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Brief Paper

A linear matrix inequality approach to synthesizing low-order suboptimal mixed $\ell_1/\mathscr{H}_p$ controllers☆

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Abstract

Mixed objective control problems have attracted much attention lately since they allow for capturing different performance specifications. However, optimal multiobjective controllers may exhibit some undesirable properties such as arbitrarily high order. This paper addresses the problem of designing stabilizing controllers that minimize an upper bound of the ℓ_1 norm of a certain closed-loop transfer function, while maintaining the $\mathscr{H}_2$ norm (mixed $\ell_1/\mathscr{H}_2$), or the $\mathscr{H}_\infty$ norm (mixed $\ell_1/\mathscr{H}_\infty$), of a different transfer function below a prespecified level. The main results show that these suboptimal controllers have the same order as the generalized plant and can be synthesized by a two-stage process, involving an LMI optimization problem and a line search over (0, 1).

Keywords: Multiobjective control; ★ norm; ℓ_1; $\mathscr{H}_\infty$; Linear matrix inequalities

1. Introduction

During the past decade a powerful robust control framework has been developed addressing issues of stability and performance in the presence of norm-bounded uncertainties. Robust stability and performance are achieved by minimizing a suitably weighted norm (either $\|\cdot\|_\infty$ or $\|\cdot\|_1$) of a closed-loop transfer function. While this framework has gained wide acceptance among control engineers, it is limited by the fact that in its context, performance is measured in the same norm used to assess stability. However, often a single norm cannot capture several, perhaps conflicting, specifications. Motivated by this shortcoming, multiple objective control problems have attracted much attention lately (Bernstein & Haddad, 1989; Kaminer, Khargonekar & Rotea, 1993; Sznaier & Bu, 1998; Salapaka, Dahleh & Voulgaris, 1995).

This paper addresses the problem of designing stabilizing controllers that minimize the ℓ_1 norm of a certain closed-loop transfer function, while maintaining the $\mathscr{H}_2$ norm (mixed $\ell_1/\mathscr{H}_2$), or the $\mathscr{H}_\infty$ norm (mixed $\ell_1/\mathscr{H}_\infty$), of a different transfer function below a prespecified level. This problem arises in the context of rejecting both bounded persistent and stochastic (or bounded energy) disturbances.

Both discrete time mixed $\ell_1/\mathscr{H}_2$ and $\ell_1/\mathscr{H}_\infty$ problems can be solved by using the Youla parametrization to cast the problem into a (infinite-dimensional) constrained convex optimization (Salapaka et al., 1995; Sznaier & Bu, 1998). However, as in the pure ℓ_1 optimal control, it has been shown in Salapaka et al. (1995) and Sznaier & Bu (1998) that the order of the controller is not bounded by the order of the plant, and could be arbitrarily high. Motivated by the complexity of these controllers, an alternative approach will be introduced in this paper, based upon recent results on synthesizing suboptimal low-order ℓ_1 controllers (Bu, Sznaier & Holmes, 1996; Nagpal, Abedor & Poolla, 1996). By using upper bounds of the ℓ_1 and $\mathscr{H}_2$ ($\mathscr{H}_\infty$) norms given in terms of Linear Matrices Inequalities, a modified problem can be obtained such that its solution is feasible and upper-bounds the original problem. The main result of this paper shows that suboptimal controllers can be synthesized by a two-stage process, involving an LMI optimization problem

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and a line search over (0, 1). Additional results include the facts that in the state-feedback case optimal performance can be achieved using static controllers, while in the output feedback case it can be achieved using controllers with McMillan degree less than or equal that of the plant. Moreover, as in the discrete time $\mathscr{H}_\infty$ case (Gahinet & Apkarian, 1994; Iwasaki & Skelton, 1994), our approach also provides an LMI-based parameterization of all output feedback suboptimal controllers, including reduced order ones for a class of systems.

2. Preliminaries

2.1. Notation and background results

ℓ^p_∞ denotes the space of bounded vector sequences $\{h(k) \in R^p\}$, equipped with the norm $\|h\|_{\ell_\infty} = \sup_i\{\sup_k |h_i(k)|\}$. Alternatively, in this space we will also consider the norm $\|h\|_{\infty,\mathrm{e}} \doteq \sup_k \{h'(k)h(k)\}^{1/2}$, i.e. the supremum over time of the pointwise *euclidean* norm of the vector $h(k)$. The operator norm induced by $\|.\|_{\infty,\mathrm{e}}$ will be denoted by $\|H\|_{1,\mathrm{e}}$, i.e. $\|H\|_{1,\mathrm{e}} \doteq \sup_{\|v\|_{\infty,\mathrm{e}} \le 1} \|H * v\|_{\infty,\mathrm{e}}$. Note that for scalar signals this norm coincides with the usual ℓ^1 norm, while in the general case we have: $(\sqrt{q})^{-1}\|H\|_1 \le \|H\|_{1,\mathrm{e}} \le \sqrt{p}\|H\|_1$

Lemma 1 (Bu et al., 1996). *Consider the proper stable FDLTI system*

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then

$$\|G\|^2_{1,\mathrm{e}} \le \|G\|^2_\star \triangleq \inf_{\alpha \in (0,\alpha_{\max})} V(\alpha)$$

where $\alpha_{\max} = 1 - \rho(A)$ *and*

$$V(\alpha) = \inf_{\sigma > 0,\ Q > 0} \gamma^2,$$

subject to

$$\begin{pmatrix} \alpha\sigma Q^{-1} & 0 & C^{\mathrm{T}} \\ 0 & (\gamma^2 - \sigma)I & D^{\mathrm{T}} \\ C & D & I \end{pmatrix} > 0, \tag{1}$$

$$\frac{1}{1-\alpha} AQA^{\mathrm{T}} - Q + BB^{\mathrm{T}} \le 0. \tag{2}$$

Moreover $V(\alpha)$ *is a quasiconvex function for* $\alpha \in (0,\alpha_{\max})$.

3. Mixed $\star/\mathscr{H}_2$ and $\star/\mathscr{H}_\infty$ performance measures

3.1. Mixed $\star/\mathscr{H}_2$ problem formulation

Consider the FDLTI system P shown in Fig. 1(a), where w represents an exogenous disturbance, z_1 and z_2 represent performance outputs, and P has the following realization:

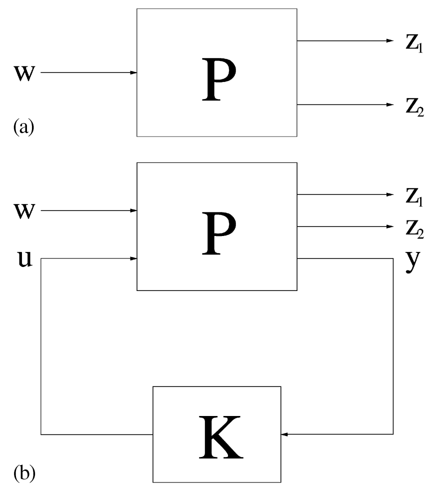

Fig. 1. (a) The generalized plant for analysis, (b) setup for synthesis.

$$P = \left[\begin{array}{c|c} A & B \\ \hline C_1 & D_1 \\ C_2 & D_2 \end{array}\right] \tag{3}$$

with A stable. The following lemma provides an upper bound of the $\mathscr{H}_2$ norm of $T_{z_2 w}$.

Lemma 2. *Given* $\alpha \in (0,\alpha_{\max})$, *let* X *and* Q_α *denote positive semi-definite solutions of*

$$AXA^{\mathrm{T}} - X + BB^{\mathrm{T}} = 0 \tag{4}$$

and (2), *respectively. Then* $Q_\alpha \ge X \ge 0$ *and* $\|T_{z_2 w}\|_2^2 \le \mathrm{Trace}(C_2 Q_\alpha C_2^{\mathrm{T}} + D_2 D_2^{\mathrm{T}})$.

Proof. It is well known (see for instance Sánchez Peña & Sznaier (1998, p. 475)) *that* $\|T_{z_2 w}\|_2^2 = \mathrm{Trace}(C_2 X C_2^{\mathrm{T}} + D_2 D_2^{\mathrm{T}})$. Let $\Delta \doteq Q_\alpha - X$. Subtracting (4) from (2) yields

$$A\Delta A^{\mathrm{T}} - \Delta + \alpha(Q_\alpha - BB^{\mathrm{T}}) \le 0.$$

Since, from (2) we have that $Q_\alpha - BB^{\mathrm{T}} \ge 0$ and since A is stable, it follows from the properties of Lyapunov equations that $\Delta \ge 0$. The rest of proof then follows immediately. □

Motivated by Lemma 2, we define the mixed $\star/\mathscr{H}_2$ performance measure as

$$J_{\star,2} \triangleq \inf_{\alpha \in (0,\alpha_{\max}),\ \sigma > 0,\ Q > 0} \gamma^2 \tag{5}$$

subject to

$$\begin{pmatrix} \alpha\sigma Q^{-1} & 0 & C_1^{\mathrm{T}} \\ 0 & (\gamma^2-\sigma)I & D_1^{\mathrm{T}} \\ C_1 & D_1 & I \end{pmatrix} > 0,$$

$$\frac{1}{1-\alpha}AQA^{\mathrm{T}} - Q + BB^{\mathrm{T}} \le 0, \tag{6}$$

$$\mathrm{Trace}(C_2QC_2^{\mathrm{T}} + D_2D_2^{\mathrm{T}}) \le 1.$$

This leads to the following mixed $\star/\mathscr{H}_2$ control problem:

Problem 1 ($\star/\mathscr{H}_2$). *Given the system shown in Fig.* 1(b), *where the plant P has the following state-space realization*:

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & 0 & D_{22} \\ C_3 & D_{31} & 0 \end{array}\right] \tag{7}$$

and where the pairs (A, B_2) *and* (C_3, A) *are stabilizable and detectable, respectively, find an internally stabilizing controller K such that* $J_{\star,2}$ *is minimized.*

For simplicity, we have assumed here that the open loop plant P is strictly proper, and without loss of generality, we have set the $\mathscr{H}_2$ constraint level at 1. The general case of a proper plant is addressed in Bu (1997). More detailed discussions on removing these assumptions by *loop-shifting* can be found in Bu (1997), Sánchez Peña and Sznaier (1998) and references therein.

3.2. Mixed $\star/\mathscr{H}_\infty$ performance measure

Lemma 3. *Consider system* (3). *For a given* $\alpha \in (0, \alpha_{\max})$, *let* $Y = Y^{\mathrm{T}} > 0$ *be any positive-definite solution to the following inequality*:

$$\begin{pmatrix} -Y & A & B & 0 \\ A^{\mathrm{T}} & (\alpha-1)Y^{-1} & 0 & C_2^{\mathrm{T}} \\ B^{\mathrm{T}} & 0 & -I & D_2^{\mathrm{T}} \\ 0 & C_2 & D_2 & -I \end{pmatrix} < 0. \tag{8}$$

Then Y also satisfies inequality (2) *and* $\|T_{z_2w}(\hat{z})\|_\infty < 1$, *where* $\hat{z} = (\sqrt{1-\alpha})z$.

Proof. Applying Schur complements to the (1,1) block of (8) yields

$$\frac{1}{1-\alpha}AYA^{\mathrm{T}} - Y + BB^{\mathrm{T}} \le 0$$

which is precisely (2). Moreover, multiplying (8) on the left and right by

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-\alpha}}I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

and using the Bounded Real Lemma (Sánchez Peña and Sznaier, 1998, p. 181) shows that

$$\left\| D_2 + \frac{1}{\sqrt{1-\alpha}}C_2\left(zI - \frac{1}{\sqrt{1-\alpha}}A\right)^{-1}B \right\|_\infty < 1.$$

Finally, the transformation $\hat{z} = (\sqrt{1-\alpha})z$ yields $\|T_{z_2w}(\hat{z})\|_\infty < 1$. □

Remark 1. Note that the transformation $z \to (\sqrt{1-\alpha})z$ maps the unit disk into a disk with radius $\delta \doteq \sqrt{1-\alpha}$. Therefore, from the Maximum Modulus Theorem it follows that $\|T(z)\|_\infty \le \|T(\hat{z})\|_\infty$. Moreover $\|T(z)\|_\infty \uparrow \|T(\hat{z})\|_\infty$ as $\alpha \downarrow 0$. This transformation is similar to the transformation used in Sznaier and Bu (1998) to decouple the mixed $\ell_1/\mathscr{H}_\infty$ problem into a convex finite-dimensional optimization and an unconstrained $\mathscr{H}_\infty$ problem.

As in the mixed $\star/\mathscr{H}_2$ case, based upon Lemma 3 we define the following mixed $\star/\mathscr{H}_\infty$ performance measure:

$$J_{\star,\infty} \triangleq \inf_{\alpha\in(0,\alpha_{\max}),\ \sigma>0,\ X>0} \{\gamma^2\} \tag{9}$$

subject to

$$\begin{pmatrix} \alpha\sigma X & 0 & C_1^{\mathrm{T}} \\ 0 & (\gamma^2-\sigma)I & D_1^{\mathrm{T}} \\ C_1 & D_1 & I \end{pmatrix} > 0,$$

$$\begin{pmatrix} -X^{-1} & A & B & 0 \\ A^{\mathrm{T}} & (\alpha-1)X & 0 & C_2^{\mathrm{T}} \\ B^{\mathrm{T}} & 0 & -I & D_2^{\mathrm{T}} \\ 0 & C_2 & D_2 & -I \end{pmatrix} < 0. \tag{10}$$

The mixed $\star/\mathscr{H}_\infty$ control problem is then formulated as:

Problem 2 ($\star/\mathscr{H}_\infty$). *Given the system shown in Fig.* 1(b), *where P is given by* (7), *find an internally stabilizing controller K such that* $J_{\star,\infty}$ *is minimized.*

3.3. State feedback controllers synthesis

In this section, we analyze the structure of the optimal solutions to Problems 1 and 2. The main result shows that for the state feedback case the optimal cost over the set of stabilizing controllers is achieved by static feedback controllers.

Theorem 4. *Consider the system P given in* (7) *and assume* $C_3 = I$ *and* $D_{31} = 0$ (*i.e. state feedback*). *The following statements are equivalent*:

(1) *There exists an FDLTI controller such that* $J_{\star,2} < \gamma^2$.

(2) *There exists a static control law* $u = Kx$ *such that* $J_{\star,2} < \gamma^2$.

(3) *The following LMIs* (*in the variables* Q, V *and* S) *admit a solution*:

$$\begin{pmatrix} \gamma^2 I & C_1 Q + D_{12} V \\ QC_1^{\mathrm{T}} + V^{\mathrm{T}} D_{12}^{\mathrm{T}} & \alpha Q \end{pmatrix} > 0,$$

$$\begin{pmatrix} S & C_2 Q + D_{22} V \\ QC_2^{\mathrm{T}} + V^{\mathrm{T}} D_{22}^{\mathrm{T}} & Q \end{pmatrix} > 0,$$

$$\begin{pmatrix} -Q + B_1 B_1^{\mathrm{T}} & AQ + B_2 V \\ QA^{\mathrm{T}} + V^{\mathrm{T}} B_2 & (\alpha - 1)Q \end{pmatrix} \leq 0, \quad \mathrm{Trace}(S) < 1.$$

Moreover, a suitable static controller is given by $K = VQ^{-1}$.

Proof. $(1 \Rightarrow 3)$ Suppose that there exists an FDLTI controller

$$K = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right]$$

that renders $J_{\star,2} < \gamma^2$. Then

$$\begin{pmatrix} T_{z_1 w} \\ T_{z_2 w} \end{pmatrix} = \left[\begin{array}{cc|c} A + B_2 D_k & B_2 C_k & B_1 \\ B_k & A_k & 0 \\ \hline C_1 + D_{12} D_k & D_{12} C_k & 0 \\ C_2 + D_{22} D_k & D_{22} C_k & 0 \end{array}\right] \triangleq \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C}_1 & 0 \\ \bar{C}_2 & 0 \end{array}\right]$$

From the definition of $J_{\star,2}$ we have that there exists a symmetric matrix $Q > 0$ such that

$$\frac{1}{1-\alpha} \bar{A} Q \bar{A}^{\mathrm{T}} - Q + \bar{B}\bar{B}^{\mathrm{T}} \leq 0, \quad \frac{1}{\alpha} \bar{C}_1 Q \bar{C}_1^{\mathrm{T}} < \gamma^2 I$$

and

$$\mathrm{Trace}(\bar{C}_2 Q \bar{C}_2^{\mathrm{T}}) < 1.$$

By Schur complements, the first two conditions are equivalent to

$$\begin{pmatrix} -Q + \bar{B}\bar{B}^{\mathrm{T}} & \bar{A}Q \\ Q\bar{A}^{\mathrm{T}} & (\alpha - 1)Q \end{pmatrix} \leq 0, \tag{11}$$

$$\begin{pmatrix} \gamma^2 I & \bar{C}_1 Q \\ Q\bar{C}_1^{\mathrm{T}} & \alpha Q \end{pmatrix} > 0. \tag{12}$$

Partition Q as

$$\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_{22} \end{pmatrix}.$$

Since $Q > 0$, it follows that $Q_{11} > 0$. Multiplying (11) on the left by

$$\begin{pmatrix} (I\ 0) & 0 \\ 0 & (I\ 0) \end{pmatrix}$$

and on the right by its transpose yields

$$\begin{pmatrix} -Q_{11} + B_1 B_1^{\mathrm{T}} & AQ_{11} + B_2 V \\ Q_{11} A^{\mathrm{T}} + V^{\mathrm{T}} B_2^{\mathrm{T}} & (\alpha - 1)Q_{11} \end{pmatrix} \leq 0$$

where $V = D_k Q_{11} + C_k Q_{12}^{\mathrm{T}}$. Multiplying (12) on the left by

$$\begin{pmatrix} I & 0 \\ 0 & (I\ 0) \end{pmatrix}$$

and on the right by its transpose yields

$$\begin{pmatrix} \gamma^2 I & C_1 Q_{11} + D_{12} V \\ Q_{11} C_1^{\mathrm{T}} + V^{\mathrm{T}} D_{12}^{\mathrm{T}} & \alpha Q_{11} \end{pmatrix} > 0.$$

Finally, from $\mathrm{Trace}(\bar{C}_2 Q \bar{C}_2^{\mathrm{T}}) < 1$ it follows that there exists a symmetric matrix S such that

$$\mathrm{Trace}(S) < 1,$$

$$\begin{pmatrix} S & C_2 Q_{11} + D_{22} V \\ Q_{11} C_2^{\mathrm{T}} + V^{\mathrm{T}} D_{22}^{\mathrm{T}} & Q_{11} \end{pmatrix} > 0.$$

$(3 \Rightarrow 2$ and $3 \Rightarrow 1)$ With the controller given by $u = VQ^{-1}x$, the closed-loop system becomes

$$\begin{pmatrix} T_{z_1 w} \\ T_{z_2 w} \end{pmatrix} = \left[\begin{array}{c|c} A + B_2 V Q^{-1} & B_1 \\ \hline C_1 + D_{12} V Q^{-1} & 0 \\ C_2 + D_{22} V Q^{-1} & 0 \end{array}\right] = \left[\begin{array}{c|c} A_f & B_1 \\ \hline C_{1f} & 0 \\ C_{2f} & 0 \end{array}\right].$$

It can be easily verified that

$$\frac{1}{\alpha} C_{1f} Q C_{1f}^{\mathrm{T}} - \gamma^2 I < 0, \quad \mathrm{Trace}(C_{2f} Q C_{2f}^{\mathrm{T}}) < 1,$$

and

$$\frac{1}{1-\alpha} A_f Q A_f^{\mathrm{T}} - Q + B_1 B_1^{\mathrm{T}} \leq 0.$$

It follows that this static controller renders $J_{\star,2} < \gamma^2$. □

As in Bu et al. (1996), the optimal solution to Problem 1 can be obtained by minimizing $V_2(\alpha)$ over $\alpha \in (0,1)$, where $V_2(\alpha)$ is defined as $V_2(\alpha) = \{\min \gamma^2\colon$ LMIs in Theorem 4 are feasible$\}$.

For completeness we quote below similar results for the mixed $\star/\mathscr{H}_\infty$ problem. The proof is omitted since it follows along the same lines as the proof of Theorem 4.

Theorem 5. *Consider the plant P given in* (7) *in the state feedback case. The following statements are equivalent*:

(1) *There exists an FDLTI controller such that* $J_{\star,\infty} < \gamma^2$.

(2) *There exists a static control law* $u = Kx$ *such that* $J_{\star,\infty} < \gamma^2$.

(3) *The following LMIs* (*in the variables Q and V*) *admit a solution*:

$$\begin{pmatrix} \gamma^2 I & C_1 Q + D_{12} V \\ QC_1^{\mathrm{T}} + V^{\mathrm{T}} D_{12}^{\mathrm{T}} & \alpha Q \end{pmatrix} > 0,$$

$$\begin{pmatrix} -Q & AQ + B_2 V & B_1 & 0 \\ QA^{\mathrm{T}} + V^{\mathrm{T}} B_2^{\mathrm{T}} & (\alpha - 1)Q & 0 & QC_2^{\mathrm{T}} + V^{\mathrm{T}} D_{22}^{\mathrm{T}} \\ B_1^{\mathrm{T}} & 0 & -I & 0 \\ 0 & C_2 Q + D_{22} V & 0 & -I \end{pmatrix} < 0.$$

A suitable static controller is given by $K = VQ^{-1}$.

4. All output feedback controllers for a class of mixed $\star/\mathscr{H}_p$ problems

In this section we establish necessary and sufficient conditions for the existence of γ-suboptimal output feedback controllers for a class of mixed $\star/\mathscr{H}_\infty$ and $\star/\mathscr{H}_2$ control problems.

Theorem 6. *Consider a discrete time FDLTI plant* P^1 *of McMillan degree n with a minimal realization*:

$$\begin{pmatrix} z_1 \\ z_2 \\ y \end{pmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & 0 \\ C_2 & D_{21} & D_{22} \\ C_3 & D_{31} & 0 \end{array}\right] \begin{pmatrix} w \\ u \end{pmatrix} \tag{13}$$

Assume that (A, B_2, C_3) *is stabilizable and detectable. The suboptimal mixed* $\star/\mathscr{H}_\infty$ *problem with parameter* γ, *i.e.* $J_{\star,\infty} < \gamma^2$, *is solvable if and only if there exist pairs of symmetric matrices* (R, S) *in* $\Re^{n\times n}$ *such that the following inequalities are feasible*:

$$N_P^{\mathrm{T}} \begin{pmatrix} \frac{1}{1-\alpha} ARA^{\mathrm{T}} - R + B_1 B_1^{\mathrm{T}} & \frac{1}{1-\alpha} ARC_2^{\mathrm{T}} + B_1 D_{21}^{\mathrm{T}} \\ \frac{1}{1-\alpha} C_2 RA^{\mathrm{T}} + D_{21} B_1^{\mathrm{T}} & -I + \frac{1}{1-\alpha} C_2 RC_2^{\mathrm{T}} + D_{21} D_{21}^{\mathrm{T}} \end{pmatrix} N_P < 0, \tag{14}$$

$$N_Q^{\mathrm{T}} \begin{pmatrix} A^{\mathrm{T}} SA - (1-\alpha)S + C_2^{\mathrm{T}} C_2 & A^{\mathrm{T}} SB_1 + C_2^{\mathrm{T}} D_{21} \\ B_1^{\mathrm{T}} SA + D_{21}^{\mathrm{T}} C_2 & -I + B_1^{\mathrm{T}} SB_1 + D_{21}^{\mathrm{T}} D_{21} \end{pmatrix} N_Q < 0, \tag{15}$$

$$\alpha\gamma^2 I - C_1 RC_1^{\mathrm{T}} > 0; \quad \begin{pmatrix} R & I \\ I & S \end{pmatrix} \geq 0 \tag{16}$$

where N_P *and* N_Q *are any matrices that span the null spaces of* $(B_2^{\mathrm{T}} \;\; D_{22}^{\mathrm{T}})$ *and* $(C_3 \;\; D_{31})$ *respectively. Moreover, the set of* γ-*suboptimal controllers of order k is nonempty if and only if* (14)–(16) *hold for some* R, S *which further satisfy the rank constraint rank*$(I - RS) \leq k$.

Proof. The proof follows along the following lines:

(1) Given any controller $K(z) = D_k + C_k(zI - A_k)^{-1}B_k$, $A_k \in \Re^{k\times k}$ write conditions (10) in terms of the closed-loop matrices, leading to an LMI of the form:

$$\Phi + P^{\mathrm{T}}\Theta Q + Q^{\mathrm{T}}\Theta^{\mathrm{T}} P < 0 \tag{17}$$

where

$$\Theta = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$$

contains all the controller parameters, the matrices P and Q depend only on the open-loop plant and where

$$\Phi = \begin{pmatrix} -X_c^{-1} & A_0 & B_0 & 0 \\ A_0^{\mathrm{T}} & (\alpha-1)X_c & 0 & C_{02}^{\mathrm{T}} \\ B_0^{\mathrm{T}} & 0 & -I_{m_1} & D_{21}^{\mathrm{T}} \\ 0 & C_{02} & D_{21} & -I_{n_2} \end{pmatrix}$$

$$A_0 = \begin{pmatrix} A & 0 \\ 0 & 0_k \end{pmatrix}; \quad B_0 = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

$$C_{02} = (C_2 \quad 0); \quad D_{02} = (0 \quad D_{22})$$

(2) partition X_c and X_c^{-1} as

$$X_c = \begin{pmatrix} S & N \\ N^{\mathrm{T}} & * \end{pmatrix}; \qquad X_c^{-1} = \begin{pmatrix} R & M \\ M^{\mathrm{T}} & * \end{pmatrix}$$

where $R, S \in \Re^{n\times n}$ and $M, N \in \Re^{n\times k}$.

(3) Proceed as in Sánchez Peña and Sznaier (1998, Section 6.4.1) to eliminate the parameters of the controller, leading to two LMIs in R and S.

A complete proof can be found in Bu (1997). A discussion on how to reconstruct the controller from the solution to the LMIs (14)–(16) can be found for instance in Gahinet and Apkarian (1994). □

[1] Formulae for the general case where $D_{12} \neq 0$ can be found in Bu (1997).

Therefore, as in the state feedback case, the optimal solution to the mixed $\bigstar/\mathscr{H}_\infty$ problem can be obtained by minimizing over $\alpha \in (0,1)$ the function

$\Gamma_\infty(\alpha) \triangleq \{\min \gamma^2$: LMIs in Theorem 6 are feasible$\}$.

Remark 2. Theorem 6 implies that if the performance measure $J_{\bigstar,\infty} < \gamma^2$ is achieved by some controller of order $k \geq n$, there exists a controller of order n also rendering $J_{\bigstar,\infty} < \gamma^2$.

The LMI-based approach introduced in Theorem 6 is also useful for synthesizing reduced-order controllers. These γ-suboptimal controllers of order $k < n$ correspond to pairs of (R,S) satisfying (14)–(16) and the additional rank constraint $\text{rank}(I - RS) = k$. Note that this additional constraint is non-convex in R and S, making the problem harder to solve. A detailed discussion on reduced-order controller design can be found in Gahinet and Apkarian (1994).

Finally, for completeness we state the necessary and sufficient conditions for the existence of γ-suboptimal output feedback controllers for mixed $\bigstar/\mathscr{H}_2$ control problems.

Theorem 7. *Consider a discrete time FDLTI plant P of McMillan degree n with the minimal realization* (13). *Assume that* (A, B_2, C_2) *is stabilizable and detectable and* $D_{22} = 0$. *The suboptimal mixed* $\bigstar/\mathscr{H}_2$ *problem with parameter* γ, *i.e.* $J_{\bigstar,2} < \gamma^2$, *is solvable if and only if there exist a pair of symmetric matrices* (R,S) *in* $\Re^{n\times n}$ *such that the following inequalities are feasible*:

$$W_1^{\mathrm{T}}\left(\frac{1}{1-\alpha}ARA^{\mathrm{T}} - R + B_1B_1^{\mathrm{T}}\right)W_1 < 0, \tag{18}$$

$$N_Q^{\mathrm{T}}\begin{pmatrix} A^{\mathrm{T}}SA - (1-\alpha)S & A^{\mathrm{T}}SB_1 \\ B_1^{\mathrm{T}}SA & -I + B_1^{\mathrm{T}}SB_1 \end{pmatrix}N_Q < 0, \tag{19}$$

$$\mathrm{Trace}(C_2RC_2^{\mathrm{T}} + D_{21}D_{21}^{\mathrm{T}}) < 1, \tag{20}$$

$$\alpha\gamma^2 I - C_1RC_1^{\mathrm{T}} > 0; \qquad \begin{pmatrix} R & I \\ I & S \end{pmatrix} \geq 0, \tag{21}$$

where W_1 *and* N_Q *are any matrices whose columns form bases of the null spaces of* B_2^{T} *and* $(C_3 \;\; D_{31})$ *respectively. Moreover, the set of* γ*-suboptimal controllers of order k is nonempty if and only if* (18)–(21) *hold for some R, S which further satisfy the rank constraint* $\text{rank}(I - RS) \leq k$.

Proof. Omitted, for space reasons, follows along the same lines as in Theorem 6. □

5. A simple example

Consider a plant with the following state-space realization:

$$A = \begin{pmatrix} 2.7 & -23.5 & 4.6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad B_1 = B_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix};$$

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 1 & -2.5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix};$$

$$D_{12} = 0; \quad D_{21} = 0.1; \quad D_{22} = 0.01; \quad D_{31} = 0.1.$$

The design objective is to minimize $\|T_{z_1w}\|_1$ subject to $\|T_{z_2w}\|_\infty \sim \leq 15$. The LMI-based method introduced in Section 4 yields $J_{\bigstar,\infty} = 4470$ achieved at $\alpha \approx 0.162$. The corresponding (second order) controller is given by

$$K = \left[\begin{array}{cc|c} -0.7927 & 0.3796 & -0.0981 \\ 2.9893 & -1.4317 & 0.1790 \\ \hline 15.7189 & 60.1616 & 17.3008 \end{array}\right]$$

and yields $\|T_{z_1w}\|_1 = 33.49$. For benchmarking purposes we also synthesized an optimal $\ell_1/\mathscr{H}_\infty$ controller using the convex optimization method described in Sznaier and Bu (1998). This method yields a seventh-order controller (after model reduction) with optimal cost $\|T_{z_1w}\|_1 = 32.03$. It is worth noticing that the different between the optimal $\|T_{z_1w}\|_1$ and the one achieved using the proposed LMI approach is rather small. Moreover, achieving the additional performance entails a

Table 1
Results for the mixed $\ell_1/\mathscr{H}_\infty$ problem

Type	Controller order	$\|T_{z_1w}\|_1$	$\|T_{z_1w}\|_\bigstar$	$\|T_{z_2w}\|_\infty$
Optimal ℓ_1	16	23.07	45.02	32.60
Optimal $\mathscr{H}_\infty$	3	83.86	128.2	9.862
Optimal $\ell_1/\mathscr{H}_\infty$	7 (red.)	32.03	47.66	15.00
Mixed $\bigstar/\mathscr{H}_\infty$	2	33.49	50.73	14.95

substantial increase in the controller order. These results are summarized in Table 1.

6. Conclusions

Multiple objective control problems have attracted much attention lately since they allow for simultaneously addressing several different, sometimes conflicting, performance specifications. In this paper we consider discrete time mixed $\ell_1/\mathscr{H}_2$ and $\ell_1/\mathscr{H}_\infty$ problems, where the ℓ_1 norm of a certain closed-loop transfer function is minimized, while maintaining the $\mathscr{H}_2$ norm (or the $\mathscr{H}_\infty$ norm) of a different transfer function below a prespecified level.

By exploiting upper bounds of the ℓ_1 and $\mathscr{H}_2$ ($\mathscr{H}_\infty$) norms given in terms of Linear Matrices Inequalities, we defined alternative performance measures for both problems. The main result of the paper shows that controllers optimizing these performance criteria can be synthesized via a two-stage process, involving an LMI optimization problem and a line search over $\alpha \in (0,1)$. Furthermore, we present necessary and sufficient conditions for existence of γ-suboptimal controllers, including reduced-order ones. As a byproduct of these conditions we established that in the state-feedback case optimal performance can be achieved with *static* controllers, while in the output feedback case the controller has the same order as the plant. These results are illustrated with a simple example in Section 5, where the proposed controller compares favorably with the exact, high-order solution. Consistent numerical experience suggests that the second step in the design process, i.e. the line search over $\alpha \in (0,1)$, is a convex optimization problem, although no formal proof of this fact exists at the present. Finally, our approach also allows for synthesizing output feedback controllers having a lower order than the plant, by imposing an additional (non-convex) rank constraint.

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