Proceedings of the American Control Conference
Albuquerque, New Mexico June 1997
0-7803-3832-4/97/$10.00 © 1997 AACC

A Linear Matrix Inequality Approach to Synthesizing Low Order Mixed $l_1/\mathcal{H}_p$ Controllers [1]

Juanyu Bu Mario Sznaier[2]
Department of Electrical Engineering
The Pennsylvania State University
University Park, PA 16802
msznaier@frodo.ee.psu.edu

Abstract

Mixed objective control problems have attracted much attention lately since they allow for capturing different performance specifications without resorting to approximations or the use of weighting functions, thus eliminating the need for trial and error type iterations. This paper addresses the problem of designing stabilizing controllers that minimize the l_1 norm of a certain closed-loop transfer function, while maintaining the $\mathcal{H}_2$ norm (mixed $l_1/\mathcal{H}_2$), or the $\mathcal{H}_\infty$ norm (mixed $l_1/\mathcal{H}_\infty$), of a different transfer function below a prespecified level. Based on a linear matrix inequality approach, the main results of this paper show that, suboptimal controllers can be synthesized by a two-stage process, involving an LMI optimization problem and a line search over $(0,1)$. Furthermore, this approach also provides an LMI-based parameterization of all suboptimal output feedback controllers, including reduced order ones, for mixed $l_1/\mathcal{H}_\infty$ and $l_1/\mathcal{H}_2$ problems.

1. Introduction

During the past decade a powerful robust control framework has been developed addressing issues of stability and performance in the presence of norm-bounded uncertainties. Robust stability and performance are achieved by minimizing a suitably weighted norm (either $\|\cdot\|_\infty$ [7, 16] or $\|\cdot\|_1$ [5, 6]) of a closed-loop transfer function. This framework has gained wide acceptance among control engineers, since it embodies many desirable design objectives.

However, this framework is limited by the fact that in its context, performance is measured in the same norm used to assess stability. It is often the case that the controller is required to meet several different, sometimes conflicting goals, such as simultaneous rejection of disturbances having different characteristics (white noise, bounded energy, persistent); good tracking of classes of inputs; satisfaction of bounds on peak values of some outputs; closed-loop bandwidth; etc. Clearly, a single norm will not be enough to address these diverse specifications. Hence, mixed objective control problems have attracted much attention lately since they allow for directly capturing different performance specifications without resorting to approximations or the use of weighting functions, thus eliminating the need for trial and error type iterations. In particular, $\mathcal{H}_2/\mathcal{H}_\infty$ (see [1, 10] and references therein) and $l_1/\mathcal{H}_\infty$ [4, 14, 15] mixed control problems have been extensively investigated since its introduction. More recently $l_1/\mathcal{H}_2$ [13] control problems have also been formulated.

This paper addresses the problem of designing stabilizing controllers that minimize the l_1 norm of a certain closed-loop transfer function, while maintaining the $\mathcal{H}_2$ norm (mixed $l_1/\mathcal{H}_2$), or the $\mathcal{H}_\infty$ norm (mixed $l_1/\mathcal{H}_\infty$), of a different transfer function below a prespecified level. This problem arises in the context of rejecting both bounded persistent and stochastic (or bounded energy) disturbances.

Both discrete time mixed $l_1/\mathcal{H}_2$ and $l_1/\mathcal{H}_\infty$ problems can be solved by using the Youla parameterization to cast the problem into a (infinite-dimensional) constrained convex optimization [13, 15]. However, as in the pure l_1 optimal control, it has been shown in [13, 15] that the order of the controller is not bounded by the order of the plant, and could be arbitrarily high. Motivated by the complexity of these controllers, an alternative approach will be introduced in this paper, based upon recent results on synthesizing low order l_1 controllers [3, 11]. By using upper bounds of the l_1 and $\mathcal{H}_2$ ($\mathcal{H}_\infty$) norms given in terms of Linear Matrices Inequalities, a modified problem can be obtained such that its solution is feasible and upper-bounds the original problem. The main result of this paper shows that suboptimal controllers can be synthesized by a two-stage process, involving an LMI optimization problem and a line search over $(0,1)$. Additional results include the facts that in the state-feedback case optimal performance can be achieved using static controllers, while in the output feedback case it can be achieved using controllers with McMillan degree less or equal than that of the plant. Moreover, as in the discrete time $\mathcal{H}_\infty$ case [8, 9], our approach also provides an LMI-based parameterization of all suboptimal output feedback controllers, including re-

[1]This work was supported in part by NSF grant ECS-9625920
[2]Corresponding author.

duced order ones. The latter can be obtained by imposing a rank condition on the positive definite solution to a set of LMIs, at the price of destroying the overall convexity of the problem. Nevertheless, as we illustrate with an example, problems having this additional constraint can still be efficiently solved.

The paper is organized as follows: In section 2 we introduce the notation to be used and some preliminary results. In section 3 and 4 we show that when suitably modified, both the mixed $l_1/\mathcal{H}_2$ and $l_1/\mathcal{H}_\infty$ can be reduced to an LMI optimization problem and a line search in $(0,1)$. These results are illustrated in 5 with some simple design examples. Finally, in section 6, we summarize our results and we present some concluding remarks. Due to space limitations, proofs for some theorems are omitted.

2. Preliminaries

2.1. Notation

Given a vector $x \in R^n$, its 1−norm is defined as $\|x\|_1 \doteq \sum_{i=0}^{n} |x_i|$ and its infinity norm as $\|x\|_\infty \doteq \max_i |x_i|$. l_1 denotes the space of absolutely summable sequences $h = \{h_i\}$ equipped with the norm $\|h\|_{l_1} \doteq \sum_{i=0}^{\infty} |h_i| < \infty$. l_∞ denotes the space of bounded sequences $h = \{h_i\}$ equipped with the norm $\|h\|_{l_\infty} \doteq \sup_{i \geq 0} |h_i| < \infty$. Similarly, l_∞^p denotes the space of bounded vector sequences $\{h(k) \in R^p\}$. In this space we define the norm $\|h\|_{l_\infty} \doteq \sup_i \|h_i(k)\|_\infty$. Alternatively, in this space we will also consider the norm: $\|h\|_{\infty,e} \doteq \sup_k \{h'(k)h(k)\}^{1/2}$, i.e. the supremum over time of the pointwise *euclidean* norm of the vector $h(k)$.

Assume now that $H : l_\infty^q \rightarrow l_\infty^p$ is a bounded linear operator defined by the usual convolution relation $y = H * u$. Its induced $l_\infty^q \rightarrow l_\infty^p$ norm will be denoted as $\|H\|_{l_\infty \rightarrow l_\infty} \doteq \|H\|_1$. It is a standard result that $\|H\|_1 = \max_i \sum_{j=1}^{q} \|H_{ij}\|_{l_1}$. Similarly, the operator norm induced by $\|.\|_{\infty,e}$ will be denoted by $\|H\|_{1,e}$, i.e. $\|H\|_{1,e} \doteq \sup_{\|v\|_{\infty,e} \leq 1} \|H * v\|_{\infty,e}$. Note that for scalar signals these norms coincide, while in the general case we have $\frac{1}{\sqrt{q}}\|H\|_1 \leq \|H\|_{1,e} \leq \sqrt{p}\|H\|_1$.

$\mathcal{H}_\infty$ denotes the space of complex valued matrix functions that are analytic outside the unit disk. The norm on $\mathcal{H}_\infty$ is defined by $\|G(z)\|_\infty \doteq ess\sup_{|z|>1} \overline{\sigma}(G(z))$, where $\overline{\sigma}$ denotes the largest singular value. By $\mathcal{H}_2$ we denote the space of complex valued matrix functions that are analytic outside the unit disk and square integrable on the unit circle, with $\|G(z)\|_2$ defined as:

$$\|G(z)\|_2^2 = \frac{1}{2\pi} \oint_{|z|=1} \mathrm{trace}(G(z)^* G(z)) \frac{dz}{z}$$

where $*$ denotes complex conjugate.

2.2. Some Useful Results

The following lemmas related to linear matrix inequalities play a central role in our approach.

Lemma 2.1 ([8, 9]) *Given a symmetric matrix $\Psi \in \Re^{m \times m}$ and two matrices P, Q of column dimension m, there exists some matrix Θ such that $\Psi + P^T \Theta^T Q + Q^T \Theta P < 0$ if and only if $W_P^T \Psi W_P < 0$ and $W_Q^T \Psi W_Q < 0$, where W_P, W_Q are any matrices whose columns form bases of the null space of P and Q respectively.*

Lemma 2.2 (Schur Complements [2, 8]) *The block matrix $\begin{pmatrix} P & M \\ M^T & Q \end{pmatrix}$ is negative definite if and only if $Q < 0$ and $P - MQ^{-1}M^T < 0$.*

Next we recall some well known results [8, 9, 10] about the discrete time $\mathcal{H}_2$-norm and $\mathcal{H}_\infty$-norm.

Lemma 2.3 *Consider a discrete time stable FDLTI system $G(z) = D + C(zI - A)^{-1}B$. Then, $\|G\|_2^2 = trace(CXC^T + DD^T)$, where $X > 0$ satisfies*

$$AXA^T - X + BB^T = 0 \tag{1}$$

Lemma 2.4 *Consider a discrete time transfer function $T(z) = D + C(zI - A)^{-1}B$. Then, $\|T\|_\infty < 1$ and A is stable if and only if there exists $X = X^T > 0$ such that* $\begin{pmatrix} -X^{-1} & A & B & 0 \\ A^T & -X & 0 & C^T \\ B^T & 0 & -I & D^T \\ 0 & C & D & -I \end{pmatrix} < 0.$

Finally we recall a result on the upper bound of the $\|\cdot\|_{1,e}$ norm of a stable FDLTI system.

Lemma 2.5 ([3]) *Consider the proper, stable FDLTI system $G(z) = D + C(zI - A)^{-1}B$. Then $\|G\|_{1,e}^2 \leq \|G\|_*^2 \triangleq \inf_{\alpha \in (0,1)} \{\gamma^2 : \exists \sigma > 0 s.t. L(Q) > 0\}$ where* $\mathrm{L(Q)} = \begin{pmatrix} \alpha\sigma Q^{-1} & 0 & C^T \\ 0 & (\gamma^2 - \sigma)I & D^T \\ C & D & I \end{pmatrix}$ *and $Q > 0$ satisfies*

$$\frac{1}{1-\alpha} AQA^T - Q + BB^T = 0 \tag{2}$$

3. Mixed $l_1/\mathcal{H}_2$ and $l_1/\mathcal{H}_\infty$ Performance Measures

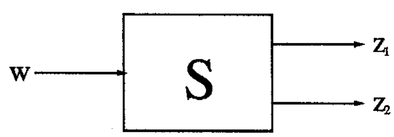

Figure 1: The generalized plant for analysis

3.1. Mixed $l_1/\mathcal{H}_2$ Problem Formulation

Consider the FDLTI system S shown in Figure 1, where w represents an exogenous disturbance and z_1, z_2

represent performance outputs. Assume S has the following state–space realization:

$$S = \left[\begin{array}{c|c} A & B \\ \hline C_1 & D_1 \\ C_2 & D_2 \end{array}\right] \quad (3)$$

Lemma 3.1 *Let Q and X denote any positive definite solutions to (1) and (2) respectively. Then $Q \geq X > 0$ and $\|T_{z_2w}\|_2^2 \leq trace(C_2QC_2^T + D_2D_2^T)$.*

Motivated by Lemma 3.1, we can define the mixed $l_1/\mathcal{H}_2$ performance mesure as $J_{1,2} \triangleq \inf_{\alpha\in(0,1)}\{\gamma^2 : \exists\sigma > 0 s.t. L(Q) > 0\}$ where $Q > 0$ satisfies (2) and $\text{trace}(C_2QC_2^T + D_2D_2^T) < \beta^2$. Based on this performance measure, the mixed $*/\mathcal{H}_2$ control problem can be formulated as follows:

Problem 1 ($*/\mathcal{H}_2$) *Given the system shown in Figure 2, and where the plant P has the following state–space realization:*

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & 0 & D_{22} \\ C_3 & 0 & 0 \end{array}\right] \quad (4)$$

find an internally stabilizing controller K such that $J_{1,2}(T_{z_1w})$ is minimized.

For simplicity, we have assumed here that the open loop plant P given in (4) is strictly proper, i.e., D_{11}, D_{21}, D_{31}, $D_{32} = 0$, and $\beta = 1$.

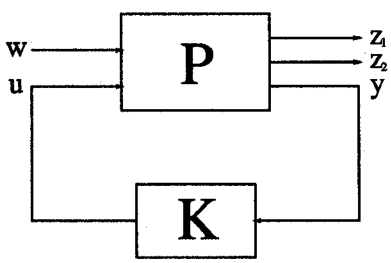

Figure 2: The setup for controller synthesis

3.2. Mixed $l_1/\mathcal{H}_\infty$ Performance Measure

Lemma 3.2 *Consider the system (3) and for a given $\alpha \in (0,1)$, denote by Q the positive definite solution to (2). Let $Y = Y^T > 0$ be any positive definite solution to the following inequality:*

$$P(Y) = \begin{pmatrix} -Y & A & B & 0 \\ A^T & (\alpha-1)Y^{-1} & 0 & C_2^T \\ B^T & 0 & -I & D_2^T \\ 0 & C_2 & D_2 & -I \end{pmatrix} < 0 \quad (5)$$

Then, $Y \geq Q > 0$ and $\|T_{z_2w}(\hat{z})\|_\infty < 1$, where $z \to \hat{z}$ is defined by the mapping $\hat{z} = \left(\sqrt{1-\alpha}\right) z$.

Remark 1 *Note that the transformation $z \to \left(\sqrt{1-\alpha}\right) z$ maps the unit disk into a disk with radius $\delta \doteq \sqrt{1-\alpha}$. Therefore, from the Maximum Modulus Theorem it follows that $\|T(z)\|_\infty \leq \|T(\hat{z})\|_\infty$. Moreover $\|T(z)\|_\infty \uparrow \|T(\hat{z})\|_\infty$ as $\alpha \downarrow 0$. This transformation is similar to the transformation used in [14] to decouple the mixed $l_1/\mathcal{H}_\infty$ problem into a convex finite–dimensional optimization and an unconstrained $\mathcal{H}_\infty$ problem.*

As in the mixed $l_1/\mathcal{H}_2$ case, based upon Lemma 3.2 we define the following mixed $l_1/\mathcal{H}_\infty$ performance measure $J_{1,\infty} \triangleq \inf_{\alpha\in(0,1)}\{\gamma^2 : \exists\sigma > 0 s.t. L(X) > 0\}$ where $X = X^T > 0$ satisfies $P(X) < 0$.

Problem 2 ($*/\mathcal{H}_\infty$) *Given the system in Figure 2, and P by (4), find an internally stabilizing controller K such that $J_{1,\infty}(T_{z_1w})$ is minimized.*

3.3. State Feedback Controllers Synthesis

In this section, we analyze the structure of the optimal solutions to Problems 1 and 2. The main result shows that for the state feedback case the optimal cost over the set of stabilizing controllers is achieved by static feedback controllers.

Theorem 3.1 *Consider the system P given in (4) and assume $C_3 = I$ (i.e. state feedback). Then, there exists a static control law $u = Kx$ rendering $J_{1,2}(T_{z_1w}) < \gamma^2$ if and only if the following LMIs (in the variables Q, V and S) admit a solution:*

$$\begin{pmatrix} \gamma^2 I & C_1Q + D_{12}V \\ QC_1^T + V^TD_{12}^T & \alpha Q \end{pmatrix} > 0$$

$$\begin{pmatrix} S & C_2Q + D_{22}V \\ QC_2^T + V^TD_{22}^T & Q \end{pmatrix} > 0$$

$$\text{trace}(S) < 1; \quad \begin{pmatrix} -Q + B_1B_1^T & AQ + B_2V \\ QA^T + V^TB_2 & (\alpha-1)Q \end{pmatrix} \leq 0$$

The static controller K is given by $K = VQ^{-1}$.

As in [3], the optimal solution to problem 1 can be obtained by minimizing a function $V_{1,2}(\alpha)$ over $\alpha \in (0,1)$, where $V_{1,2}(\alpha)$ is defined as $V_{1,2}(\alpha) = \{\min\gamma^2 :$ LMIs in Theorem 3.1 are feasible$\}$.

Theorem 3.2 *Consider the system P given in (4) in the state feedback case $C_3 = I$. Then, there exists a static control law $u = Kx$, $K = VQ^{-1}$, such that $J_{1,\infty}(T_{z_1w}) < \gamma^2$, where V, Q satisfy:*

$$\begin{pmatrix} \gamma^2 I & C_1Q + D_{12}V \\ QC_1^T + V^TD_{12}^T & \alpha Q \end{pmatrix} > 0$$

$$\begin{pmatrix} -Q & AQ+B_2V & B_1 & 0 \\ (AQ+B_2V)^T & (\alpha-1)Q & 0 & (C_2Q+D_{22}V)^T \\ B_1^T & 0 & -I & 0 \\ 0 & C_2Q+D_{22}V & 0 & -I \end{pmatrix} < 0$$

4. All Output Feedback Controllers for Mixed $*/\mathcal{H}_\infty$ Problems

In this section we establish necessary and sufficient conditions for the existence of γ-suboptimal output feedback controllers for mixed $*/\mathcal{H}_\infty$ and $*/\mathcal{H}_2$ control problems.

Theorem 4.1 *Consider a discrete time FDLTI plant P of McMillan degree n with a minimal realization:*

$$\begin{pmatrix} z_1 \\ z_2 \\ y \end{pmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & D_{22} \\ C_3 & D_{31} & 0 \end{array}\right] \begin{pmatrix} w \\ u \end{pmatrix} \qquad (6)$$

Assume that (A, B_2, C_2) is stabilizable and detectable. Then, $J_{1,\infty} < \gamma^2$ if and only if there exist pairs of symmetric matrices (R, S) in $\Re^{n\times n}$ and $\sigma > 0$ such that:

$$\mathrm{N_P^T}\begin{pmatrix} \beta ARA^T - R + B_1B_1^T & \beta ARC_2^T + B_1D_{21}^T \\ \beta C_2RA^T + D_{21}B_1^T & -I + \beta C_2RC_2^T + D_{21}D_{21}^T \end{pmatrix}\mathrm{N_P} < 0$$

$$\mathrm{N_Q^T}\begin{pmatrix} A^TSA - \frac{1}{\beta}S + C_2^TC_2 & A^TSB_1 + C_2^TD_{21} \\ B_1^TSA + D_{21}^TC_2 & -I + B_1^TSB_1 + D_{21}^TD_{21} \end{pmatrix}\mathrm{N_Q} < 0$$

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix} \geq 0; \quad \mathrm{W_3^T}(\alpha\sigma \mathrm{I} - \mathrm{C_1RC_1^T})\mathrm{W_3} > 0$$

$$\begin{pmatrix} \alpha S & C_1^T \\ C_1 & \sigma I \end{pmatrix} > 0; \quad \gamma^2 - \sigma > 0$$

where $\beta = \frac{1}{1-\alpha}$, N_P, N_Q, and W_3 are any matrices whose columns form bases of the null spaces of $(\,B_2^T \quad D_{22}^T\,)$, $(\,C_3 \quad D_{31}\,)$, and D_{12}^T respectively. Moreover, the set of γ-suboptimal controllers of order k is nonempty if and only if the above inequalities hold for some R, S which further satisfy the rank constraint rank $(I - RS) \leq k$.

Remark 2 *Theorem 4.1 implies that if the performance measure $J_{1,\infty} < \gamma^2$ is achieved by some controller of order $k \geq n$, there exists a controller of order n also rendering $J_{1,\infty} < \gamma^2$. It follows that in the output feedback case optimal performance can be always achieved with controllers having the same order as the generalized plant.*

The LMI-based approach introduced in Theorem 4.1 is also useful for synthesizing reduced–order controllers. These γ-suboptimal controllers of order $k < n$ correspond to pairs of (R,S) satisfying conditions in Theorem 4.1 and the additional rank constraint $rank(I - RS) = k$. Note that this additional constraint is non–convex in R and S, making the problem harder to solve. A detailed discussion on reduced–order controller design can be found in [8]. For completeness, we state necessary and sufficient conditions for the existence of γ-suboptimal output feedback controllers for mixed $*/\mathcal{H}_2$ problems.

Theorem 4.2 *Consider the plant P of McMillan degree n with the minimal realization (6). Assume that (A, B_2, C_2) is stabilizable and detectable. Then, $J_{1,2} < \gamma^2$ if and only if there exist pairs of symmetric matrices (R, S) in $\Re^{n\times n}$, $\sigma > 0$, and a symmetric matrix Λ such that:*

$$\mathrm{W_1^T}\left(\frac{1}{1-\alpha}\mathrm{ARA^T} - \mathrm{R} + \frac{1}{\alpha}\mathrm{B_1B_1^T}\right)\mathrm{W_1} < 0$$

$$N_Q^T\begin{pmatrix} A^TSA - (1-\alpha)S & A^TSB_1 \\ B_1^TSA & -\alpha I + B_1^TSB_1 \end{pmatrix}N_Q < 0$$

$$\mathrm{W_2^T}(\Lambda - \mathrm{D_{21}D_{21}^T} - \mathrm{C_2RC_2^T})\mathrm{W_2} > 0$$

$$\begin{pmatrix} N_Q^T & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} S & 0 & C_2^T \\ 0 & I & D_{21}^T \\ C_2 & D_{21} & \Lambda \end{pmatrix}\begin{pmatrix} N_Q & 0 \\ 0 & I \end{pmatrix} > 0$$

$$\mathrm{W_3^T}(\alpha\sigma\mathrm{I} - \mathrm{C_1RC_1^T})\mathrm{W_3} > 0; \quad \begin{pmatrix} \alpha S & C_1^T \\ C_1 & \sigma I \end{pmatrix} > 0$$

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix} \geq 0; \quad \gamma^2 - \sigma > 0; \quad \mathrm{trace}(\Lambda) < 1.$$

where $N_P = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$, N_Q, and W_3 are any matrices whose columns form bases of the null spaces of $(\,B_2^T \quad D_{22}^T\,)$, $(\,C_3 \quad D_{31}\,)$, and D_{12}^T respectively. Moreover, the set of γ-suboptimal controllers of order k is nonempty if and only if the above conditions hold for some R, S which further satisfy the rank constraint rank $(I - RS) \leq k$.

5. A Simple Example

We use the next example to illustrate the synthesis of reduced-order output feedback controllers for mixed $*/\mathcal{H}_\infty$ problems. Consider the plant given by the following state–space realization:

$$\mathrm{A} = \begin{pmatrix} 2.7 & -23.5 & 4.6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \mathrm{B_1} = \mathrm{B_2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \mathrm{D_{12}} = 0;$$

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 1 & -2.5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \begin{pmatrix} D_{21} \\ D_{22} \\ D_{31} \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.01 \\ 0.1 \end{pmatrix}$$

The design objective is to minimize $\|T_{z_1w}\|_1$ subject to $\|T_{z_2w}\|_\infty \sim\leq 15$. By using the LMI–based parameterization introduced in Section 4, the optimal performance measure $J_{1,\infty}$ over the set of all controllers is obtained by minimizing the function $\Gamma_{1,\infty}(\alpha)$. Figure 3 shows the plot of $\Gamma_{1,\infty}(\alpha)$ versus α. Note that this plot is convex, with a minimum $J_{1,\infty} = 4470$ achieved for $\alpha \approx 0.162$. The corresponding reduced-order controller is given by

$$K = \left[\begin{array}{cc|c} -0.7927 & 0.3796 & -0.0981 \\ 2.9893 & -1.4317 & 0.1790 \\ \hline 15.7189 & 60.1616 & 17.3008 \end{array}\right]$$

and yields $\|T_{z_1w}\|_1 = 33.49$. For benchmarking purposes we also synthesized an optimal $l_1/\mathcal{H}_\infty$ controller using the convex optimization method described in [15]. This method yields a 7th order controller (after model reduction) with optimal cost $\|T_{z_1w}\|_1 = 32.03$. It is worth noticing that the different between the optimal $\|T_{z_1w}\|_1$ and the one achieved using the proposed LMI approach is rather small. Moreover, achieving the additional performance entails a substantial increase in the controller order. These results are summarized in Table 1.

Type	order	$\|\|T_{z_1 w}\|\|_1$	$\|\|T_{z_1 w}\|\|_*$	$\|\|T_{z_2 w}\|\|_\infty$
optimal l_1	16	23.07	45.02	32.60
optimal $\mathcal{H}_\infty$	3	83.86	128.2	9.862
optimal $l_1/\mathcal{H}_\infty$	7	32.03	47.66	15.00
optimal $*/\mathcal{H}_\infty$	2	33.49	50.73	14.95

Table 1: Results for the mixed $l_1/\mathcal{H}_\infty$ problem

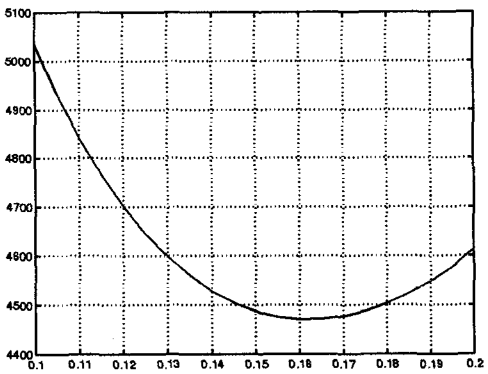

Figure 3: Evaluation of the function $\Gamma_{1,\infty}(\alpha)$ vs. α

6. Conclusions

Multiple objective control problems have attracted much attention lately since they allow for simultaneously addressing several different, sometimes conflicting performance specifications. In this paper we consider discrete time mixed $l_1/\mathcal{H}_2$ and $l_1/\mathcal{H}_\infty$ problems, where the l_1 norm of a certain closed–loop transfer function is minimized, while maintaining the $\mathcal{H}_2$ norm (or the $\mathcal{H}_\infty$ norm) of a different transfer function below a prespecified level.

By exploiting upper bounds of the l_1 and $\mathcal{H}_2$ ($\mathcal{H}_\infty$) norms given in terms of Linear Matrices Inequalities, we defined alternative performance measures for both problems. The main result of the paper shows that controllers optimizing these performance criteria can be synthesized via a two–stage process, involving an LMI optimization problem and a line search over $\alpha \in (0,1)$. Furthermore, we present necessary and sufficient conditions for existence of γ-suboptimal controllers, including reduced–order ones. As a byproduct of these conditions we establish that in the state–feedback case optimal performance can be achieved with *static* controllers, while in the output feedback case the controller has the same order as the plant. These results are illustrated through some simple examples in Section 5, where the proposed controllers compare favorably with the exact, high order solutions. Consistent numerical experience suggests that the second step in the design process, i.e. the line search over $\alpha \in (0,1)$, is a convex optimization problem, although no formal proof of this fact exists at the present. Finally, our approach also allows for synthesizing output feedback controllers having a lower order than the plant, by imposing an additional rank constraint, at the price of losing overall convexity.

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