for this modified problem provides a lower bound $\underline{\mu}$ of the optimal cost; however, the controller obtained this way is infeasible for the original problem. A feasible controller can be recovered by simply discarding the inputs and outputs associated with the delays. This controller yields a cost $\overline{\mu}$ that is an upper bound of the true cost. It can be shown that, under mild conditions, the lower bound always converges to the true cost. The convergence properties of the upper bound are harder to ascertain. It is shown in [4] that, when the optimal solution is such that the *first* n_u rows of the optimal closed-loop (where n_u is the number of controls) achieve the optimal norm, then $\overline{\mu} \to \mu^o$, the optimal cost. Under this condition, there exists a sequence of optimal closed-loop systems ϕ_N that converges strongly to the optimal solution. Hence, the convergence properties are strongly dependent on the *ordering* of the inputs and the outputs. A critical step in the optimization is to reorder inputs and outputs in such a way that the set of input-output pairs of minimum order corresponds to the first n_u inputs and outputs. Therefore, while this method has the advantages of avoiding order inflation (in some cases yielding exact solutions) and providing more insights into the structure of the optimal solutions, it may require several iterations of reordering inputs and outputs. Moreover, it shares with the SISO case the disadvantage that the controller can have arbitrarily high complexity. This poses a problem in many modern control applications

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involve high–order plant models. In these cases implementation considerations dictate that the order of the controller must be kept reasonably low, comparable to that of the plant.

In this paper we propose an alternative, suboptimal solution to l^1 problems, based upon the minimization of an upper bound of the cost, which is obtained by overbounding the closed–loop origin reachable set with a family of ellipsoids. This approach is motivated by a similar approach used in [11] in the context of continuous time $\mathcal{L}^1$ control theory. As in there, the main results of this paper show that, for the state–feedback case the optimal controller is static, and for the output–feedback case it has the same order as that of the augmented plant. Moreover, the synthesis procedure only entails solving a Linear Matrix Inequality optimization problem, followed by a line search over $(0,1)$, which can be efficiently solved with existing algorithms [3, 11]. Compared with the continuous time version of the problem [11], the discrete–time results presented here, while involving more complicated expressions, have the advantage that the scalar minimization is limited to the interval (0,1) (versus $(0,\infty)$).

The main drawback of the method is that at the present time there is no analytical expression for the gap between the upper bound minimized and the true l^1 norm. In moderately large systems this gap can be assessed by solving the optimal l^1 problem using the DA method (see also [13] for an alternative) and comparing the optimal cost with the suboptimal cost obtained using the proposed method. As we illustrate with an example taken form literature, the order of the optimal l^1 controller may be several times the order of the plant, necessitating some type of model reduction, and this model resuction may not be trivial. For instance, in our example, the optimal l^1 controller has order of 16, and model reduction to order 14 already yields a larger cost than that obtained using the suboptimal controller. Moreover, attempting to reduce the order of the controller below 12 produces unstable closed–loop systems. Thus, we believe that the proposed method offers a valuable alternative to Delay Augmentation, especially in cases where the DA method results in large optimization problems or in high order controllers.

2 Preliminaries

2.1 Notation and Definitions

Given a vector $x \in R^n$ its 1–norm is defined as $\|x\|_1 \doteq \sum_{i=0}^{n}|x_i|$ and its infinity norm as $\|x\|_\infty \doteq \max_i |x_i|$. l^1 denotes the space of absolutely summable sequences $h = \{h_i\}$ equipped with the norm $\|h\|_{l^1} \doteq \sum_{i=0}^{\infty}|h_i| < \infty$. l_∞ denotes the space of bounded sequences $h = \{h_i\}$ equipped with the norm $\|h\|_{l_\infty} \doteq \sup_{i\geq 0}|h_i| < \infty$. Similarly, l_∞^p denotes the space of bounded vector sequences $\{h(k) \in R^p\}$. In this space we define the norm $\|h\|_{l_\infty} \doteq \sup_i \|h_i(k)\|_\infty$. Alternatively, in this space we will also consider the norm: $\|h\|_{\infty,e} \doteq \sup_k \{h'(k)h(k)\}^{1/2}$, i.e. the supremum over time of the pointwise *euclidean* norm of the vector $h(k)$.

Assume now that $H : l_\infty^q \rightarrow l_\infty^p$ is a bounded linear operator defined by the usual convolution relation $y = H * u$. Its induced $l_\infty^q \rightarrow l_\infty^p$ norm will be denoted as $\|H\|_{l_\infty \rightarrow l_\infty} \doteq \|H\|_1$. It is a standard result that: $\|H\|_1 = \max_i \sum_{j=1}^{q}\|H_{ij}\|_{l^1}$. Similarly, the operator norm induced by $\|.\|_{\infty,e}$ will be denoted by $\|H\|_{1,e}$, i.e. $\|H\|_{1,e} \doteq \sup_{\|v\|_{\infty,e}\leq 1}\|H * v\|_{\infty,e}$. Note that for scalar signals these norms coincide, while in the general case we have: $\frac{1}{\sqrt{q}}\|H\|_1 \leq \|H\|_{1,e} \leq \sqrt{p}\|H\|_1$. Next we recall some results connecting the l_∞ to l_∞ induced norm with positively invariant sets.

Definition 1 *Consider the discrete-time dynamic system*

$$x(k+1) = Ax(k) + Bd(k) \tag{1}$$

where $x(k) \in R^n$ and $d(k) \in R^q$, $\|d(k)\|_\infty \leq 1$. A convex, compact set P containing the origin is said to be positively invariant for this system if for all $x \in P$ we have $Ax + Bd \in P$.

Definition 2 *Consider the system (1). Given a sequence $d = \{d(0), d(1), \dots\}$ and an initial condition x_o, denote by $\phi(k, x_o, d(\cdot))$ the corresponding trajectory. The origin–reachable set R_∞ is defined as $R_\infty \doteq \{\xi : \xi = \phi(k, 0, d)\}$ for some finite time k and some sequence $d(k)$, $\|d(k)\|_\infty \leq 1$.*

It can be easily shown that the set R_∞ is the smallest invariant set containing the origin in its interior. Moreover, consider a stable system having a state–space realization (A, B, C, D) and define the following set: $\Xi(\mu) = \{\xi: |C\xi| \leq \mu\bar{1} - \delta\}$, where $\bar{1} \doteq [\,1\ 1\ \dots\ 1\,]^T \in R^p$ and $\delta \in R^p$ is the vector whose i-th component is given by $\delta_i \doteq \|D_i\|_1$. Then $\|(A, B, C, D)\|_1 \leq \mu$ if and only if $R_\infty \subseteq \Xi(\mu)$ [2].

This result can be used to synthesize (sub)optimal l_1 controllers by selecting a performance level μ and then finding the largest invariant set S (along with the corresponding control action) contained in $\Xi(\mu)$ (see [2] for details). This approach has been successfully used to synthesize static non–linear optimal l^1 controllers [1]. However, attempting to proceed in a similar way to synthesize *linear* controllers leads to non–differentiable, non–convex optimization problems.

In order to circumvent this difficulty, following an idea introduced in [12] and used in [11] in the context of

[1] optimal controllers can be synthesized by finding the smallest μ such that the set S is non–empty.

continuous–time $\mathcal{L}^1$ control, in this paper we will bound R_∞ by a family of invariant ellipsoids. An upper bound of the $||.||_{1,e}$ norm can then be found by finding the element of this family having the tightest fit in $\Xi(\mu)$. Furthermore, suboptimal controllers can then be found by optimizing this upper bound. As we show in the sequel, both the resulting analysis and synthesis problems can be solved by combining an LMI feasibility problem with a scalar convex optimization in $(0,1)$.

2.2 Computing Bounds on the $||.||_{1,e}$ Norm

Motivated by the discussion in the last section, we consider next the problem of computing the tightest upper bound of $||.||_{1,e}$ based on invariant ellipsoids. We will first consider the case of strictly proper systems.

Lemma 1 *Consider the stable strictly proper discrete–time system:* $G = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$. *Given* $0 < \alpha < 1$, *define* $f(\alpha) \doteq \inf_{Q>0} \{\bar{\sigma}[CQC^T]\}$, *where* $Q > 0$ *solves the the following Linear Matrix Inequality:*

$$\begin{bmatrix} -(Q - \frac{1}{\alpha}BB^T) & AQ \\ QA^T & (\alpha-1)Q \end{bmatrix} \leq 0 \quad (2)$$

Then the following properties hold:

1. *The set* $\{x : x^TQ^{-1}x \leq 1\}$ *is invariant for G.*
2. $||G||_{1,e} \leq f(\alpha)^{1/2}$.
3. $f(\alpha)$ *is convex for* $\alpha \in (0,1)$.

This lemma suggests that an upper bound of $||G||_{1,e}$ can be computed by minimizing $f(\alpha)$. Following the approach in [11] we will define this upper bound as the $*$–norm of G, i.e.: $||G||_* \doteq \min_\alpha f(\alpha)^{1/2}$. Note that from Lemma 1 it follows that $||G||_*$ can be efficiently computed by combining an LMI optimization with a scalar convex optimization in $(0,1)$.

Lemma 2 *Consider the stable proper discrete–time system:* $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. *Given* $0 < \alpha < 1$, *let*

$$\mathrm{V}(\alpha) = \inf_{\sigma>0, Q>0} \eta \;\; \text{s.t.} \;\; \begin{bmatrix} \sigma Q^{-1} & 0 & C^T \\ 0 & (\eta-\sigma)I & D^T \\ C & D & I \end{bmatrix} > 0$$

where $Q > 0$ *solves the the following LMI:*

$$\begin{bmatrix} -(Q - \frac{1}{\alpha}BB^T) & AQ \\ QA^T & (\alpha-1)Q \end{bmatrix} \leq 0 \quad (3)$$

Then the following properties hold:

1. *The set* $\{x : x^TQ^{-1}x \leq 1\}$ *is invariant for G.*
2. $||G||_{1,e} \leq V(\alpha)^{1/2}$.
3. $V(\alpha)$ *is quasiconvex for* $\alpha \in (0,1)$.

Hence, the $*$–norm of a proper system G can be defined as $||G||_* \triangleq \inf_{0<\alpha<1} V(\alpha)^{1/2}$.

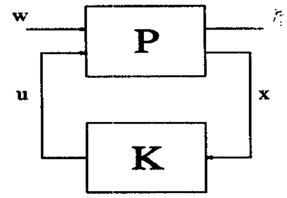

Figure 1: Block diagram of the closed–loop system

3 Controller Synthesis: The Full State Feedback Case

In this section we consider the problem of synthesizing full state feedback controllers that minimize the $*$-norm of the closed loop system. The main result of this section shows that these controllers are *static* and can be found by combining an LMI optimization with a scalar optimization in $(0,1)$.

Theorem 1 *Assume that the system shown in Figure 1 has the following state space realization:*

$$\begin{bmatrix} x(k+1) \\ \hline \xi(k) \\ x(k) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ I & 0 & 0 \end{array}\right] \begin{bmatrix} x(k) \\ \hline w(k) \\ u(k) \end{bmatrix} \quad (4)$$

Then, the following statements are equivalent:

1. *There exists a finite-dimensional, full state feedback internally stabilizing LTI controller such that* $||T_{\xi w}||_* \leq \gamma$.
2. *There exists a static control law* $u = Kx$ *such that* $||T_{\xi w}||_* \leq \gamma$.
3. *There exists a scalar* α, $0 < \alpha < 1$ *such that the following LMI's (in Q and V) are feasible:*

$$\begin{bmatrix} Q & QC_1^T + V^TD_{12}^T \\ C_1Q + D_{12}V & \gamma^2 I \end{bmatrix} > 0$$
$$\begin{bmatrix} -(Q - \frac{1}{\alpha}B_1B_1^T) & AQ + B_2V \\ QA^T + V^TB_2^T & (\alpha-1)Q \end{bmatrix} \leq 0 \quad (5)$$

An internally stabilizing static controller such that $||T_{zw}||_* \leq \gamma$ *is given by* $K = VQ^{-1}$.

Hence, the optimal $*$–norm problem with full state feedback control, i.e., the problem of minimizing $||T_{zw}||_*$ using full state feedback, reduces to the problem of minimizing γ subject to (5) followed by a line search over $\alpha \in (0,1)$. It is worth noticing that if the optimal value γ_{opt} is achieved for some α_{opt}, the closed–loop system will have a guaranteed stability measure of $\rho(A) < \sqrt{1-\alpha_{opt}}$. Moreover, numerical evidence suggests that the line search of α is indeed a convex minimization, but no formal proof of this fact is available at the present time.

Next we illustrate these results with a simple example taken from literature. Consider the third order system of Example 1 in [7]. A state-space realization of the plant is given by:

$$\left[\begin{array}{ccc|cc} 2.7 & -23.5 & 4.6 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 1 & -2.5 & \kappa & 0 & 0 \end{array}\right]$$

where $\kappa > 0$ is a parameter. As shown in [7], for $\kappa > 3.5$ the optimal l^1 state-feedback controller is static. However, for $1.5 < \kappa < 3.5$, the optimal state-feedback controller is dynamic, leading to an optimal closed-loop of the form: $\Phi = \lambda + \phi_k(2)\lambda^2 + \phi_k(N_\kappa)\lambda^{N_\kappa}$. Moreover, it can be shown that as $\kappa \downarrow 1.5$, $N_\kappa \uparrow \infty$ and $\|\Phi\|_1 \downarrow 3$. Table 1 shows a comparison of the optimal l^1 norm corresponding to different values of κ versus the l^1 norm achieved by the *static* controller synthesized using the proposed method. It is worth noticing that the gap remains constant at about 15%, even when the order of the optimal controller approaches ∞. Moreover, for $\kappa = 1.501$, Delay Augmentation leads to a Linear Programming problem having an 103×5 constraint matrix.

Table 2 shows the results obtained when attempting to reduce the order of the controller corresponding to $\kappa = 1.501$ using balanced truncation. While the controller can be easily reduced to 15th order, reduction to 14th order already yields worse performance than that obtained with the suboptimal static controller. Attempting to reduce the order below 12 yields controllers that do not stabilize the plant. Similar results were obtained when using Hankel norm [8] and balanced stochastic truncation [9] model reduction.

κ	N_{l^1}	$\|\Phi\|_1$	$\|\Phi_{static}\|_1$	gap	γ
2	2	4.21	4.65	10%	5.2
1.51	11	3.04	3.48	14%	3.91
1.501	16	3.01	3.46	15%	3.90
1.500	∞	3.00	3.46	15%	3.90

Table 1: Optimal versus static-feedback l^1 norm. N_{l^1} denotes the order of the optimal l^1 controller

N	16	15	14	12	11	subopt
$\|\Phi\|_1$	3.01	3.08	3.62	4.71	unst.	3.46

Table 2: Performance of reduced-order controllers, $\kappa = 1.501$ ($\|\Phi_{static}\|_1 = 3.46$)

4 The Output Feedback Case

In this section we consider the output feedback case. The main result of the section shows that output feedback controllers can be designed by solving an optimal (in *-norm sense) estimation problem and an optimal control problem. To establish this result, we will consider first the problem of designing optimal estimators.

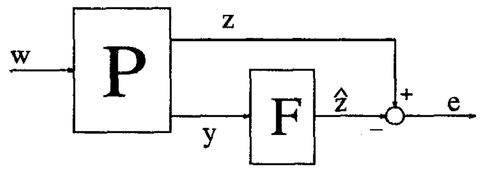

Figure 2: Filtering setup

4.1 Optimal Filtering in the *-Norm

Consider the setup shown in Figure 2. The problem of interest is to design a filter that minimizes the *-norm of T_{ew}, the transfer function from the driving noise to the estimation error. For simplicity, we will assume that we are interested in strictly proper filters and that the process and measurement noises are independent.

Theorem 2 *Consider the setup shown in Figure 2 where P has the following state-space realization:*

$$\left[\begin{array}{c} x(k+1) \\ \hline z(k) \\ y(k) \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C_1 & 0 \\ C_2 & D \end{array}\right] \left[\begin{array}{c} x(k) \\ \hline w(k) \end{array}\right] \qquad (6)$$

Assume that the following conditions hold: (1) (C_2, A) is detectable and (A, B) is stabilizable; (2) $BD^T = 0$; (3) D has linearly independent rows, i.e. $DD^T > 0$. Then the following statements are equivalent:

1. *There exists a strictly proper finite dimensional filter such that $\|T_{ew}\|_* \leq \gamma$.*
2. *There exists $\alpha \in (0,1)$ so that the Riccati equation* $\frac{1}{1-\alpha}A\left[Q_\alpha^{-1} + \frac{\alpha}{1-\alpha}C_2^T(DD^T)^{-1}C_2\right]^{-1}A^T - Q_\alpha + \frac{1}{\alpha}BB^T = 0$ *admits a stabilizing solution $Q_\alpha > 0$ and such that $\bar{\sigma}(C_1 Q_\alpha C_1^T) \leq \gamma^2$. Moreover, in this case the following filter:*

$$F_\alpha = \left[\begin{array}{c|c} A - \frac{\alpha}{1-\alpha}AQ_\alpha C_2^T R_\alpha^{-1} C_2 & \frac{\alpha}{1-\alpha}AQ_\alpha C_2^T R_\alpha^{-1} \\ \hline C_1 & 0 \end{array}\right]$$

where $R_\alpha = DD^T + \frac{\alpha}{1-\alpha}C_2 Q_\alpha C_2^T$ renders $\|T_{ew}\|_* \leq \gamma$.

4.2 Output Feedback Control Using Strictly Proper Controllers

Theorem 3 *Consider the system shown in Figure 3, and assume P has the following state space realization:*

$$\left[\begin{array}{c} x(k+1) \\ \hline z(k) \\ y(k) \end{array}\right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \left[\begin{array}{c} x(k) \\ \hline w(k) \\ u(k) \end{array}\right] \qquad (7)$$

Additionally, assume that (1) (C_2, A) is detectable and (A, B_1), (A, B_2) both are stabilizable; (2) $B_1 D_{21}^T = 0$; (3) D_{21} has linearly independent rows, i.e., $D_{21}D_{21}^T > 0$. Then the following statements are equivalent:

1. *There exists an internally stabilizing strictly proper LTI controller such that $\|T_{zw}\|_* < \gamma$.*
2. *There exists $0 < \alpha < 1$ such that the following LMIs (in Q_1, Q_2^{-1} and V) are feasible:*

$$\begin{aligned}
&\begin{bmatrix} -Q_1 + \frac{1}{\alpha}B_1B_1^T & AQ_1 + B_2V & A \\ Q_1A^T + V^TB_2^T & (\alpha-1)Q_1 & (\alpha-1)I \\ A^T & (\alpha-1)I & (\alpha-1)Q_2^{-1} \end{bmatrix} \leq 0 \\
&\begin{bmatrix} Q_1 & I & Q_1C_1^T + V^TD_{12}^T \\ I & Q_2^{-1} & C_1^T \\ C_1Q_1 + D_{12}V & C_1 & \gamma^2 I \end{bmatrix} > 0
\end{aligned} \tag{8}$$

where Q_2 solves the following Riccati equation:

$$\begin{aligned} \tfrac{-\alpha}{(1-\alpha)^2} AQ_2C_2^T\left(D_{21}D_{21}^T + \tfrac{\alpha}{1-\alpha}C_2Q_2C_2^T\right)^{-1}C_2Q_2A^T \\ + \tfrac{1}{1-\alpha}AQ_2A^T - Q_2 + \tfrac{1}{\alpha}B_1B_1^T = 0 \end{aligned} \tag{9}$$

Moreover, in this case a stabilizing controller rendering $\|T_{zw}\|_* \leq \gamma$ is given by:

$$K_\alpha = \left[\begin{array}{c|c} A + B_2F + LC_2 & -L \\ \hline F & 0 \end{array}\right] \tag{10}$$

where $F = V(Q_1 - Q_2)^{-1}$ and $L = -\frac{\alpha}{1-\alpha}AQ_2C_2^T\left(D_{21}D_{21}^T + \frac{\alpha}{1-\alpha}C_2Q_2C_2^T\right)^{-1}$.

Remark 1 *It is interesting to analyze the structure of the conditions in Theorem 3. These conditions establish the existence of a separation-like principle, where the existence of a controller depends on the existence of a solution to a filtering Riccati equation, a condition requiring the existence of a state feedback controller capable of rendering $\|T_{zw}\|_* \leq \gamma$ and a condition coupling the solution of the control and filtering problems.*

These results can be used to synthesize a controller that minimizes $\|T_{zw}\|_*$ by minimizing γ (as a function of α) subject to the feasibility of (8). As in the state feedback case, this entails combining a LMI feasibility problem with a scalar minimization in $(0,1)$. Moreover, consistent numerical evidence suggests that the function of α to be minimized is convex, although no formal proof of this fact is available.

4.3 Output Feedback Example

Consider again the simple example used in section 3, and assume that the only measurement available to the controller is the first state, corrupted by noise, i.e. the plant is given by:

$$\mathrm{P} = \left[\begin{array}{ccc|ccc} 2.7 & -23.5 & 4.6 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & -2.5 & \kappa & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0.1 & 0 \end{array}\right]$$

Table 3 shows a comparison of $\|\Phi\|_{l_1}$, the closed-loop l^1 norm achieved by the optimal l^1 controller versus $\|\Phi_{out}\|_{l_1}$, the norm achieved by the 3^{rd} order output feedback optimal *-norm controller for different values of κ. Note that as $\kappa \downarrow 1.5$, the order of the optimal l^1 controller $N_\kappa \to \infty$, while the maximum gap between the optimal and suboptimal norms remains below 15%. These results are consistent with the results obtained in section 3 for the full-state feedback controller.

5 The General Case

As in the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control theory, the assumption $BD^T = 0$ can be relaxed in the more general case of optimal *-norm control design. For completeness, we state the following theorem, which gives results for the general case of output feedback control in the *-norm, where the controllers are not limited to strictly proper.

Theorem 4 *Consider the setup of Figure 3, where P is given by*

$$\left[\begin{array}{c} x(k+1) \\ \hline z(k) \\ y(k) \end{array}\right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \left[\begin{array}{c} x(k) \\ \hline w(k) \\ u(k) \end{array}\right] \tag{11}$$

1. *(C_2, A) is detectable and (A, B_1), (A, B_2) both are stabilizable;*
2. *D_{21}has linearly independent rows, i.e. $D_{21}D_{21}^T > 0$ is invertible.*

Then, the following statements are equivalent:

1. *There exists a proper, finite-dimensional, LTI controller K that internally stabilizes the system and renders $\|T_{ew}\|_* < \gamma$.*
2. *There exists $\alpha \in (0,1)$ such that the following LMI (in the variables $Q_1 = Q_1^T$, σ, D_k, and V) admits a solution:*

$$\begin{bmatrix} -Q_1 & B_1{+}B_2D_kD_{21} & AQ_1{+}B_2V & A{+}B_2D_kC_2 \\ (B_1{+}B_2D_kD_{21})^T & -\alpha\sigma I & 0 & 0 \\ (AQ_1 + B_2V)^T & 0 & (\alpha-1)Q_1 & (\alpha-1)I \\ (A{+}B_2D_kC_2)^T & 0 & (\alpha-1)I & (\alpha-1)\sigma Q_2^{-1} \end{bmatrix} \leq 0$$

$$\begin{bmatrix} Q_1 & I & 0 & (C_1Q_1{+}D_{12}V)^T \\ I & \sigma Q_2^{-1} & 0 & (C_1{+}D_{12}D_kC_2)^T \\ 0 & 0 & (\gamma^2-\sigma)I & (D_{11}{+}D_{21}D_kD_{12})^T \\ C_1Q_1{+}D_{12}V & C_1{+}D_{12}D_kC_2 & D_{11}{+}D_{12}D_kD_{21} & I \end{bmatrix} > 0$$

where Q_2 solves the following Riccati equation:

$$\frac{1}{1-\alpha}\mathrm{E}\left[\mathrm{Q}_2^{-1} + \frac{\alpha}{1-\alpha}\mathrm{Q}_2\mathrm{C}_2^{\mathrm{T}}\left(\mathrm{D}_{21}\mathrm{D}_{21}^{\mathrm{T}}\right)^{-1}\mathrm{C}_2\right]^{-1}\mathrm{E}^{\mathrm{T}} - \mathrm{Q}_2 + \frac{1}{\alpha}\mathrm{F}_1\mathrm{F}_1^{\mathrm{T}} = 0$$

where $E = A - B_1D_{21}^T\left(D_{21}D_{21}^T\right)^{-1}C_2$ and $F_1 = B_1 - B_1D_{21}^T\left(D_{21}D_{21}^T\right)^{-1}D_{21}$.

Moreover, the controller that achieves $\|T_{ew}\|_* < \gamma$ is given by $K_\alpha = K\left(I + D_{22}K\right)^{-1}$ where

$$\mathrm{K} = \left[\begin{array}{c|c} A + B_2F + LC_2 - B_2D_kC_2 & -L + B_2D_k \\ \hline F - D_kC_2 & D_k \end{array}\right] \tag{12}$$

where $F = (\sigma V - D_kC_2Q_2)(\sigma Q_1 - Q_2)^{-1}$, $L = -\left(\frac{\alpha}{1-\alpha}AQ_2C_2^T + B_1D_{21}^T\right)\left(D_{21}D_{21}^T + \frac{\alpha}{1-\alpha}C_2Q_2C_2^T\right)^{-1}$.

κ	N_κ	$\|\Phi\|_{l_1}$	γ^o	$\|\Phi_{out}\|_{l_1}$	gap
2	23	14.27	15.02	15.27	7%
1.51	25	10.98	11.29	12.18	11%
1.501	40	10:59	11.23	12.08	14%
1.500	∞	10.45	11.22	12.07	15%

Table 3: optimal versus output feedback l_1 norm

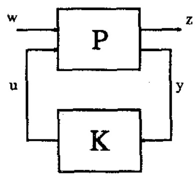

Figure 3: Block diagram for output feedback

6 Conclusions

In this paper we propose an alternative, suboptimal solution to l^1 problems, based upon the minimization of an upper bound of the cost, obtained by overbounding the closed–loop origin reachable set by a family of ellipsoids. The main result of the paper shows that, for the state–feedback case the optimal controller is static, and for the output–feedback case it has the same order as that of the augmented plant. The synthesis procedure only entails solving a Linear Matrix Inequality optimization problem, and a scalar minimization in $(0, 1)$. Both problems can be efficiently solved with existing algorithms [3, 11]. Moreover, for the state feedback case the objective function of the scalar minimization is convex, further simplifying the problem. While consistent numerical experience suggests that this also holds in the general output feedback case, no formal proof of the fact is available at the present time.

The main drawback of the method is the fact that at the present time there is no analytical expression for the gap between the upper bound minimized and the true l^1 norm. In moderately large systems this gap can be assessed by solving the optimal l^1 problem using the DA method (see also [13] for an alternative) and comparing the optimal cost with the suboptimal cost obtained using the proposed synthesis method. As we illustrate with an example taken form the literature, the optimal l^1 controller may be high order (several times the order of the plant) necessitating some type of model reduction, and this step may not be trivial. Thus, we believe that the proposed method offers a valuable alternative to Delay Augmentation, specially in cases where it results in large optimization problems or in high order controllers. Moreover, in cases where the number of inputs or outputs is not small, DA will result in larger LP problems, and it may require a large number of trial and error type iterations (reordering inputs and outputs) before converging.

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